The snake eyes puzzle

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The 'Snake-eyes' puzzle set by Daniel Reeves on Manifold Markets[1] is as follows:

> You're offered a gamble where a pair of six-sided dice are rolled and unless they come up snake eyes you get a bajillion dollars. If they do come up snake eyes, you're devoured by snakes.
>
> So far it sounds like you have a 1/36 chance of dying, right?
>
> Now the twist. First, I gather up an unlimited number of people willing to play the game. I take 1 person from that pool and let them play. Then I take 2 people and have them play together, where they share a dice roll and either get the bajillion dollars each or both get devoured. Then I do the same with 4 people, and then 8, 16, and so on.
>
> At some point one of those groups will be devoured by snakes and then I stop.
>
> **Is the probability that you'll die, given that you're chosen to play, still 1/36?**

Daniel also gives some additional clarifications:

1 The game is not adversarial and the dice rolls are independent and truly random.

2 Choosing each group also happens uniformly randomly and without replacement.

3 The question is about the unrealistic case of an unbounded number of people but we can cap it and say that if no one has died after N rounds then the game ends and no one dies. We just need to then find the limit as N goes to infinity, in which case the probability that no one dies goes to zero.

4 We're asking for a conditional probability: given that you're chosen to play, what is the probability that you die? I.e., what fraction of people chosen to play die [in expectation]?

[1]https://manifold.markets/dreev/is-the-probability-of-dying-in-the

5 Importantly, in the finite version it's possible for no one to die. But the probability of that approaches zero as the size of the pool approaches infinity.

6 What if "the fraction of people chosen to play who die in expectation" is different from the conditional probability? I'm still mulling whether those may be different but the answer, as clarified early on in the comments, is that if so, we're talking about the conditional probability. That's because we're treating this as a decision theory problem: assuming you want to play the one-shot version, do you still want to play the doubling-groups version?

7 What if the most correct answer is "undefined"? If we went by just the title of this market, that would be NO, but from the beginning the description specified that NO requires the probability to be strictly greater than 1/36, which "undefined" is not. I failed to consider the possible answer of "undefined" when creating this market! So if that's the answer it's going to be hard to have a satisfactory resolution but I think N/A will be the least unsatisfactory in that case.

8 "At some point one of those groups will be devoured by snakes and then I stop" has an implicit "unless I roll snake eyes forever". i.e., we are not conditioning on the game ending with snake eyes. The probability of an infinite sequences of non-snake-eyes is zero and that's the sense in which it's correct to say "at some point snake eyes will happen" but non-snake-eyes forever is possible in the technical sense of "possible".

My solution

One thing to get out of the way immediately is that there is some ambiguity in the word 'chosen' in the problem statement: 'chosen' can refer to being picked in a single round of the iterative game, or 'chosen' can refer to being one of the people who are chosen across all rounds in the iterative game. We can express the associated probabilities as

$$Pr(death|chosen\ at\ round\ i) \tag{1}$$

(the conditional probability of death given that you are chosen at some specific round i in the iterative game) and

$$Pr(death|\ chosen) \tag{2}$$

(the conditional probability of death given that you will be chosen at some round in the game). Letting $p = 1/36$ we can give mathematical expressions for both of these probabilities. The first probability is simply p. I think it is clear that this is not the meaning for 'chosen' that Daniel intends: instead the snake-eyes puzzle asks us to estimate the conditional probability of death given that you are chosen at some round in the game.

This probability depends in an interesting way on your knowledge about the game. To illustrate this, assume you magically know that the game is going to end with snake-eyes at round n (and, because I'm a computer scientist, I'll take the first round to be round 0). If it happens that $n = 0$ (the game ends with snake-eyes in the very first round) then since you know you will be chosen at some round in the game and that the first is the only round, you know you will be chosen in the first round, when snake-eyes comes up. Since the iterative game ends when snake-eyes comes up, you also know that snake-eyes will come up in the first round, and so your probability of dying is 1. Similarly, if you know that $n = 1$ (there are two rounds, round 0 and round 1) and the game will end with snake-eyes at round 1 then you know that your probability of dying is 2/3 (since there are 2 rounds in this game, with a single player chosen in the first round and surviving and with 2 players chosen in the second round and dying: and one of these 3 players is you).

These examples show that conditional probability in Equation 2 is necessarily different from that in Equation 1. There is no paradox in this: these probabilities are different because they are conditional on different prior information, Equation 1 on the information that you are chosen in a specific round i and Equation 2 on the information that you are chosen at some round before the game ends.

But when will the game end?

The above explanation for the difference between these two probabilities, and for why that difference is not paradoxical or a contradiction, depends on having perfect prior knowledge of n, the number of rounds in the game. We do not, of course, have that perfect knowledge, because the round at which the game will end is a random variable (each possible value of $n \geq 0$ has a certain probability of being the round on which a given game will end). We can deal with this problem as we do with all random variables: by calculating, for each possible value of $n \geq 0$, the probability that the game will end at round n (that is, the discrete probability distribution for the variable n).

Letting E be a random variable representing the ending round of a snake-eyes game (that is, the first round where we hit snake-eyes on our dice roll), what is the probability of $E = n$ (of a snake-eyes game finishing at the nth round)? Letting the first round be $n = 0$, it is

$$Pr(E = n) = (1 - p)^n p$$

since for the game to end at round n, we must have n rounds where we don't hit snake-eyes (at probability $1 - p$ for each round) and then a round where snake-eyes comes up (with probability p). This is known as the geometric probability distribution (the discrete analog of the exponential distribution, which approaches the exponential in the limit) with parameter p: we say the random variable E follows this distribution. Note that letting L represent the length of a game (the number of rounds in the game) is given by $Pr(L = n) = (1 - p)^{n-1}p$, since every game has at least one round.

Similarly letting R_i be a random variable representing the number of remaining rounds to be played in a snake-eyes game given that the game has been played up to round i, what is the probability of $R_i = n$? Just as before, this is

$$Pr(R_i = n) = (1 - p)^n p = Pr(R_0 = n) = Pr(E = n)$$

for all i, since for the game to have n more rounds, we must have n rounds where we don't hit snake-eyes (at probability $1 - p$ for each round) and then a round where snake-eyes comes up (with probability p). The fact that the distribution of R_i is the same for all i means that the geometric distribution is 'memoryless': the probability distribution for the number of remaining rounds until the first success is the same no matter how many rounds have already been played.

We can also calculate the probability of a snake-eyes game going on for more than n rounds (assuming you have enough people willing to take part): this is simply the chance of reaching n rounds without hitting snake-eyes, given by

$$Pr(E > n) = (1 - p)^n)$$

Note that, because $Pr(E > n) > 0$ for all n, a snake-eyes game can potentially have more rounds than any number n you care to pick (as long as you have an unbounded set of people willing to play). Informally, we might say that 'an iterative snake-eyes game can be infinitely long'. This doesn't mean that, in the space of all possible snake-eyes games, there actually is some infinitely long game (this makes no sense: infinity isn't a number). Instead, saying that the game can be infinitely long is a shorthand for saying 'an iterative snake-eyes game can last more than n rounds, for any n you care to pick' or equivalently 'there is no upper bound on the length of a snake-eyes game: the game is unbounded'.

Finally, even though the snake-eyes game can potentially continue indefinitely, the expected (mean) number of rounds in a snake-eyes game is finite and equal to $1/p$ (because this is the expected value of a random variable following a geometric distribution with parameter p, and the number of rounds in a snake-eyes game is such a variable). The expected number of remaining rounds in a snake-eyes game given that i rounds have already been played is similarly $1/p$, and does not change no matter how many rounds have already been played. Similarly, the median number of rounds in a snake-eyes game is the median value of such a geometric distribution, and is

$$l = \frac{1}{-\log_2(1 - p)}$$

and the chance of a snake-eyes game being shorter than this value is 0.5. There is a direct analogy with processes like radioactive decay here: for an element with half-life λ, the probability for an atom of that element undergoing radioactive decay in the next period λ is 0.5, and that probability is the same no matter how much time has already passed. Similarly, the probability of a snake-eyes game ending in the next l rounds is 0.5, and that probability is the same no matter how many rounds have been played; and so l is the 'half-life' of a snake-eyes game.[2]

[2]The snake-eyes game also reminds me of Greg Egan's great sci-fi story 'Into Darkness' : `https://i.warosu.org/data/sci/img/0145/68/1655241504685.pdf`

But what are the chance I'll die?

What is the probability of death for players in a snake-eyes game? We can calculate this by considering the proportion of people who die in such a game: the total number who die divided by the total number of players in that game. This proportion depends on when the game ends and on what happens when the game ends. To express this probability we can consider three distinct situations (three different types of snake-eyes games):

Unbounded: There is an unbounded number of players willing to play the game, and so an iterative snake-eyes game will only end when snake-eyes are actually rolled at some round n (running out of players will never happen and so the only way the game can end is with snake-eyes).

Bounded death: There is some maximum number of players M willing to play the game, and so a game can end in two ways: when snake-eyes are actually rolled at some round n *or* when there are no more players left at some round n. When the game ends (because snake-eyes is rolled or because there are no players left to pick) the players chosen at the final round n die.

Bounded escape: There is some maximum number of players M willing to play the game, and so the game can end in two ways: when snake-eyes are actually rolled at some round n *or* when there are no more players left at some round n. If the game ends without rolling snake-eyes (because there are no players left to pick) the players in the final round do not die.

In the sections below I consider the probability of death in the above three versions of the snake-eyes game: it turns out that the probability of death is greater than 1/2 in the 'unbounded' game, greater than 1/4 in the 'bounded death' game, and $p = 1/36$ in the 'bounded escape' game. The answer to Daniel's question therefore depends on which type of snake-eyes game he is asking about.

What about an infinitely long game?

Some may consider that there is another option that isn't listed above: a game that continues for an infinite number of rounds and never ends. However, this doesn't make mathematical or logical sense: the fact that the number of rounds in a snake-eyes game is unbounded does not mean that there actually is some game with length $n = \infty$ (because ∞ is not a number), and the fact that the snake-eyes game is defined by what happens at the last round (death by snake-eyes) means that a game that never ends is not snake-eyes.[3]

[3] For people who are mathematically careful, notice that the set of possible lengths of snake-eyes games is countably infinite and so is equal to the transfinite number $\aleph_0$, the smallest infinity. However $\aleph_0$ is not a member of the integer field $\mathbb{N}$ (e.g. $\aleph_0 + f = \aleph_0$ for $f \in \mathbb{N}$) and so is not a 'number' in that sense.

The 'unbounded' game

Consider the probability of death in an unbounded game: one that you know will end with snake-eyes being rolled at some round n (and this is the only way the game can end: we will never run out of players). Since the game finishes at round n and round n is a snake-eyes round, the 2^n players chosen in that round die; the total number of players from round 0 to round n is

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

and so the proportion of chosen players who die is

$$\frac{2^n}{2^{n+1} - 1} > \frac{1}{2}$$

irrespective of the length of the game: so in version of the game, the probability of death, given that you are chosen to play the game, is greater than $1/2$. The actual value of this probability can be expressed as

$$Pr(death|\ chosen) = \sum_{n=0}^{\infty} \frac{2^n}{2^{n+1} - 1}(1-p)^n p$$

(the proportion of deaths in a game that ends at round n, weighted by the chance of a game ending in that round) and since with $p = 1/36$ the partial sum

$$\sum_{n=0}^{N} \frac{2^n}{2^{n+1} - 1}(1-p)^n p$$

converges to a limit given by the constant $D \approx .5218873$ as $N \to \infty$ and we see that

$$Pr(death|\ chosen) \approx D$$

in the unbounded game.

The 'bounded death' game

Consider the probability of death in an bounded game with only M players available: one where the game will either end at snake-eyes or when there are no more players left, and where the players in the final round of the game always die (whether or not snake-eyes was rolled). Let m be the round at which we run out of players, which means that

$$2^m < M < 2^{m+1}$$

and in this situation a game can end at snake-eyes for any round n from 0 to $m-1$, with

$$\frac{2^n}{2^{n+1} - 1} > \frac{1}{2}$$

of the chosen players dying in each game, as before.

If the game does not end at any of those rounds, it ends at round m with the number of players chosen at that round being between 1 and 2^m, and so the average number of players chosen at this round is 2^{m-1}. All of the players in the last round die whether snake-eyes are rolled or not, and so on average

$$\frac{2^{m-1}}{2^{m+1}} > \frac{1}{4}$$

players will die if this round is reached. Since the probability of death if the game ends before this round is greater than 1/2, we thus see that in this 'bounded death' game, the probability of death in this game is necessarily greater than 1/4.

The actual value of this probability can be expressed as

$$Pr(death|\ chosen) = \frac{2^{m-1}}{2^{m+1}}(1-p)^m + \sum_{n=0}^{m-1} \frac{2^n}{2^{n+1}-1}(1-p)^n p$$

(the proportion of deaths in a game that ends at round $n < m$, weighted by the chance of a game ending in that round, plus the proportion of deaths in a game that ends at round m, weighted by the chance of getting to that round). As we saw earlier, with rising m the sum here approaches the limit given by the constant $D \approx .5218873$, and so for a large pool of players M we expect

$$Pr(death|\ chosen) \approx D$$

as before.

The 'bounded escape' game

In both the original unbounded game, and in a bounded game that ends in death, it looks like half the players are guaranteed to die. Oh the humanity! What can we do to save ~~our bet on 1/36~~ all those people?

One way to save most of our players is to modify the bounded game so that, if a game hits round m and runs out of players, then all the players at round m escape and survive. We can run games as normal otherwise so if a game ends at $n < m$ then more than half of the players in that game will die; it turns out that this change to what happens when we hit round m is enough to reduce the expected chance of death overall to something around 1/36.

To see why this small change in outcomes at round m (which for large m is unlikely to be reached often) alters the probability of death from above 1/2 to around 1/36, consider the probability of death in an bounded game with exactly $2^{m+1} - 1$ players available, where the game will either end at snake-eyes or when there are no more players left. In this situation a game can end at snake-eyes for any round n from 0 to $m - 1$, with the players chosen at that round dying. If the game does not end at any of those rounds, it ends at round m with either snake-eyes being rolled (with probability p) and so all players at that round dying, or without snake-eyes, in which case nobody dies. Notice that in this game the probability of a player dying, no matter what round they are picked in, is p : and so in this version of the game, the chance of death given that you are chosen to play is p irrespective of the value of m (or of $2^{m+1} - 1$, the total number of players available).

Which game are we playing again?

So what is the answer to Daniel's question 'Is the probability that you'll die, given that you're chosen to play, still 1/36'? It depends on which game we're in. If we're in the unbounded snake-eyes game, our chance of death is approximately $D > 1/2$; in the 'bounded death' version of the game, it approaches D and is greater than $1/4$; in the 'bounded escape' version, the chance of death is around $p = 1/36$. Which game is Daniel actually asking about? Since the question explicitly say that

> First, I gather up an unlimited number of people willing to play the game [. . .]
> At some point one of those groups will be devoured by snakes and then I stop.

it is clear that the question is directly asking about the unbounded game (since the question assumes an unlimited number of people to play the game); and so the answer to Daniel's question is **No**.

Notice, however, that though the market as stated asks about the unbounded game, Daniel's clarification 3 in the FAQ says

> The question is about the unrealistic case of an unbounded number of people but we can cap it and say that if no one has died after N rounds then the game ends and no one dies. We just need to then find the limit as N goes to infinity, in which case the probability that no one dies goes to zero.

This clarification asserts that the solution to the unbounded game can be obtained by finding the solution to the 'bounded escape' game, and suggests that, to answer the market question, we just need to find the solution in the 'bounded escape' case. There at least five problems here. The first is that the unbounded game is no more unrealistic than other abstract puzzles of this type (how realistic is the Sleeping Beauty problem? Newcomb's Problem?) and so there is no reason we need to consider a more 'realistic' alternative game. The second is that the unbounded game already has a clear answer: we don't need to consider more 'realistic' versions of the game to get that answer. The third is that the solution in the unbounded case is clearly not the same as the solution in the 'bounded escape' case: and so even though the probability of death in the bounded escape game is $P(death) = 1/36$, that doesn't change the answer to the market question, which is $P(death) > 1/2$. The fourth is that the solution in the bounded escape game doesn't require us to take limits: the probability of death in that version of the game is

1/36 no matter what round m we bound the game at. Finally the fifth is that we have no reason to pick the 'bounded escape' game as our realistic replacement for the unbounded game: we could equally pick the 'bounded death' game. Given these problems it is clear this 'clarification' is incoherent (it doesn't clarify anything) and so we can ignore it in answering the market question.

Some minor points

One thing that has come up repeatedly both in Daniel's presentation of the problem, in his write-up, and in other $p = 1/36$ comments, is the issue of infinity. In Daniel's write-up, for example, we have:

> *We want the probability that you die given that you are chosen to play, $Pr(death|chosen)$. It seems like we can ignore the 0% chance of rolling not-snake-eyes forever and say that eventually about half the people who are chosen die, but let's Bayes it out carefully:*
>
> $$Pr(death|chosen) = \frac{Pr(chosen|death)Pr(death)}{Pr(chosen)} = \frac{1 - Pr(death)}{Pr(chosen)}$$
>
> *In the uncapped case, that conditional probability is undefined. You're part of an infinite pool so you have a 0% chance of being chosen and a 0% chance of dying. The probability we want is 0/0. *robot-with smoke-coming-out-of-its-ears-emoji* Since we can't directly calculate the probability in the infinite case, we have to take a limit.*

There is a basic mistake here, however; to see the mistake consider an analogous question as an 'intuition pump'. Suppose I am going to choose an integer in some random way, and I want to estimate the probability $Pr(even|chosen)$: the probability that an integer is even given that it is chosen. Suppose also that the way that I randomly choose this integer doesn't place any upper bound on its value, so I have an infinite range of possible integers to consider (one way to do this would be to use the snake-eyes approach and throw dice until snake-eyes comes up: then my randomly chosen integer is the number of throws until I hit snake-eyes and has no upper bound, so for every integer across the infinite range there is some probability that I will pick that integer). In this uncapped case, can we conclude that $Pr(even|chosen)$ is undefined, because I have an infinite pool of integers to choose from? Clearly not: the chance of a randomly selected integer being even has to be 1/2, since half of all integers are even[4]. The flaw arises from the statement

[4] Going further, the idea that a randomly chosen integer has a $1/x$ probability of being evenly divisible by x is fundamental to many mathematical proofs, especially in Number Theory; and if we have to reject that idea and say that this probability is undefined whenever the choice is drawn from an infinite pool, then all those proofs are rejected too

that " You're part of an infinite pool so you have a 0% chance of being chosen and a 0% chance of dying". This is not correct: with no upper bound on the set of integers, the probability of a given integer being chosen at random *approaches* 0, but is never equal to 0; and so 'with an infinite pool' you do not have a 0 chance of being chosen (and so do not have a 0 chance of dying).

More generally, the problem here (and in a lot of other discussions of this 'infinite case') is the idea that infinity is an actual number: that just as you can be in a pool containing exactly m members, you can be in a pool of containing exactly ∞ members. This is wrong: the number of members in any pool must be an integer (because the number of members is a count), and ∞ is not an integer.

0.1 Why does the market think P(death) is 1/36?

The preferred answer to the question 'Is the probability that you'll die, given that you're chosen to play, still 1/36? ' on Manifold Markets is, at the moment **Yes** (86% at the moment); and so the majority money opinion is that the probability of death in the iterative snake-eyes game is equal to that in the one-shot game. Most likely this is because, as Daniel writes,

> The dice rolls are independent and whenever you're chosen, whatever happened in earlier rounds is irrelevant. Your chances of death are the chances of snake eyes on your round: 1/36.

The answer derived above, however, is **No**: the probability that you'll die, given that you're chosen to play, is $Pr(death|\ chosen) > 1/2$, not 1/36. What explains this difference between the majority money's view and the result derived above? The difference arises because the information in the question, 'given that you're chosen to play', has a different meaning in the one-shot and the iterative game. In the one shot-game, if it's given that you are chosen to play, then you are playing in this round: and so you have a 1/36 chance of death. In the iterative game, however, if it's given that you are chosen to play then you are playing in some round in the game, but you don't know which round; and since the number of players chosen doubles per round, you are most likely to be chosen in the last round (which contains more players than all previous rounds combined). The fact that the information 'given that you're chosen to play' has a different meaning in the two games means that probabilities conditional on that information are also different between the two games. The majority money opinion that the probability of death 'given that you're chosen to play' is the same in the one-shot and the iterative game most likely rests on the mistaken assumption that the information conveyed by this statement is the same in both games.