Теория наилучших оценок

7

Информация фишера

7.1

$X = (X_1 \ldots X_n)$ – выборка из распр. $P \in \mathcal{P}$ – домин. сем. с $p_\theta(x)$

$$L_x(\theta) = \prod_{i=1}^{n} p_\theta(x_i) \qquad l_x(\theta) = \sum_{i=1}^{n} \log p_\theta(x_i)$$

$U_x(\theta) = \dfrac{\partial\, l_x(\theta)}{\partial \theta}$ – вклад выборки X в оценку парам θ

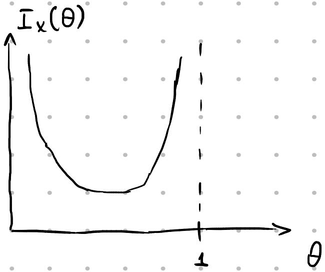
$I_x(\theta) = D_\theta\, U_x(\theta)$ – инф. Фишера о парам. θ, содерж. в X

Пример $\quad X_1 \ldots X_n \sim \mathrm{Bern}(\theta)$

$$\triangle \quad U_x(\theta) = \frac{\partial}{\partial\theta} \log\left(\theta^{\sum x_i}(1-\theta)^{n-\sum x_i}\right) = \frac{\partial}{\partial\theta}\left[\sum x_i \log\theta + \left(n-\sum x_i\right)\log(1-\theta)\right] =$$

$$= \frac{\sum x_i}{\theta} - \frac{n-\sum x_i}{1-\theta} = \frac{\sum x_i - \theta\sum x_i - n\theta + n\sum x_i}{\theta(1-\theta)} = \frac{\sum x_i - n\theta}{\theta(1-\theta)}$$

$$I_x(\theta) = D_\theta U_x(\theta) = \frac{D\sum x_i}{\theta^2(1-\theta)^2} = \frac{n}{\theta(1-\theta)}$$



Утв. 1) $E_\theta\, U_x(\theta) = 0$ [E1 – E4]

2) $I_x(\theta) = E_\theta\, U_x^2(\theta)$ – очев из 1)

3) $I_x(\theta) = n\, i(\theta)$ где $i(\theta) = I_{x_1}(\theta)$ – инф. ф. от 1 эл-та

4) $-E_\theta \dfrac{\partial^2 L_x(\theta)}{\partial \theta^2}$ (если сущ.)

▲ 1) $$U_x(\theta) = \frac{\partial L_x(\theta)}{\partial \theta} = \frac{\partial}{\partial \theta} \sum_{i=1}^{n} \log p_\theta(x_i) = \sum_{i=1}^{n} \frac{\partial}{\partial \theta} L_{x_i}(\theta) = \sum_{i=1}^{n} U_{x_i}(\theta)$$

$$E_\theta\, U_{x_1}(\theta) = \int_{\mathbb{R}} p_\theta(x) \frac{\partial}{\partial \theta} \log p_\theta(x)\, dx = \int_{\mathbb{R}} \frac{\partial p_\theta(x)}{\partial \theta} dx = [E1 - E4] =$$

$$= \frac{\partial}{\partial \theta} \int_{\mathbb{R}} p_\theta(x)\, dx = \frac{\partial}{\partial \theta} 1 = 0$$

3) $I_x(\theta) = D_\theta U_x(\theta) = D_\theta \sum_{i=1}^{n} U_{x_i}(\theta) \overset{\text{т.к. нез.}}{=} \sum_{i=1}^{n} D_\theta U_{x_i}(\theta) = n\, i(\theta)$

4) $$\frac{\partial^2}{\partial\theta^2} L_x(\theta) = \frac{\partial^2}{\partial\theta^2} \log p_\theta(x) = \frac{\partial}{\partial\theta} \frac{\partial p_\theta(x)/\partial\theta}{p_\theta(x)} =$$

$$= \frac{\frac{\partial^2 p_\theta(x)}{\partial\theta^2}}{p_\theta(x)} - \Bigg(\underbrace{\frac{\frac{\partial p_\theta(x)}{\partial\theta}}{p_\theta(x)}}_{U_x(\theta)}\Bigg)^2$$

Возьмём м.о. от обоих частей.

$$E_\theta \frac{\partial^2}{\partial\theta^2} \log p_\theta(x_1) = E_\theta \frac{\partial^2 p_\theta(x)}{\partial\theta^2} \frac{1}{p_\theta(x)} - \overbrace{E_\theta U_x^2(\theta)}^{I_x(\theta)}$$

$$\int p_\theta(x) \frac{\partial^2 p_\theta(x)/\partial\theta^2}{p_\theta(x)}\, dx = \frac{\partial^2}{\partial\theta^2} \int p_\theta(x)\, dx = \frac{\partial^2}{\partial\theta^2} 1 = 0$$