Singular Value Decomposition

SVD

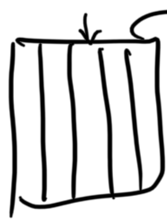
Matrix factorization

$$X = USV^T \qquad N > D$$

$N \times D$

$N \times N$ $\qquad$ $D \times D$

$U \rightarrow$ left singular matrix



left singular vector.

$$U^{-1} = U^T \qquad U^T U = I$$

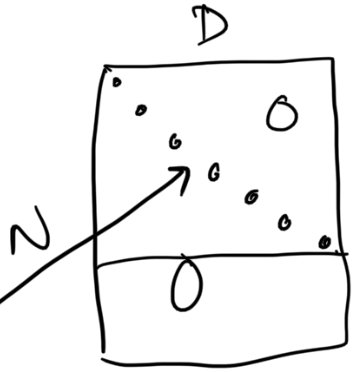
$$U_{:i}^T U_{:j} = 1 \quad \text{if } i = j$$
$$= 0 \quad \text{if } i \neq j$$

$U \rightarrow$ orthonormal matrix

$V \rightarrow$ right singular matrix.

V – orthonormal matrix.

$S \rightarrow$



D non-zero values

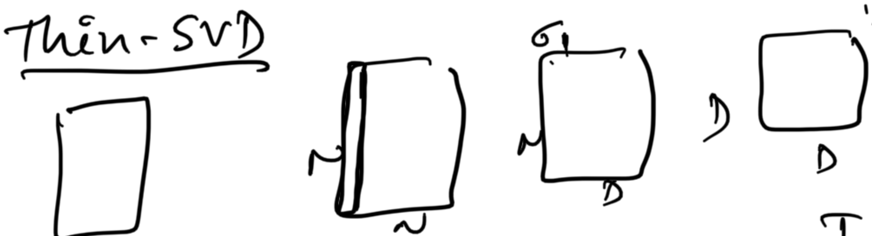
Singular values

How to perform SVD?

σ_1 - 1 s.v

σ_2 - 2 s.v

Thin-SVD



$X = \tilde{U} \; S \; V^T$

$$\boxed{\sigma_1 > \sigma_2 > \dots > \sigma_D}$$

$$\underbrace{\boxed{\tilde{X}}}_{N \times D} = \underbrace{U_{:1}}_{N \times 1} * \sigma_1 * \underbrace{V_{:1}^T}_{1 \times D}$$

$$\boxed{\tilde{X} = U_L * S_L * V_L^T}$$

L

$\tilde{X}$ is an approximate version of X

Example $N = 10^6$

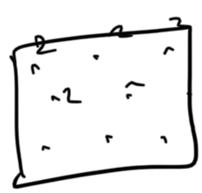
$D = 10^5$

$X = 10^6 \times 10^5$

$L = 20$

$$\boxed{\begin{aligned} U_L &\quad 10^6 \times 20 \\ S_L &\cdot\quad 20 \\ V_L &= 10^5 \times 20 \end{aligned}}$$

$\tilde{X}$



Assume that X is centered

$X = USV^T$

$X = U S V$

$$X^T X = (USV^T)^T (USV^T)$$

$$= V S^T U^T U S V^T$$

$$= V S^T S V^T$$

$$\begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{bmatrix}$$

$$= V D V^T$$

$$(X^T X) V = V D V^T V$$

$$\begin{bmatrix} \sigma_1^2 & & \\ & \sigma_2^2 & \\ & & \sigma_3^2 \end{bmatrix}$$

$$(X^T X) V = V D$$

$$\boxed{(X^T X) V_{:i} = V_{:i} \sigma_i^2}$$

Columns of V are the eigen-vectors of $X^T X$ and D is the eigen value

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$$X X^T = (USV^T)(USV^T)^T$$

Columns of U are the eigen vectors of $X X^T$

$\frac{1}{N} X^T X$ is the covariance matrix of the data.

V contains the principal components.

$$\tilde{X} = \underbrace{\tilde{U}}_{N\times L}\ \underbrace{\tilde{S}}_{L\times L}\ \underbrace{\tilde{V}^T}_{L\times D}$$

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rank.

$$\begin{bmatrix} 3 & 3 & 3 \\ 7 & 7 & 7 \\ 2 & 2 & 2 \\ 9 & 9 & 9 \end{bmatrix} \qquad r = 1$$

$$\begin{bmatrix} 3 & 1 & 2 \\ 7 & 4 & 3 \\ 2 & 9 & -7 \\ 9 & 2 & 7 \end{bmatrix} \qquad r = 2$$

for a $N \times D$ matrix

if $r = D$ full-rank matrix.

if $r < D$ reduced-rank matrix

SVD with L singular vectors

rank L approx of matrix X

$$X = \tilde{U} \, \tilde{S} \, \tilde{V}^T$$

4×3 $\quad$ 4×2 $\quad$ 2×2 $\quad$ 2×3

$L = 2$

$\tilde{X}$ will have rank 2

rank-L approx Method 1 $\rightarrow \tilde{X}_1$

method 2 $\rightarrow \tilde{X}_2$

SVD $\rightarrow \tilde{X}_{SVD}$

$$\| X - \tilde{X}_1 \|_F$$

$$\| X - \tilde{X}_2 \|_F$$

$$\vdots$$

$\| \ldots \tilde{X} \ldots \|_F \rightarrow$ smallest

$\| X - A_{SVD} \|_F$