Homework 1
"Applied Methods of Linear Algebra"
variant 50

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Task 1

Description

Find a singular value decomposition for the matrix

$$A = \begin{pmatrix} 20 & -110 & -6 & 13 \\ -62 & -16 & -48 & -118 \\ -52 & 64 & 84 & 46 \end{pmatrix}$$

Solution

Singular value decomposition:

$$A_{m\times n} = U_{m\times m}\cdot\Sigma_{m\times n}\cdot {V^*}_{n\times n}$$

where U and V are complex unitary matrices, Σ is a rectangular diagonal matrix with non-negative real numbers on the diagonal. As A is real, U and V are real orthogonal.

The first step is calculating eigenvalues of $A\cdot A^*$:

$$det\Big(A\cdot A^* - \lambda\cdot I\Big) = det\begin{pmatrix} 12705-\lambda & -726 & -7986 \\ -726 & 20328-\lambda & -7260 \\ -7986 & -7260 & 15972-\lambda \end{pmatrix} = \big(\underbrace{27225}_{=\lambda_1}-\lambda\big)\cdot\big(\underbrace{17424}_{=\lambda_2}-\lambda\big)\cdot\big(\underbrace{4356}_{=\lambda_3}-\lambda\big)$$

The second step is calculating corresponding eigenvectors:

$$\Big(A\cdot A^* - \lambda_i\cdot I\Big)\cdot x_i = 0 \quad\Rightarrow\quad \begin{cases} v_1 = \big(-1, -2, 2\big)^T \\ v_2 = \big(-2, 2, 1\big)^T \\ v_3 = \big(2, 1, 2\big)^T \end{cases}$$

The third step is orthonormalizing these eigenvectors. Let us use Gram-Schmidt process:

$$u_i = v_i - \sum_{k=1}^{i-1}\frac{(v_i, u_k)}{(u_k, u_k)}\cdot u_k \qquad e_i = \frac{u_i}{||u_i||}$$

$$\begin{cases} e_1 = \Big({}^{-1}/_3, {}^{-2}/_3, {}^{2}/_3\Big)^T \\ e_2 = \Big({}^{-2}/_3, {}^{2}/_3, {}^{1}/_3\Big)^T \\ e_3 = \Big({}^{2}/_3, {}^{1}/_3, {}^{2}/_3\Big)^T \end{cases} \quad\Rightarrow\quad U = \begin{pmatrix} {}^{-1}/_3 & {}^{-2}/_3 & {}^{2}/_3 \\ {}^{-2}/_3 & {}^{2}/_3 & {}^{1}/_3 \\ {}^{2}/_3 & {}^{1}/_3 & {}^{2}/_3 \end{pmatrix}$$

Then, the same calculations should be made with $A^*\cdot A$:

$$det\Big(A^*\cdot A - \lambda\cdot I\Big) = det\begin{pmatrix} 6948-\lambda & -4536 & -1512 & 5184 \\ -4536 & 16452-\lambda & 6804 & 3402 \\ -1512 & 6804 & 9396-\lambda & 9450 \\ 5184 & 3402 & 9450 & 16209\cdot\lambda \end{pmatrix} = \big(\lambda-\underbrace{27225}_{=\lambda_1}\big)\cdot\big(\lambda-\underbrace{17424}_{=\lambda_2}\big)\cdot\big(\lambda-\underbrace{4356}_{=\lambda_3}\big)\cdot\lambda$$

$$\Big(A^*\cdot A - \lambda_i\cdot I\Big)\cdot x_i = 0 \quad\Rightarrow\quad \begin{cases} v_1 = \big(0, 6, 6, 7\big)^T \\ v_2 = \big(6, -7, 0, 6\big)^T \\ v_3 = \big(-7, -6, 6, 0\big)^T \\ v_4 = \big(-6, 0, -7, 6\big)^T \end{cases}$$

$$\begin{cases} e_1 = \Big(0, {}^{6}/_{11}, {}^{6}/_{11}, {}^{7}/_{11}\Big)^T \\ e_2 = \Big({}^{-6}/_{11}, {}^{7}/_{11}, 0, {}^{-6}/_{11}\Big)^T \\ e_3 = \Big({}^{-7}/_{11}, {}^{-6}/_{11}, {}^{6}/_{11}, 0\Big)^T \\ e_3 = \Big({}^{-6}/_{11}, 0, {}^{-7}/_{11}, {}^{6}/_{11}\Big)^T \end{cases} \quad\Rightarrow\quad V = \begin{pmatrix} 0 & {}^{-6}/_{11} & {}^{-7}/_{11} & {}^{-6}/_{11} \\ {}^{6}/_{11} & {}^{7}/_{11} & {}^{-6}/_{11} & 0 \\ {}^{6}/_{11} & 0 & {}^{6}/_{11} & {}^{-7}/_{11} \\ {}^{7}/_{11} & {}^{-6}/_{11} & 0 & {}^{6}/_{11} \end{pmatrix}$$

The last step is getting Σ:

$$\Sigma = \begin{pmatrix} \sqrt{\lambda_1} & 0 & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 & 0 \\ 0 & 0 & \sqrt{\lambda_3} & 0 \end{pmatrix} = \begin{pmatrix} 165 & 0 & 0 & 0 \\ 0 & 132 & 0 & 0 \\ 0 & 0 & 66 & 0 \end{pmatrix}$$

Answer

Singular value decomposition:

$$\underbrace{\begin{pmatrix} 20 & -110 & -6 & 13 \\ -62 & -16 & -48 & -118 \\ -52 & 64 & 84 & 46 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} {}^{-1}/_{3} & {}^{-2}/_{3} & {}^{2}/_{3} \\ {}^{-2}/_{3} & {}^{2}/_{3} & {}^{1}/_{3} \\ {}^{2}/_{3} & {}^{1}/_{3} & {}^{2}/_{3} \end{pmatrix}}_{U} \cdot \underbrace{\begin{pmatrix} 165 & 0 & 0 & 0 \\ 0 & 132 & 0 & 0 \\ 0 & 0 & 66 & 0 \end{pmatrix}}_{\Sigma} \cdot \underbrace{\begin{pmatrix} 0 & {}^{6}/_{11} & {}^{6}/_{11} & {}^{7}/_{11} \\ {}^{-6}/_{11} & {}^{7}/_{11} & 0 & {}^{-6}/_{11} \\ {}^{-7}/_{11} & {}^{-6}/_{11} & {}^{6}/_{11} & 0 \\ {}^{-6}/_{11} & 0 & {}^{-7}/_{11} & {}^{6}/_{11} \end{pmatrix}}_{V^*}$$

Task 2

Description

Find a full rank decomposition and the pseudoinverse of the matrix

$$A = \begin{pmatrix} -1 & 4 & 9 \\ -2 & 7 & 12 \\ 4 & -11 & -6 \\ 3 & -8 & -3 \end{pmatrix}$$

Solution

Full rank decomposition:

$$A_{m,n} = F_{m,rank(A)} \cdot G_{rank(A),n}$$

Gaussian elimination:

$$A = \begin{pmatrix} -1 & 4 & 9 \\ -2 & 7 & 12 \\ 4 & -11 & -6 \\ 3 & -8 & -3 \end{pmatrix} \to \begin{pmatrix} -1 & 4 & 9 \\ 0 & -1 & -6 \\ 0 & 5 & 30 \\ 0 & 4 & 24 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 15 \\ 0 & 1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad F = \begin{pmatrix} -1 & 4 \\ -2 & 7 \\ 4 & -11 \\ 3 & -8 \end{pmatrix}, G = \begin{pmatrix} 1 & 0 & 15 \\ 0 & 1 & 6 \end{pmatrix}$$

Therefore:

$$A^+ = G^+ \cdot F^+$$

As far as G is a full row rank matrix:

$$\exists G^+ = G^* \cdot (G \cdot G^*)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 15 & 6 \end{pmatrix} \cdot \left[\begin{pmatrix} 1 & 0 & 15 \\ 0 & 1 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 15 & 6 \end{pmatrix} \right]^{-1} =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 15 & 6 \end{pmatrix} \cdot \frac{1}{262} \begin{pmatrix} 37 & -90 \\ -90 & 226 \end{pmatrix} = \frac{1}{262} \cdot \begin{pmatrix} 37 & -90 \\ -90 & 262 \\ 15 & 6 \end{pmatrix}$$

As far as F is a full column matrix:

$$\exists F^+ = (F^* \cdot F)^{-1} \cdot F^* = \left[\begin{pmatrix} -1 & -2 & 4 & 3 \\ 4 & 7 & -11 & -8 \end{pmatrix} \cdot \begin{pmatrix} -1 & 4 \\ -2 & 7 \\ 4 & -11 \\ 3 & -8 \end{pmatrix} \right]^{-1} \cdot \begin{pmatrix} -1 & -2 & 4 & 3 \\ 4 & 7 & -11 & -8 \end{pmatrix} =$$

$$= \frac{1}{52} \cdot \begin{pmatrix} 125 & 43 \\ 43 & 15 \end{pmatrix} \cdot \begin{pmatrix} -1 & -2 & 4 & 3 \\ 4 & 7 & -11 & -8 \end{pmatrix} = \frac{1}{52} \cdot \begin{pmatrix} 47 & 51 & 27 & 31 \\ 17 & 19 & 7 & 9 \end{pmatrix}$$

Thus:

$$A^+ = G^+ \cdot F^+ = \frac{1}{262} \cdot \begin{pmatrix} 37 & -90 \\ -90 & 262 \\ 15 & 6 \end{pmatrix} \cdot \frac{1}{52} \cdot \begin{pmatrix} 47 & 51 & 27 & 31 \\ 17 & 19 & 7 & 9 \end{pmatrix} = \frac{1}{13624} \cdot \begin{pmatrix} 209 & 177 & 369 & 337 \\ 224 & 388 & -596 & -432 \\ 807 & 879 & 447 & 519 \end{pmatrix}$$

Answer

Full rank decomposition:

$$\underbrace{\begin{pmatrix} -1 & 4 & 9 \\ -2 & 7 & 12 \\ 4 & -11 & -6 \\ 3 & -8 & -3 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} -1 & 4 \\ -2 & 7 \\ 4 & -11 \\ 3 & -8 \end{pmatrix}}_{F} \cdot \underbrace{\begin{pmatrix} 1 & 0 & 15 \\ 0 & 1 & 6 \end{pmatrix}}_{G}$$

Pseudoinverse of A:

$$A^+ = G^+ \cdot F^+ = \frac{1}{13624} \cdot \begin{pmatrix} 209 & 177 & 369 & 337 \\ 224 & 388 & -596 & -432 \\ 807 & 879 & 447 & 519 \end{pmatrix}$$

Task 3

Description

Find the minimal length least squares solution of the system of linear equations

$$\begin{cases} 4x-10y+11z-12t = 4 \\ 3x-7y+9z-13t = 7 \\ -x+5y+9z+15t = 6 \\ -2x+8y+7z+14t = 3 \end{cases}$$

Solution

Let us check consistence of the given system with the Gaussian elimination:

$$A|b = \left(\begin{array}{cccc|c} 4 & -10 & 11 & -12 & 4 \\ 3 & -7 & 9 & -13 & 7 \\ -1 & 5 & 9 & 15 & 6 \\ -2 & 8 & 7 & 14 & 3 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -5 & -9 & -15 & -6 \\ 0 & 8 & 36 & 32 & 25 \\ 0 & 10 & 47 & 48 & 28 \\ 0 & -2 & -11 & -16 & -15 \end{array}\right) \to$$

$$\to \left(\begin{array}{cccc|c} 1 & 0 & 37/2 & 25 & 63/2 \\ 0 & 1 & 11/2 & 8 & 15/2 \\ 0 & 0 & -8 & -32 & -47 \\ 0 & 0 & -8 & -32 & -35 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 37/2 & 25 & 63/2 \\ 0 & 1 & 11/2 & 8 & 15/2 \\ 0 & 0 & 1 & 4 & 47/8 \\ 0 & 0 & 0 & 0 & 12 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & -49 & 0 \\ 0 & 1 & 0 & -14 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

Thus, the given system is inconsistent, i.e. vector $u = A^+ \cdot b$ is the minimal length squares solution (proved at the lectures). Using the algorithm from the previous task:

$$F = \begin{pmatrix} 4 & -10 & 11 \\ 3 & -7 & 9 \\ -1 & 5 & 9 \\ -2 & 8 & 7 \end{pmatrix}, \quad G = \begin{pmatrix} 1 & 0 & 0 & -49 \\ 0 & 1 & 0 & -14 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

$$F^+ = (F^* \cdot F)^{-1} \cdot F^* = \frac{1}{16} \cdot \begin{pmatrix} -48 & 85 & 73 & 60 \\ -16 & 27 & -23 & 20 \\ 4 & -6 & 6 & -4 \end{pmatrix}, \quad G^+ = G^* \cdot (G \cdot G^*)^{-1} = \frac{1}{2614} \cdot \begin{pmatrix} 213 & -686 & 196 \\ -686 & 2418 & 56 \\ 196 & 56 & 17 \\ -49 & -14 & 4 \end{pmatrix}$$

$$A^+ = G^+ \cdot F^+ = \frac{1}{2614} \cdot \begin{pmatrix} 1536 & -1593 & 32503 & -1724 \\ -5536 & 6640 & -105356 & 6976 \\ -10236 & 18070 & 13112 & 12812 \\ 2592 & -4567 & -3231 & -3236 \end{pmatrix}$$

$$u = A^+ \cdot b = \frac{1}{2614} \cdot \begin{pmatrix} 184839 \\ -586872 \\ 202714 \\ -50695 \end{pmatrix}$$

Answer

The minimal length least squares solution:

$$u = \frac{1}{2614} \cdot \begin{pmatrix} 184839 \\ -586872 \\ 202714 \\ -50695 \end{pmatrix}$$

Task 4

Description

Find an interpolation polynomial that passes through the four points whose coordinates form the columns of the matrix

$$P = \begin{bmatrix} -2 & 0 & 1 & 3 \\ 12 & -8 & 0 & 6 \end{bmatrix}$$

Solution

Let us find a Lagrange interpolation polynomial:

$$L(x) = \sum_{i=0}^{n} y_i \frac{\upsilon\big(x_0, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n\big)}{\upsilon\big(x_0, \ldots, x_n\big)}, \quad \upsilon\big(x_0, \ldots, x_n\big) = det \begin{pmatrix} 1 & x_0 & \cdots & {x_0}^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & {x_n}^n \end{pmatrix} = \prod_{0 \leqslant i < j \leqslant n} (x_j - x_i)$$

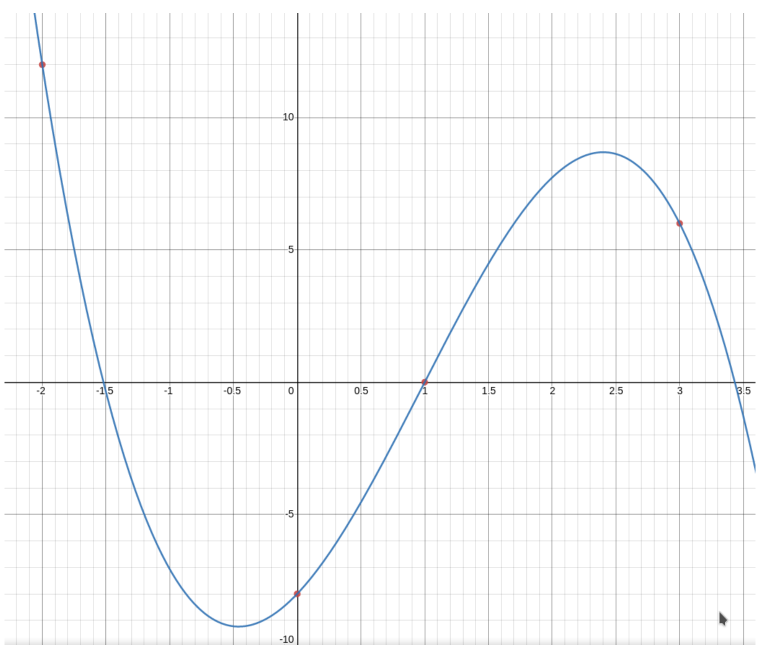
In out case:

$$\begin{aligned} L(x) &= 12 \cdot \frac{\upsilon\big(x, 0, 1, 3\big)}{\upsilon\big(-2, 0, 1, 3\big)} - 8 \cdot \frac{\upsilon\big(-2, x, 1, 3\big)}{\upsilon\big(-2, 0, 1, 3\big)} + 0 \cdot \frac{\upsilon\big(-2, 0, x, 3\big)}{\upsilon\big(-2, 0, 1, 3\big)} + 6 \cdot \frac{\upsilon\big(-2, 0, 1, x\big)}{\upsilon\big(-2, 0, 1, 3\big)} = \\ &= 12 \cdot \frac{(-x)(1-x)(3-x)}{2 \cdot 3 \cdot 5} - 8 \cdot \frac{(x+2)(1-x)(3-x)}{2 \cdot 1 \cdot 3} + 6 \cdot \frac{(x+2)x(x-1)}{5 \cdot 3 \cdot 2} = \\ &= -\frac{2}{5}x^3 + \frac{8}{5}x^2 - \frac{6}{5}x - \frac{4}{3}x^3 + \frac{8}{3}x^2 + \frac{20}{3}x - 8 + \frac{1}{5}x^3 + \frac{1}{5}x^2 - \frac{2}{5}x = \\ &= -\frac{23}{15}x^3 + \frac{67}{15}x^2 + \frac{76}{15}x - 8 \end{aligned}$$

Answer

The required polynomial:

$$L(x) = -\frac{23}{15}x^3 + \frac{67}{15}x^2 + \frac{76}{15}x - 8$$



Picture 4.1: Sketch of the polynomial (blue curve) with the 4 given (red) points

Task 5

Description

Find a (parametric) equation defining the Bezier curve defined by the four points whose coordinates form the columns of the matrix. Sketch the curve on the coordinate plane

$$P = \begin{bmatrix} 2 & 3 & 7 & 9 \\ 2 & 0 & 2 & 6 \end{bmatrix}$$

Solution

Let us use a general formula for Bezier curves:

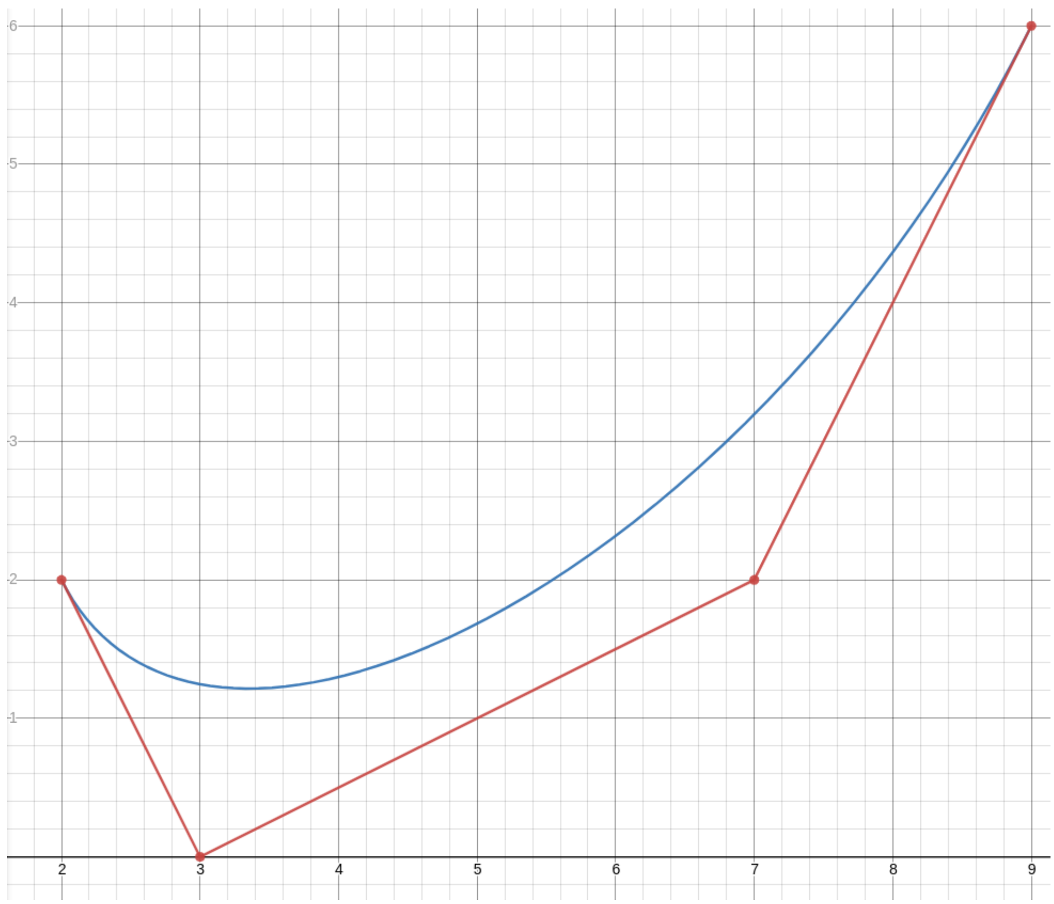
$$B(t) = \sum_{k=0}^{n} C_n^k \cdot t^k \cdot (1-t)^{n-k} \cdot A_k = C_3^0 \cdot t^0 \cdot (1-t)^3 \cdot \begin{pmatrix} 2 \\ 2 \end{pmatrix} + C_3^1 \cdot t^1 \cdot (1-t)^2 \cdot \begin{pmatrix} 3 \\ 0 \end{pmatrix} + C_3^2 \cdot t^2 \cdot (1-t)^1 \cdot \begin{pmatrix} 7 \\ 2 \end{pmatrix} + C_3^3 \cdot t^3 \cdot (1-t)^0 \cdot \begin{pmatrix} 9 \\ 6 \end{pmatrix} =$$

$$= \begin{pmatrix} 2-6t+6t^2-2t^3 \\ 2-6t+6t^2-2t^3 \end{pmatrix} + \begin{pmatrix} 9t^3-18t^2+9t \\ 0 \end{pmatrix} + \begin{pmatrix} 21t^2-21t^3 \\ 6t^2-6t^3 \end{pmatrix} + \begin{pmatrix} 9t^3 \\ 6t^3 \end{pmatrix} = \begin{pmatrix} -5t^3+9t^2+3t+2 \\ -2t^3+12t^2-6t+2 \end{pmatrix}$$

Answer

The required parametric equation:

$$B(t) = \begin{pmatrix} -5t^3+9t^2+3t+2 \\ -2t^3+12t^2-6t+2 \end{pmatrix}$$



Picture 5.1: Sketch of the (blue) Bezier curve with forming (red) intervals

Task 6

Description

For the polynomial x^3-3x^2+3x-5 find the best approximation with respect to the max-norm by a polynomial of degree 2 on a line segment $[-2;2]$

Solution

For function f defined on the segment [-1; 1] Chebyshev norm represents deviation from zero, giving the maximum absolute value of f on the segment:

$$|f|_0 = \underset{[-1;1]}{max} |f(x)|$$

Therefore, for the sake of approximating the given polynomial we need to minimize $\left|x^3-3x^2+3x-5 - g(x)\right|_0$, where $g(x)$ is the target approximation function.

According to a theorem from the lectures, the least deviating from zero polynomial on the segment $[-1;1]$ is $\widetilde{T}_n(x) = \frac{1}{2^{n-1}}T_n(x)$. However, the approximation we are looking for is defined on the segment $[-2;2]$. Consequently, we need to make an axis transformation $y=\frac{x}{2}$:

$$\widetilde{T}_3(y) = \frac{1}{4}T_3(y) = \frac{1}{2}\cdot y\cdot T_2(y)-\frac{1}{4}\cdot T_1(y) = \frac{1}{2}\cdot y\cdot\Big(2\cdot y\cdot T_1(y)-T_0(y)\Big)-\frac{1}{4}\cdot y = y^3-\frac{1}{2}\cdot y-\frac{1}{4}\cdot y = \frac{1}{8}\cdot x^3-\frac{3}{8}\cdot x$$

$$\frac{1}{8}\cdot x^3-\frac{3}{8}\cdot x = k\cdot\Big(x^3-3x^2+3x-5 - g(x)\Big)$$

$$\frac{1}{8}\cdot x^3-\frac{3}{8}\cdot x = k\cdot x^3-k\cdot(3+a_2)\cdot x^2+k\cdot(3-a_1)\cdot x-k\cdot(5+a_0)$$

$$\begin{cases} k = \frac{1}{8} \\ a_2 = -3 \\ a_1 = 6 \\ a_0 = -5 \end{cases}$$

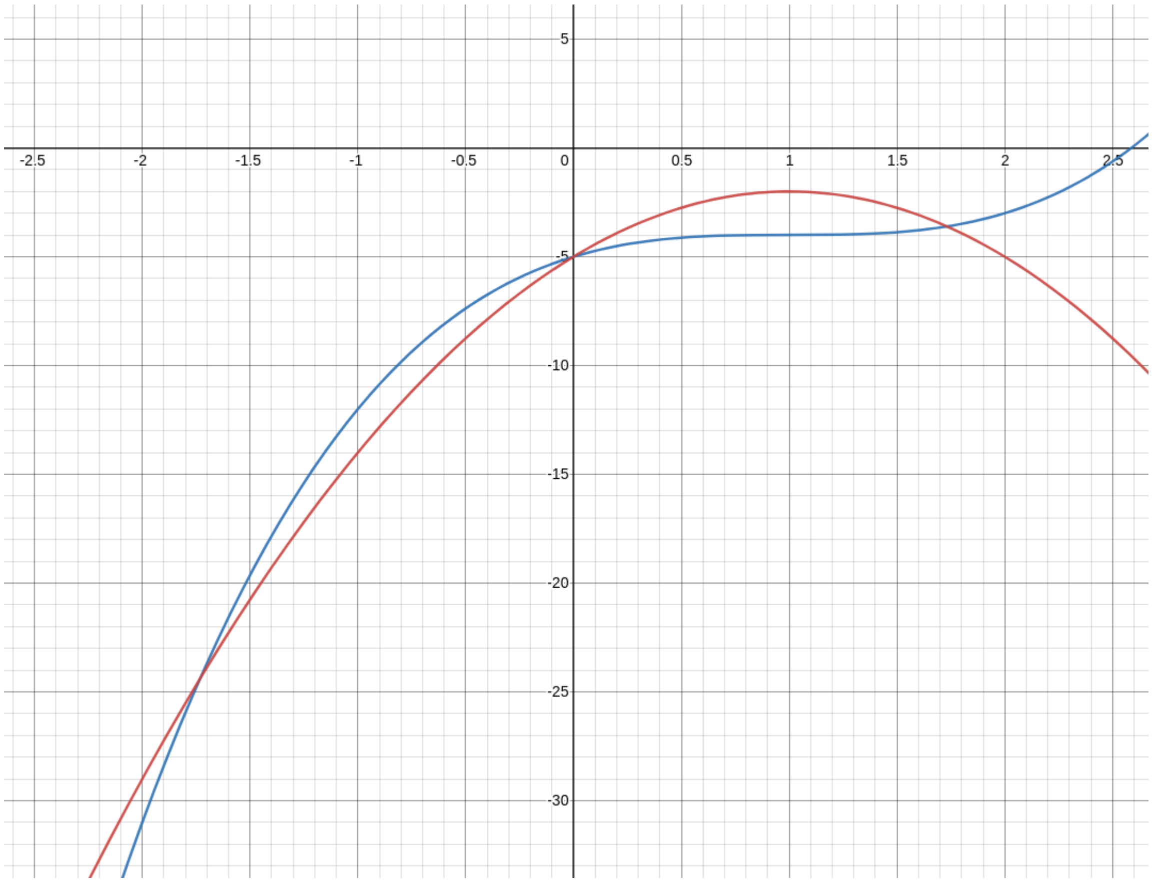
Thus, the approximation function looked for is:

$$g(x) = -3x^2+6x-5$$

Answer

The required polynomial is:

$$g(x) \ = \ -3x^2+6x-5$$



Picture 6.1: Sketch of the approximation polynomial (red curve) with the original one (blue curve)

Task 7

Description

Find a polynomial of degree $\leqslant 3$ that approximates the function $f(x) = \sqrt{3x+1}$ on the segment $[1;5]$ in the norm $|h|_T = \sqrt{\int\limits_1^5 \frac{h(x)^2}{\sqrt{1-{(2x-6)^2}/{16}}}\cdot dx}$

Solution

Put $y = \frac{1}{2}\cdot x - \frac{3}{2}$. Then:

$$|h|_T = \sqrt{\int\limits_{-1}^{1} \frac{h(2\cdot y+3)^2}{\sqrt{1-y^2}}\cdot 2\cdot dy}$$

$$f(y) = \sqrt{6\cdot y+10}$$

Put $h_1(y) = \sqrt{2}\cdot h(2\cdot y+3)$. Then:

$$|h|_T = \sqrt{\int\limits_{-1}^{1} \frac{h_1(y)^2}{\sqrt{1-y^2}}\cdot dy}$$

According to a theorem from the lectures:

- a scalar product of continuous functions on the segment $[-1;1]$ is

$$\langle g_1, g_2\rangle = \int_{-1}^{1} \frac{g_1(x)\cdot g_2(x)}{\sqrt{1-x^2}}\cdot dx$$

- the corresponding norm is

$$||g|| = \sqrt{\langle g, g\rangle} = \sqrt{\int_{-1}^{1} \frac{g(x)^2}{\sqrt{1-x^2}}\cdot dx}$$

- the orthogonality relations are

$$\langle T_m, T_n\rangle = \begin{cases} 0, & m\neq n \\ \frac{\pi}{2}, & m=n\neq 0 \\ \pi, & m=n=0 \end{cases}$$

- the best approximation of a function g by polynomial of degree $\leqslant n$ is

$$\widetilde{g}(x) = \sum_{i=0}^{n} \frac{\langle T_i, g\rangle}{\langle T_i, T_i\rangle}\cdot T_i(x)$$

Therefore, the best approximation of function $f(y)$ is

$$\widetilde{f}(y) = \sum_{i=0}^{3} \frac{\langle T_i, f\rangle}{\langle T_i, T_i\rangle}\cdot T_i(y) = \frac{1}{\pi}\cdot 1\cdot \int_{-1}^{1} \frac{1\cdot\sqrt{6\cdot y+10}}{\sqrt{1-y^2}} + \frac{2}{\pi}\cdot y\cdot \int_{-1}^{1} \frac{y\cdot\sqrt{6\cdot y+10}}{\sqrt{1-y^2}} + \frac{2}{\pi}\cdot(2y^2-1)\cdot \int_{-1}^{1} \frac{(2y^2-1)\cdot\sqrt{6\cdot y+10}}{\sqrt{1-y^2}} +$$

$$+ \frac{2}{\pi}\cdot(4y^3-3y)\cdot \int_{-1}^{1} \frac{(4y^3-3y)\cdot\sqrt{6\cdot y+10}}{\sqrt{1-y^2}} \approx \frac{9.68845}{\pi} + \frac{2\cdot 1.54867}{\pi}\cdot y + \frac{2\cdot(-0.127198)}{\pi}\cdot(2y^2-1) + \frac{2\cdot 0.0210437}{\pi}\cdot(4y^3-3y) \approx$$

$$\approx 0.053587\cdot y^3 - 0.161954\cdot y^2 + 0.945723\cdot y + 3.164906$$

Thus, the best approximation of function $f(x)$ is

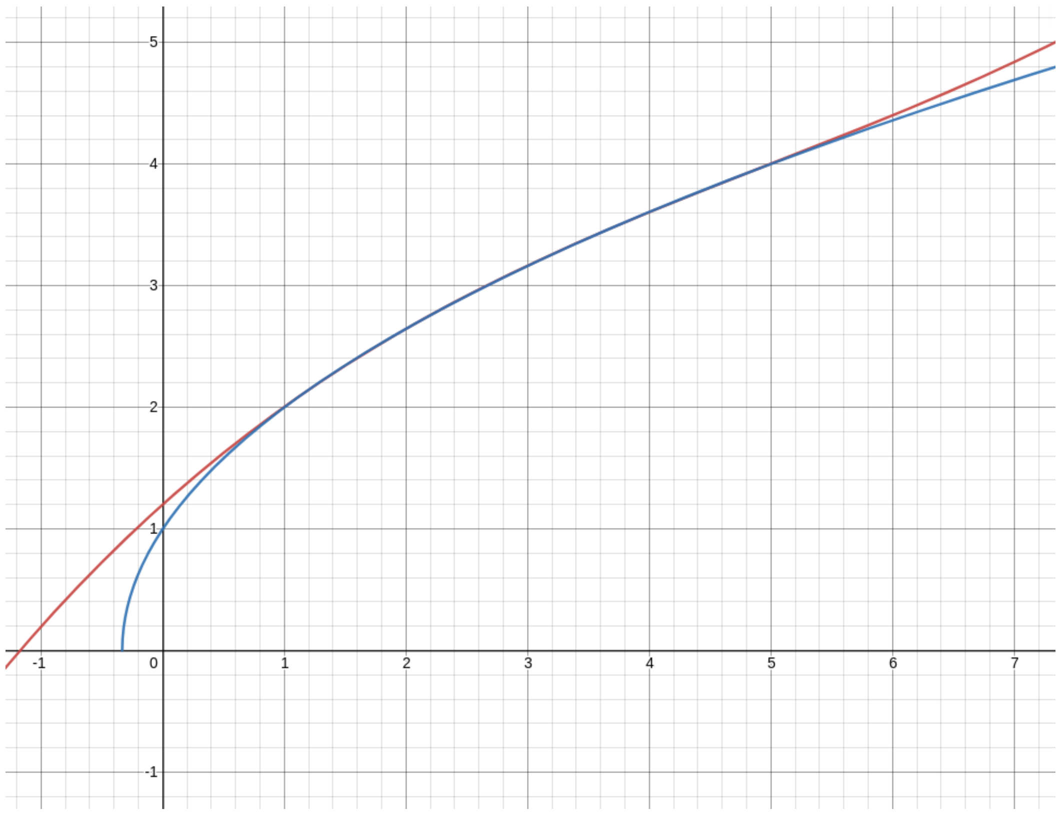
$$\widetilde{f}(x) = 0.053587\cdot\left(\frac{1}{2}\cdot x - \frac{3}{2}\right)^3 - 0.161954\cdot\left(\frac{1}{2}\cdot x - \frac{3}{2}\right)^2 + 0.945723\cdot\left(\frac{1}{2}\cdot x - \frac{3}{2}\right) + 3.164906 =$$

$$= 0.006698\cdot x^3 - 0.100774\cdot x^2 + 0.896649\cdot y + 1.201069$$

Answer

The required approximation polynomial is:

$$\widetilde{f}(x) = 0.006698 \cdot x^3 - 0.100774 \cdot x^2 + 0.896649 \cdot y + 1.201069$$



Picture 7.1: Sketch of the approximation polynomial (red curve) with the original function (blue curve)

Task 8

Description

Find all the values of q such that the equation $2x^2+xy\cdot(4q+4)+y^2\cdot(1-8q)+z^2\cdot(2-8q) = 1$ defines a unit circle with respect to some norm. Find the norm of the vector $(1,1,1)$ as a function of q.

Solution

A subset $B\subset\mathbb{R}^n$ is a unit circle with respect to some norm ν iff the following 5 properties hold:

- B is bounded
- B contains some (small) Euclidean ball
- B is closed
- B is convex
- B is center symmetrical

The given equation is a quadratic form defining a second order surface. The only class of second order surface which can define a unit circle is an ellipsoid (other classes are not bounded).

$$\underbrace{(x,y,z)}_{X^T}\cdot\underbrace{\begin{pmatrix}2 & 2q+2 & 0\\ 2q+2 & 1-8q & 0\\ 0 & 0 & 2-8q\end{pmatrix}}_{A}\cdot\begin{pmatrix}x\\ y\\ z\end{pmatrix}+2\cdot\underbrace{(0,0,0)}_{b}\cdot\begin{pmatrix}x\\ y\\ z\end{pmatrix}+\underbrace{(-1)}_{a_{44}} = 0$$

$$B = \begin{pmatrix}A & b^T\\ b & a_{44}\end{pmatrix}$$

A quadratic form defines an ellipsoid iff:

$$\left.\begin{cases}I_3 = detA \neq 0\\ I_2 = M_{A}{}^{1,2}_{1,2}+M_{A}{}^{1,3}_{1,3}+M_{A}{}^{2,3}_{2,3} > 0\\ I_1\cdot I_3 = trA\cdot detA > 0\\ K_4 = detB < 0\end{cases}\right| \Rightarrow \left.\begin{cases}32\cdot\left(q-\frac{1}{4}\right)\cdot\left(q-\frac{-6+\sqrt{34}}{2}\right)\cdot\left(q-\frac{-6-\sqrt{34}}{2}\right) \neq 0\\ 60\cdot\left(q-1\right)\cdot\left(q-\frac{1}{15}\right) > 0\\ 512\cdot\left(q-\frac{5}{16}\right)\cdot\left(q-\frac{1}{4}\right)\cdot\left(q-\frac{-6+\sqrt{34}}{2}\right)\cdot\left(q-\frac{-6-\sqrt{34}}{2}\right) < 0\\ 32\cdot\left(q-\frac{1}{4}\right)\cdot\left(q-\frac{-6+\sqrt{34}}{2}\right)\cdot\left(q-\frac{-6-\sqrt{34}}{2}\right) > 0\end{cases}\right| \Rightarrow$$

$$\Rightarrow \left.\begin{cases}q\in\left(-\infty;\infty\right)\Big\backslash\left\{\frac{1}{4},\frac{-6+\sqrt{34}}{2},\frac{-6-\sqrt{34}}{2}\right\}\\ q\in\left(-\infty;\frac{1}{15}\right)\bigcup\left(1;\infty\right)\\ q\in\left(\frac{-6-\sqrt{34}}{2};\frac{-6+\sqrt{34}}{2}\right)\bigcup\left(\frac{1}{4};\frac{5}{16}\right)\\ q\in q\in\left(\frac{-6-\sqrt{34}}{2};\frac{-6+\sqrt{34}}{2}\right)\bigcup\left(\frac{1}{4};\infty\right)\end{cases}\right| \Rightarrow\quad q\in\left(\frac{-6-\sqrt{34}}{2};\frac{-6+\sqrt{34}}{2}\right)$$

Put $\nu\left(X\right)$ is the corresponding norm. Let us consider a vector $X_0 = \alpha\cdot(1,1,1)^T$, such that:

$$\nu\left((1,1,1)^T\right) = \nu\left(\frac{X_0}{\alpha}\right) = \frac{1}{|\alpha|}\cdot\nu\left(X_0\right) = \frac{1}{|\alpha|}$$

As far as $\nu\left(X_0\right) = 1$, it must satisfy the given equation, which defines the unit circle:

$$2\cdot\alpha^2+\alpha^2\cdot(4q+4)+\alpha^2\cdot(1-8q)+\alpha^2\cdot(2-8q) = 1$$

$$\alpha = \pm\frac{1}{\sqrt{9-12q}}$$

Thus:

$$\nu\Big((1,1,1)^T\Big) = \frac{1}{|\alpha|} = \sqrt{9-12q}$$

Answer

The required values of q:

$$q \in \left(\frac{-6-\sqrt{34}}{2}; \frac{-6+\sqrt{34}}{2}\right)$$

The required norm of vector $(1,1,1)^T$:

$$\nu\Big((1,1,1)^T\Big) = \sqrt{9-12q}$$