Dhirubhai Ambani Institute of Information and
Communication Technology
Gandhinagar, Gujarat 382007

PC433 BTP-I
PC434 BTP-II

Under the guidance of
Prof. Arpita Mal

by

Nikita Shah

Student ID: 202101209

1 Proof:

Given: Let $n \in \mathbb{N}$ (the set of natural numbers).

$$\boldsymbol{X} = \begin{bmatrix} x_{00} & x_{01} & \cdots & x_{0(n-1)} \\ x_{10} & x_{11} & \cdots & x_{1(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ x_{(2^n-1)0} & x_{(2^n-1)1} & \cdots & x_{(2^n-1)(n-1)} \end{bmatrix}, \quad \text{Dimensions of } \boldsymbol{X} = 2^n \times n.$$

The (i, j)-th entry of matrix $\boldsymbol{X}$ is defined as follows:

$$x_{ij} = \begin{cases} -1 & \text{if } i < 2^{n-1} \text{ and } j = 0, \\ (-1)^{\left(\lfloor \frac{i}{2^{n-1-j}} \rfloor \mod 2\right)+1} & \text{if } i < 2^{n-1} \text{ and } j \neq 0, \\ 1 & \text{if } X_i \geq 2^{n-1} \text{ and } j = 0, \\ (-1)^{\left(\lfloor \frac{i}{2^{n-1-j}} \rfloor \mod 2\right)} & \text{if } i < 2^{n-1} \text{ and } j \neq 0, \end{cases}$$

$$\boldsymbol{Y} = \begin{bmatrix} y_{00} & y_{01} & \cdots & y_{0(n-2)} & 1 \\ y_{10} & y_{11} & \cdots & y_{1(n-2)} & 1 \\ \vdots & \vdots & \ddots & \vdots & \\ y_{(2^{(n-1)}-1)0} & y_{(2^{(n-1)}-1)1} & \cdots & y_{(2^{(n-1)}-1)(n-2)} & 1 \end{bmatrix}, \quad \text{Dimensions of } \boldsymbol{Y} = 2^{n-1} \times n.$$

The (i, j)-th entry of matrix $\boldsymbol{Y}$ is defined as follows:

$$y_{ij} = \begin{cases} (-1)^{\left(\lfloor \frac{i}{2^{n-2-j}} \rfloor \mod 2\right)+1} & \text{if } j \neq 0, \\ 1 & \text{if } j = n-1, \end{cases}$$

$$\boldsymbol{B} = \begin{bmatrix} b_{00} & b_{01} & \cdots & b_{0(n^2-1)} \\ b_{10} & b_{11} & \cdots & b_{1(n^2-1)} \\ \vdots & \vdots & \ddots & \vdots \\ b_{(2^{2n-1}-1)0} & b_{(2^{2n-1}-1)1} & \cdots & b_{(2^{2n-1}-1)(n^2-1)} \end{bmatrix}, \quad \text{Dimensions of } \boldsymbol{B} = 2^{2n-1} \times n^2.$$

The (i, j)-th entry of matrix $\boldsymbol{B}$ is defined as follows:

$$b_{ij} = x_{pk} * y_{ql}$$

$$b_{ij} = \text{(p, k)-th entry of } \boldsymbol{X} * \text{(q, l)-th entry of } \boldsymbol{Y}$$

$$p = \left\lfloor \frac{i}{2^{n-1}} \right\rfloor, \quad q = i \mod 2^{n-1}, \quad k = j \mod n, \quad l = \left\lfloor \frac{j}{n} \right\rfloor$$

To Prove: Prove that $R_i = -R_{i+2^{2n-2}}$ $\forall i \in \mathbb{W}$ and $i < 2^{2n-2}$, where R_i is the i^{th} row of $\boldsymbol{B}$.

Proof.

$$\begin{gathered}
p^{\text{th}} \text{ row of } \boldsymbol{X} \;\mapsto\; x_p := \big(x_{p0}, x_{p1}, \cdots, x_{p(n-1)}\big) \\
q^{\text{th}} \text{ row of } \boldsymbol{Y} \;\mapsto\; y_q := \big(y_{q0}, y_{q1}, \cdots, y_{q(n-2)}, 1\big) \\
x_p = -x_{p+2^{n-1}}
\end{gathered}$$

$$\begin{gathered}
i^{\text{th}} \text{ row of } \boldsymbol{B} \;\mapsto\; R_i := y_q \cdot x_p \\
p = \left\lfloor \frac{i}{2^{n-1}} \right\rfloor, \quad q = i \mod 2^{n-1}
\end{gathered}$$

$$\begin{gathered}
(i + 2^{2n-2})^{\text{th}} \text{ row of } \boldsymbol{B} \;\mapsto\; R_{i+2^{2n-2}} := y_{q'} \cdot x_{p'} \\
p' = \left\lfloor \frac{i + 2^{2n-2}}{2^{n-1}} \right\rfloor = 2^{n-1} + \left\lfloor \frac{i}{2^{n-1}} \right\rfloor = 2^{n-1} + p \\
q' = \left(i + 2^{2n-2}\right) \mod 2^{n-1} = i \mod n = q \\
\therefore R_{i+2^{2n-2}} := y_{q'} \cdot x_{p'} = y_q \cdot x_{p+2^{n-1}} \\
\therefore R_{i+2^{2n-2}} := -y_q \cdot x_p \\
\therefore R_{i+2^{2n-2}} = -R_i \\
\therefore R_i = -R_{i+2^{2n-2}}
\end{gathered}$$

□

Conclusion: Thus, rows R_i and $R_{i+2^{2n-2}}$ of $\boldsymbol{B}$ are linearly dependent.

Example: Let $n = 2$

$$\boldsymbol{X} = \begin{bmatrix} -1 & -1 \\ -1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{Dimensions of } \boldsymbol{X} = 4 \times 2.$$

$$\boldsymbol{Y} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{Dimensions of } \boldsymbol{Y} = 2 \times 2.$$

$$\boldsymbol{B} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \quad \text{Dimensions of } \boldsymbol{B} = 8 \times 4.$$

Here, $2^{2n-2} = 4$. As can be seen, $R_0 = -R_4, R_1 = -R_5$

$R_2 = -R_6, R_3 = -R_7$.