Dhirubhai Ambani Institute of Information and
Communication Technology
Gandhinagar, Gujarat 382007

PC433 BTP-I
PC434 BTP-II

Under the guidance of
Prof. Arpita Mal

by

Nikita Shah

Student ID: 202101209

1 Proof:

Given: Let $n \in \mathbb{N}, n > 2$.

$$\boldsymbol{X} = \begin{bmatrix} x_{00} & x_{01} & \cdots & x_{0(n-1)} \\ x_{10} & x_{11} & \cdots & x_{1(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ x_{(2^n-1)0} & x_{(2^n-1)1} & \cdots & x_{(2^n-1)(n-1)} \end{bmatrix}, \quad \text{Dimensions of } \boldsymbol{X} = 2^n \times n.$$

The (i,j)-th entry of matrix $\boldsymbol{X}$ is defined as follows:

$$x_{ij} = \begin{cases} -1 & \text{if } i < 2^{n-1} \text{ and } j = 0, \\ (-1)^{\left(\lfloor \frac{i}{2^{n-1-j}} \rfloor \mod 2\right)+1} & \text{if } i < 2^{n-1} \text{ and } j \neq 0, \\ 1 & \text{if } X_i \geq 2^{n-1} \text{ and } j = 0, \\ (-1)^{\left(\lfloor \frac{i}{2^{n-1-j}} \rfloor \mod 2\right)} & \text{if } i < 2^{n-1} \text{ and } j \neq 0, \end{cases}$$

$$\boldsymbol{Y} = \begin{bmatrix} y_{00} & y_{01} & \cdots & y_{0(n-2)} & 1 \\ y_{10} & y_{11} & \cdots & y_{1(n-2)} & 1 \\ \vdots & \vdots & \ddots & \vdots & \\ y_{(2^{(n-1)}-1)0} & y_{(2^{(n-1)}-1)1} & \cdots & y_{(2^{(n-1)}-1)(n-2)} & 1 \end{bmatrix}, \quad \text{Dimensions of } \boldsymbol{Y} = 2^{n-1} \times n.$$

The (i,j)-th entry of matrix $\boldsymbol{Y}$ is defined as follows:

$$y_{ij} = \begin{cases} (-1)^{\left(\lfloor \frac{i}{2^{n-2-j}} \rfloor \mod 2\right)+1} & \text{if } j \neq 0, \\ 1 & \text{if } j = n-1, \end{cases}$$

$$\boldsymbol{B} = \begin{bmatrix} b_{00} & b_{01} & \cdots & b_{0(n^2-1)} \\ b_{10} & b_{11} & \cdots & b_{1(n^2-1)} \\ \vdots & \vdots & \ddots & \vdots \\ b_{(2^{2n-1}-1)0} & b_{(2^{2n-1}-1)1} & \cdots & b_{(2^{2n-1}-1)(n^2-1)} \end{bmatrix}, \quad \text{Dimensions of } \boldsymbol{B} = 2^{2n-1} \times n^2.$$

The (i,j)-th entry of matrix $\boldsymbol{B}$ is defined as follows:

$$b_{ij} = x_{pk} * y_{ql}$$

$$b_{ij} = \text{(p, k)-th entry of } \boldsymbol{X} * \text{(q, l)-th entry of } \boldsymbol{Y}$$

$$p = \left\lfloor \frac{i}{2^{n-1}} \right\rfloor, \quad q = i \mod 2^{n-1}, \quad k = j \mod n, \quad l = \left\lfloor \frac{j}{n} \right\rfloor$$

To Prove: Prove that $R_i - R_{i+step} = R_{i+2\cdot step} - R_{i+3\cdot step}\ \forall i \in \mathbb{W}$, $i+3\cdot step < 2^{2n-2}$ and $p \mod 2 = 0$, where R_i is the i^{th} row of $\boldsymbol{B}$ and $step = 2^{n-1}$.

Proof.

$$p^{\text{th}} \text{ row of } \boldsymbol{X} \ \mapsto\ x_p := (x_{p0}, x_{p1}, \cdots, x_{p(n-1)})$$
$$q^{\text{th}} \text{ row of } \boldsymbol{Y} \ \mapsto\ y_q := (y_{q0}, y_{q1}, \cdots, y_{q(n-2)}, 1)$$
$$x_p = -x_{p+2^{n-1}}$$

$$i^{\text{th}} \text{ row of } \boldsymbol{B} \ \mapsto\ R_i := y_q \cdot x_p$$
$$p = \left\lfloor \frac{i}{2^{n-1}} \right\rfloor, \quad q = i \mod 2^{n-1}$$

$$(i + 2^{n-1})^{\text{th}} \text{ row of } \boldsymbol{B} \ \mapsto\ R_{i+2^{n-1}} := y_{q_1} \cdot x_{p_1}$$
$$p_1 = \left\lfloor \frac{i + 2^{n-1}}{2^{n-1}} \right\rfloor = 1 + \left\lfloor \frac{i}{2^{n-1}} \right\rfloor = 1 + p$$
$$q_1 = \left(i + 2^{n-1}\right) \mod 2^{n-1} = i \mod 2^{n-1} = q$$

$$\therefore R_i - R_{i+2^{n-1}} := y_q \cdot x_p - y_{q_1} \cdot x_{p_1}$$

$$\therefore R_i - R_{i+2^{n-1}} := y_q \cdot (x_p - x_{p_1}) \tag{1}$$

$$(i + 2 \cdot 2^{n-1})^{\text{th}} \text{ row of } \boldsymbol{B} \ \mapsto\ R_{i+2\cdot 2^{n-1}} := y_{q_2} \cdot x_{p_2}$$
$$p_2 = \left\lfloor \frac{i + 2 \cdot 2^{n-1}}{2^{n-1}} \right\rfloor = 2 + \left\lfloor \frac{i}{2^{n-1}} \right\rfloor = 2 + p$$
$$q_2 = \left(i + 2.2^{n-1}\right) \mod 2^{n-1} = i \mod 2^{n-1} = q$$

$$(i + 3 \cdot 2^{n-1})^{\text{th}} \text{ row of } \boldsymbol{B} \ \mapsto\ R_{i+3\cdot 2^{n-1}} := y_{q_3} \cdot x_{p_3}$$
$$p_3 = \left\lfloor \frac{i + 2 \cdot 2^{n-1}}{2^{n-1}} \right\rfloor = 3 + \left\lfloor \frac{i}{2^{n-1}} \right\rfloor = 3 + p$$
$$q_3 = \left(i + 3.2^{n-1}\right) \mod 2^{n-1} = i \mod 2^{n-1} = q$$

$$\therefore R_{i+2\cdot 2^{n-1}} - R_{i+3\cdot 2^{n-1}} := y_q \cdot (x_{p_2} - x_{p_3}) \tag{2}$$

We need to prove that

$$R_i - R_{i+2^{n-1}} = R_{i+2\cdot 2^{n-1}} - R_{i+3\cdot 2^{n-1}}$$
$$\therefore y_q \cdot (x_p - x_{p_1}) = y_q \cdot (x_{p_2} - x_{p_3})$$
$$\therefore x_p - x_{p_1} = x_{p_2} - x_{p_3}$$
$$\therefore x_p - x_{p+1} = x_{p+2} - x_{p+3} \quad \forall p \mod 2 = 0 \text{ and } p < 2^{n-1} - 3$$

$$\therefore (-1)^{\left(\lfloor \frac{p}{2^{n-1-j}} \rfloor \mod 2\right)+1} - (-1)^{\left(\lfloor \frac{p+1}{2^{n-1-j}} \rfloor \mod 2\right)+1}$$
$$= (-1)^{\left(\lfloor \frac{p+2}{2^{n-1-j}} \rfloor \mod 2\right)+1} - (-1)^{\left(\lfloor \frac{p+3}{2^{n-1-j}} \rfloor \mod 2\right)+1} \forall 0 \leq j < n, n \in \mathbb{N} \quad (3)$$

$$\text{For } j > n-3, \left(\lfloor \frac{p}{2^{n-1-j}} \rfloor \mod 2\right) = \left(\lfloor \frac{p+2}{2^{n-1-j}} \rfloor \mod 2\right) = 0$$

and

$$\left(\lfloor \frac{p+1}{2^{n-1-j}} \rfloor \mod 2\right) = \left(\lfloor \frac{p+3}{2^{n-1-j}} \rfloor \mod 2\right) = 1$$
$$\therefore L.H.S.\ (3) = -2 \text{ and } R.H.S.\ (3) = -2$$
$$\therefore L.H.S.\ (3) = R.H.S.\ (3)$$

For $0 < j \leq n-3$, $(p+1) \mod 2^{n-1-j} \neq 0$ and $(p+3) \mod 2^{n-1-j} \neq 0$.
$\therefore \left(\lfloor \frac{p}{2^{n-1-j}} \rfloor \mod 2\right) = \left(\lfloor \frac{p+1}{2^{n-1-j}} \rfloor \mod 2\right)$ and $\left(\lfloor \frac{p+2}{2^{n-1-j}} \rfloor \mod 2\right) = \left(\lfloor \frac{p+3}{2^{n-1-j}} \rfloor \mod 2\right)$.

$$\therefore L.H.S.\ (3) = 0 \text{ and } R.H.S.\ (3) = 0$$
$$\therefore L.H.S.\ (3) = R.H.S.\ (3)$$

Thus, (3) holds for $\forall 0 \leq j < n$, $n \in \mathbb{N}$.
Hence, proved.

□

Conclusion: We have shown linear dependencies amongst rows R_i, $R_{i+2^{n-1}}$, $R_{i+2.2^{n-1}}$ and $R_{i+3.2^{n-1}}$ of $\boldsymbol{B}$ for given constraints.

Example: Let $n = 3$

$$\boldsymbol{X} = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \quad \text{Dimensions of } \boldsymbol{X} = 8 \times 3.$$

$$\boldsymbol{Y} = \begin{bmatrix} -1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{Dimensions of } \boldsymbol{Y} = 4 \times 3.$$

Dimensions of $\boldsymbol{B} = 8 \times 4$.

Here, $2^{n-1} = 4$ and $R_i = -R_{i+16}$
$\therefore R_0 - R_4 = R_8 - R_{12} \implies R_{16} - R_{20} = R_{24} - R_{28}$,
$R_1 - R_5 = R_9 - R_{13}$,
$R_2 - R_4 = R_8 - R_{14}$,
$R_3 - R_7 = R_{11} - R_{15}$
and similarly the rest of the row dependencies can be shown.