Почему $h = n^{-1/5}$ — опт?

$$MSE(h) = \int_{\mathbb{R}} E\left(\tilde{p}_h(x) - p(x)\right)^2 dx$$

Пусть носитель $p(x)$ — конечный интервал

$$\gamma = \int_{\mathbb{R}} \left(p''(x)\right)^2 dx < +\infty$$

q - ядро, сим. т.ч. $\alpha = \int_{\mathbb{R}} q^2(x)\, dx < +\infty$

$$\beta = \int_{\mathbb{R}} x^2 q(x)\, dx < +\infty$$

Утв. Тогда оптимальная ширина ядра $h_n^* \sim n^{-1/5}$

$$MSE(h_n^*) \sim n^{-4/5}$$

▲ Идея док-ва:

$$p_h(x) = E\tilde{p}_h(x) = E\,\frac{1}{nh}\sum_{i=1}^{n} q\left(\frac{x-X_i}{h}\right) = \frac{1}{h_n} E\, q\left(\frac{x-X_1}{h_n}\right) =$$

$$= \frac{1}{h_n}\int_{\mathbb{R}} p(y)\, q\Big(\underbrace{\frac{x-y}{h_n}}_{z}\Big)\, dy = \int_{\mathbb{R}} q(z)\, p(x - zh_n)\, dz =$$

$$y = x - zh_n$$

$$= \int_{\mathbb{R}} q(z)\left(p(x) - zh_n p'(x) + \frac{1}{2} z^2 h_n^2 p''(x)\right) dz =$$

$$= p(x) \quad - \underset{\text{т.к. } q\text{-сим.}}{0} \quad + \frac{1}{2}\beta h_n^2 p''(x)$$

$$\tilde{p}_h(x) = p_h(x) + \frac{\xi_n(x)}{\sqrt{n h_n}}$$ ← по какой-то теореме из презентации

$$E\xi_n(x) = 0 \qquad \xi_n(x) \to N(0, \alpha p(x))$$

$$MSE(h) = \int_{\mathbb{R}} E\left(\tilde{p}_h(x) - p(x)\right)^2 dx = \int_{\mathbb{R}} E\left(p(x) + \frac{1}{2}\beta h_n^2 p''(x) + \frac{\xi_n(x)}{\sqrt{n h_n}} - p(x)\right)^2 dx =$$

$$= \frac{1}{4}\beta^2 h_n^4 \int_{\mathbb{R}} \left(p''(x)\right)^2 dx + \int_{\mathbb{R}} 2\,\frac{1}{2}\beta h_n^2 p''(x) \underbrace{E\frac{\xi_n(x)}{\sqrt{n h_n}}}_{0} dx + \int_{\mathbb{R}} E\frac{\xi_n^2(x)}{n h_n} dx \sim$$

$$\sim \frac{1}{4}\beta^2 h_n^4 \gamma + \frac{\alpha}{n h_n} \to \min_\alpha$$

т.к. $\int_{\mathbb{R}} \alpha p(x)\, dx = \alpha$

$$\frac{\partial MSE}{\partial h_n} = \beta^2 h_n^3 \gamma - \frac{\alpha}{n h_n^2} = 0$$

$$h_n = \alpha^{1/5} \underline{\underline{n^{-1/5}}} \beta^{-2/5} \gamma^{-1/5}$$ ← асимптотика

$$MSE(h_n^*) = \frac{5}{4}\alpha^{4/5} \beta^{2/5} \gamma^{1/5} \underline{\underline{n^{-4/5}}}$$ ← скорость сходимости