1. Случайная величина $\xi \sim R(0,\theta)$. По выборке объёма n найдены оценки параметра θ:

$$\tilde{\theta}_1 = 2\bar{X}, \quad \tilde{\theta}_2 = X_{min}, \quad \tilde{\theta}_3 = X_{max}, \quad \tilde{\theta}_4 = \left(X_1 + \frac{\sum_{k=2}^{n} X_k}{n-1}\right)$$

а) Проверить оценки на несмещённость и состоятельность. Если необходимо, исправить оценки.

1) $\xi \sim R(0;\theta) \Rightarrow p(x) = \frac{1}{\theta}\{(0;\theta)\}$

$$M\xi = \int_0^\theta x P(x)dx = \frac{1}{\theta}\frac{\theta^2}{2} = \frac{\theta}{2}$$

$$M\xi^2 = \int_0^\theta x^2 P(x)dx = \frac{1}{\theta}\frac{\theta^3}{3} = \frac{\theta^2}{3}$$

$$D\xi = M\xi^2 - (M\xi)^2 = \frac{\theta^2}{3} - \frac{\theta^2}{4} = \frac{\theta^2}{12}$$

$$\tilde{\theta}_1 = 2\bar{X} = \frac{2}{n}\sum_{k=1}^{n} X_k$$

Несмещённость: $M\tilde{\theta}_1 = \frac{2}{n} M\left(\sum_{k=1}^{n} X_k\right) =$

$$= \left\{\begin{matrix} X_k - \text{случ. вел.} \\ X_k \sim R(0;\theta) \end{matrix}\right\} = \frac{2}{n} n M\xi = 2M\xi = 2\cdot\frac{\theta}{2} = \theta$$

$\Rightarrow$ несмещённая $(\forall\theta > 0)$

Состоятельность: $D\tilde{\theta}_1 = D\left(\frac{2}{n}\sum_{k=1}^{n} X_k\right) =$

$$= \frac{4}{n^2} D\left(\sum_{k=1}^{n} X_k\right) = \{\text{незави.}\} = \frac{4}{n^2} n D\xi = \frac{\theta^2}{3n} \xrightarrow[\forall\theta>0]{n\to\infty} 0$$

$\Rightarrow$ по достаточному условию оценка явл. состоятельной

2) $\tilde{\theta}_2 = X_{min}$; $X_{min} \sim \underbrace{\left(1-(1-F(y))^n\right)}_{\Phi(y)}$, где

$$F(y) = \int_{-\infty}^{y} p(t)dt = \int_0^y \frac{1}{\theta}dt = y/\theta$$

$$p_2(x) = \Phi'(x) = \frac{n}{\theta}\left(1-\frac{y}{\theta}\right)^{n-1}\{(0;\theta)\}$$

$$M\tilde{\theta}_2 = \int_0^\theta x \frac{n}{\theta}\left(1-\frac{x}{\theta}\right)^{n-1} dx = \left\{ \begin{matrix} 1-\frac{x}{\theta}=t \\ dx=-\theta dt \\ x=\theta(1-t) \end{matrix} \right\} =$$

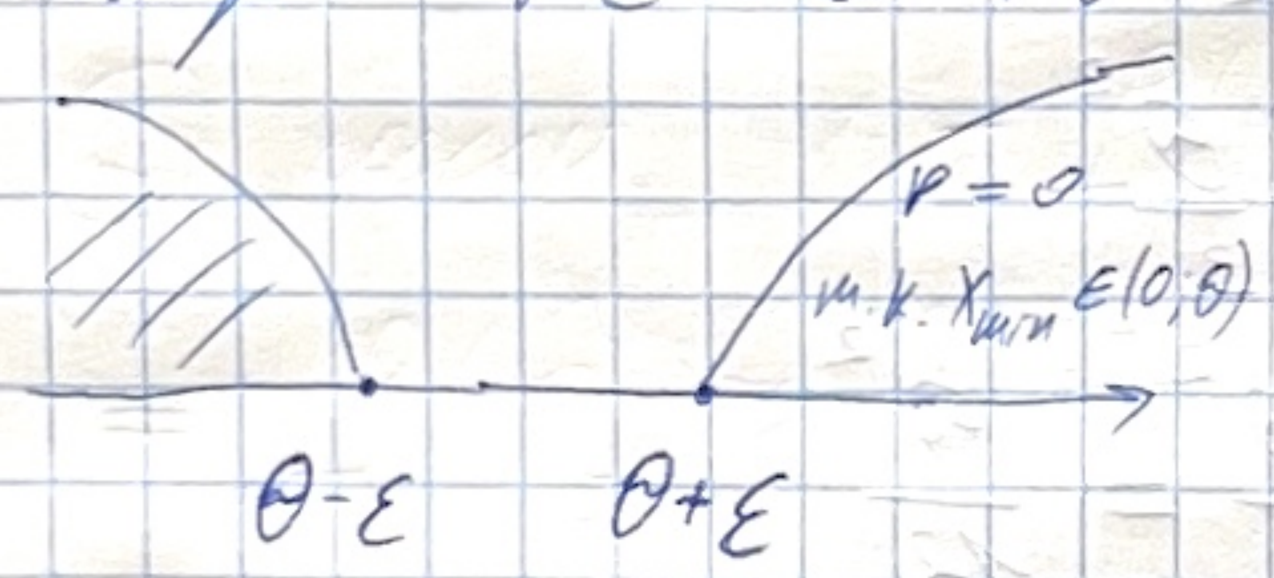
$$= \frac{n}{\theta}\int_1^0 \theta(1-t)t^{n-1}(-\theta dt) = n\theta\int_0^1 \left(t^{n-1}-t^n\right)dt =$$

$$= n\theta\left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{\theta}{n+1}$$ – не явл. несмещ.

$\tilde{\tilde{\theta}}_2 = (n+1)\tilde{\theta}_2$ – исправленная оценка

2.1) состоятельность $\tilde{\theta}_2$:

опр: $\forall \varepsilon > 0 \; \forall \theta > 0: \; P(|\tilde{\theta}_2-\theta|\geq\varepsilon)\xrightarrow{n\to\infty}0$



$P(|\tilde{\theta}_2-\theta|\geq\varepsilon) =$ [redacted] $=$

$= P(\tilde{\theta}_2 \leq \theta-\varepsilon) =$

$= P(\tilde{\theta}_2<\theta-\varepsilon)+P(\tilde{\theta}_2=\theta-\varepsilon) = \Phi(\theta-\varepsilon)$ (т.к. непр. случ. вел.)

$= 1-\left(1-\frac{\theta-\varepsilon}{\theta}\right)^n = 1-\left(\frac{\varepsilon}{\theta}\right)^n$ $\exists\varepsilon \; \exists\theta: \theta>\varepsilon$ и $P\xrightarrow[n\to\infty]{}1\neq0$

$\Rightarrow$ не явл. состоятельной

2.2) состоятельность $\tilde{\tilde{\theta}}_2$:

скачка нет

$P(|\tilde{\tilde{\theta}}_2-\theta|\geq\varepsilon)\geq P(\tilde{\tilde{\theta}}_2\geq\theta+\varepsilon) = 1-\tilde{\Phi}(\theta+\varepsilon) =$

$= 1-P((n+1)\tilde{\theta}_2<\theta+\varepsilon) = \left(1-\frac{\theta+\varepsilon}{\theta(n+1)}\right)^n \xrightarrow{n\to\infty} e^{-\frac{\theta+\varepsilon}{\theta}}>0$

$\Rightarrow \exists\varepsilon,\theta>0: \; P\xrightarrow{n\to\infty}C>0 \Rightarrow$ не явл. состоятельной

3) $\tilde{\theta}_3 = X_{max}$, $X_{max}\sim\Psi(x)=(F(x))^n=\left(\frac{x}{\theta}\right)^n$

$p_3(x)=\Psi'(x)=n\left(\frac{x}{\theta}\right)^{n-1}\frac{1}{\theta}\{(0,\theta)\}$

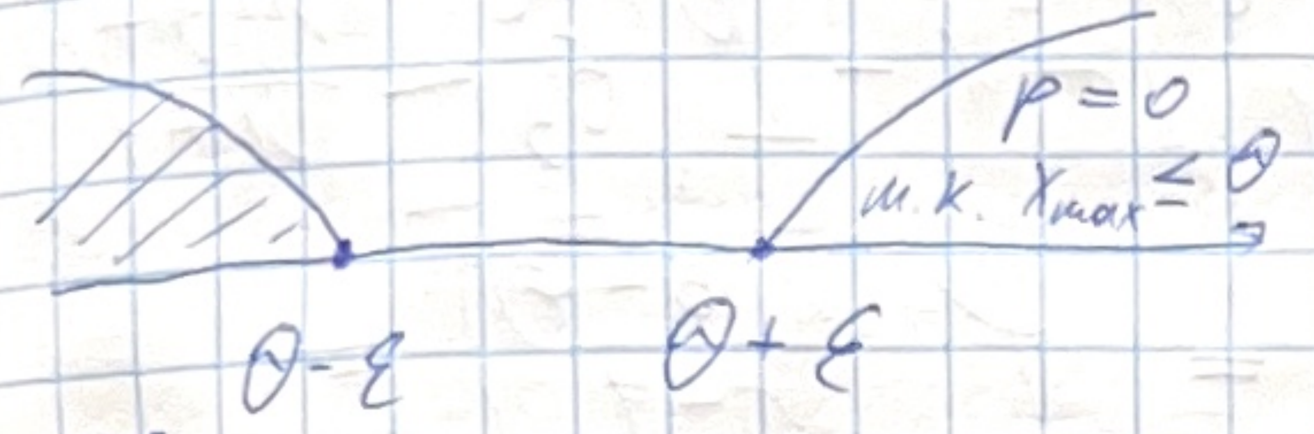
Несмещенность: $M\tilde{\theta}_3 = \int_0^\theta x\frac{n}{\theta}\left(\frac{x}{\theta}\right)^{n-1}dx =$

$= \frac{n}{\theta^n}\int_0^\theta x^n dx = \frac{n\cdot\theta^{n+1}}{\theta^n\cdot (n+1)} = \frac{n}{n+1}\theta$ – не явл. несмещенной

$\tilde{\tilde{\theta}}_3 = \frac{n+1}{n}\tilde{\theta}_3$ – исправленная оценка

3.1) Состоятельность $\tilde{\theta}_3$

$P(|\tilde{\theta}_3-\theta| \geq \varepsilon) =$



$= P(\tilde{\theta}_3 \leq \theta-\varepsilon) =$

скачка нет

$= P(\tilde{\theta}_3 < \theta-\varepsilon)$. Если θ, ε такие, что $\theta-\varepsilon \leq 0$, то $P(\tilde{\theta}_3 < \theta-\varepsilon) = 0$ т.к. $X_{max} \in [0;\theta]$ и $P = 0 \xrightarrow[n\to\infty]{} 0$. Иначе

$P(|\tilde{\theta}_3-\theta| \geq \varepsilon) = \Psi(\theta-\varepsilon) = \left(\frac{\theta-\varepsilon}{\theta}\right)^n = \left(1-\frac{\varepsilon}{\theta}\right)^n \longrightarrow$

$\xrightarrow[n\to\infty]{} 0$ при $\theta-\varepsilon > 0$. $\left(0 < 1-\frac{\varepsilon}{\theta} < 1\right)$

Итак: θ, ε: $\theta-\varepsilon \leq 0$ $P = 0 \longrightarrow 0$

$\theta-\varepsilon > 0$ $P \xrightarrow[n\to\infty]{} 0$

$\Rightarrow \forall\theta>0\ \forall\varepsilon>0\ P(|\tilde{\theta}_3-\theta| \geq \varepsilon) \xrightarrow{n\to\infty} 0$

$\Rightarrow$ оценка явл. состоятельной

3.2) Состоятельность $\tilde{\tilde{\theta}}_3$

$P(|\tilde{\tilde{\theta}}_3-\theta| \geq \varepsilon) = P(\tilde{\tilde{\theta}}_3 \leq \theta-\varepsilon) + P(\tilde{\tilde{\theta}}_3 \geq \theta+\varepsilon) =$

$= P\left(\tilde{\theta}_3 \leq \frac{n(\theta-\varepsilon)}{n+1}\right) + 1 - P(\tilde{\tilde{\theta}}_3 < \theta+\varepsilon) =$

$= P\left(\tilde{\theta}_3 < \frac{n(\theta-\varepsilon)}{n+1}\right) + 1 - P\left(\tilde{\theta}_3 < \frac{n(\theta+\varepsilon)}{n+1}\right)$

Если $\theta-\varepsilon \leq 0$, то $P\left(\tilde{\theta}_3 < \frac{n(\theta-\varepsilon)}{n+1}\right) = 0$ и

$P(|\tilde{\tilde{\theta}}_3-\theta| \geq \varepsilon) = 1 - \left(\frac{n(\theta+\varepsilon)}{(n+1)\theta}\right)^n$

т.к. $\theta-\varepsilon \leq 0$, то $\theta \leq \varepsilon \Rightarrow \theta+\varepsilon \geq 2\theta$

и $\forall n \geq 1$ $\hookrightarrow$ $1+\frac{1}{n} = \frac{n+1}{n} \leq 2 \Rightarrow \frac{n}{n+1} \geq \frac{1}{2}$

$\Rightarrow \frac{(\theta+\varepsilon)n}{n+1} \geq 2\theta \cdot \frac{1}{2} = \theta \Rightarrow P\left(X_{max} < \frac{n(\theta+\varepsilon)}{n+1}\right) = 1$

т.к. $X_{max} < \theta \Rightarrow$

$\Rightarrow$ при $\theta - \varepsilon \le 0$ $P(|\tilde{\tilde{\theta}}_3 - \theta| \ge \varepsilon) = 0 + 1 - 1 = 0 \to$
$\to 0$ при $n \to \infty$

Если $\theta - \varepsilon > 0$, то

$$P(|\tilde{\tilde{\theta}}_3 - \theta| \ge \varepsilon) = P\left(\tilde{\theta}_3 < \frac{n(\theta-\varepsilon)}{n+1}\right) + 1 - P\left(\tilde{\theta}_3 < \frac{n(\theta+\varepsilon)}{n+1}\right)$$

① рассм. $P\left(\tilde{\theta}_3 < \frac{n(\theta-\varepsilon)}{n+1}\right) = F\left(\frac{n(\theta-\varepsilon)}{n+1}\right) =$

$$= \left(\frac{n(\theta-\varepsilon)}{(n+1)\theta}\right)^n = \left(\frac{\theta-\varepsilon}{\theta}\right)^n \left(\frac{n}{n+1}\right)^n = \left(1 - \frac{\varepsilon}{\theta}\right)^n \frac{1}{\left(1+\frac{1}{n}\right)^n} \to$$

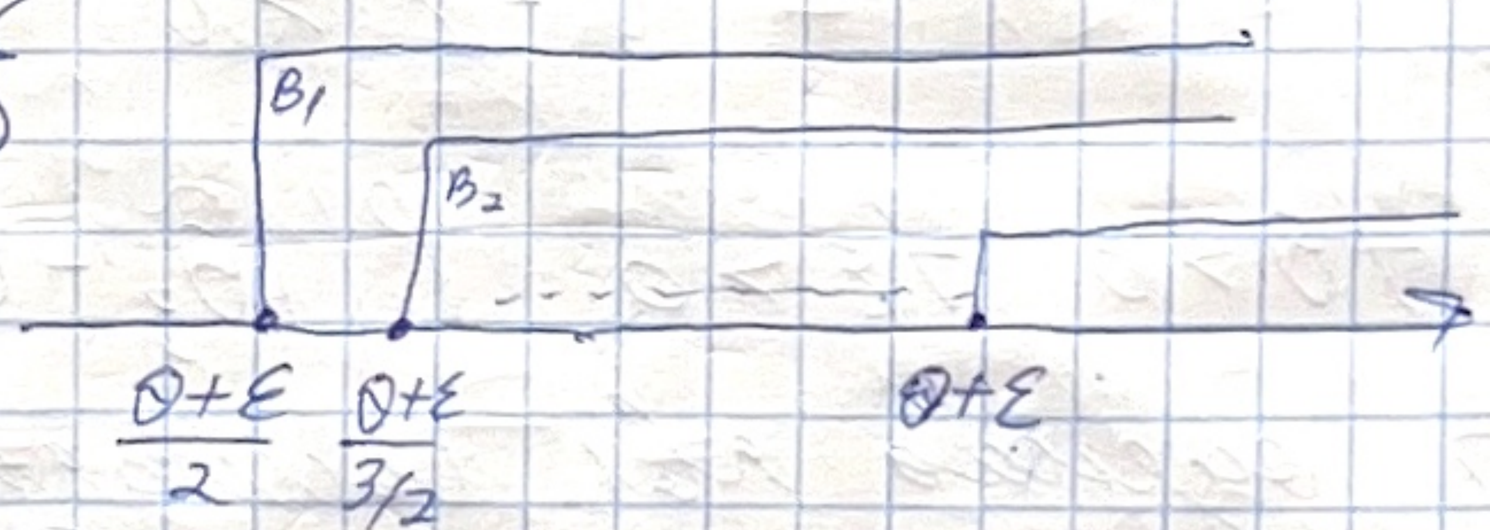
$$\xrightarrow[n\to\infty]{} 0 \cdot \frac{1}{e} = 0 \quad \text{т.к. } 0 < 1 - \frac{\varepsilon}{\theta} < 1$$

② рассм. $1 - P\left(\tilde{\theta}_3 < \frac{n(\theta+\varepsilon)}{n+1}\right) =$

$$= P\left(\tilde{\theta}_3 \ge \frac{n(\theta+\varepsilon)}{n+1}\right) = P\left(\tilde{\theta}_3 \ge \frac{\theta+\varepsilon}{1+\frac{1}{n}}\right)$$

Рассм. послед. вложенных событий B_i

$B_1 = \left\{\omega : \xi_3(\omega) \ge \frac{\theta+\varepsilon}{2}\right\}$



$B_2 \to \tilde{\theta}_3 \ge \frac{\theta+\varepsilon}{3/2}$

$B_3 \to \tilde{\theta}_3 \ge \frac{\theta+\varepsilon}{4/3}$

$B_{n+1} \subset B_n$

$B = \{\omega : \xi_3(\omega) \ge \theta + \varepsilon\} = \bigcap_{i=1}^{\infty} B_i$

$B_n \to \tilde{\theta}_3 \ge \frac{\theta+\varepsilon}{1+\frac{1}{n}}$

$B_{n+1} \to \tilde{\theta}_3 \ge \frac{\theta+\varepsilon}{1+\frac{1}{n+1}}$

По св-ву непрерывности вероятности

$$P(B) = \lim_{n\to\infty} P(B_n)$$

. . . .

$$P(B) = P(\tilde{\theta}_3 \ge \theta + \varepsilon) \underset{\text{т.к. } x_{max} < \theta}{=} 0 = \lim_{n\to\infty} P\left(\tilde{\theta}_3 \ge \frac{\theta+\varepsilon}{1+\frac{1}{n}}\right)$$

$$\Rightarrow P(|\tilde{\tilde{\theta}}_3 - \theta| \ge \varepsilon) \xrightarrow{n\to\infty} 0 \cdot \frac{1}{e} + 0 = 0$$

Итак: если $\theta - \varepsilon > 0$, то $P(|\tilde{\tilde{\theta}}_3 - \theta| \ge \varepsilon) \xrightarrow{n\to\infty} 0$

если $\theta - \varepsilon \le 0$, то $P(|\tilde{\tilde{\theta}}_3 - \theta| \ge \varepsilon) \xrightarrow{n\to\infty} 0$

$\Rightarrow$ оценка $\widehat{\theta}_3$ – состоятельна

4) $\widetilde{\theta}_4 = X_1 + \frac{1}{n-1}\sum_{k=2}^{n} X_k$

Несмещённость: $M\widetilde{\theta}_4 = MX_1 + \frac{1}{n-1} M\left(\sum_{k=2}^{n} X_k\right) =$

$= M\xi + \frac{1}{n-1}(n-1)M\xi = 2M\xi = 2\cdot\frac{\theta}{2} = \theta \Rightarrow$

$\Rightarrow$ несмещённая

Состоятельность:

X_1 – случ. вел. $X_1 \sim R(0;\theta)$, $X_1 \xrightarrow{P} X_1$

ЗБЧ Хинчина: X_k – независ. одинаково распред. случ. вел. с $MX < \infty$ (Mx сущ.)

$\Rightarrow \frac{1}{n-1}\sum_{k=2}^{n} X_k \xrightarrow{P} M\xi$

По арифметическим св-вам сходимости
$\left(\xi_k \xrightarrow{P} \xi,\ \eta_k \xrightarrow{P} \eta \Rightarrow \xi_k + \eta_k \xrightarrow{P} \xi + \eta\right)$

$\widetilde{\theta}_4 \xrightarrow{P} X_1 + M\xi$ – случ. вел., отличная от const ($\neq \theta$) $\Rightarrow$ не явл. состоятельной.

б) Какая из исправленных оценок более эффективна?

$\widetilde{\theta}_2 = (n+1)X_{min}$, $\widetilde{\theta}_3 = \frac{n+1}{n}\widehat{\theta}_3 = \frac{n+1}{n}X_{max}$

$X_{min} \sim 1 - \left(1 - \frac{x}{\theta}\right)^n$ $\qquad X_{max} \sim \left(\frac{x}{\theta}\right)^n$

$p_{min} = \frac{n}{\theta}\left(1 - \frac{x}{\theta}\right)^{n-1} \{(0;\theta)\}$ $\qquad p_{max} = \frac{n}{\theta}\left(\frac{x}{\theta}\right)^{n-1} \{(0;\theta)\}$

Вычислим дисперсии исправленных оценок:

$$D\tilde{\theta}_2 = (n+1)^2 D x_{min}$$

$$M x_{min} = \int_0^\theta x \frac{n}{\theta}\left(1-\frac{x}{\theta}\right)^{n-1} dx = \frac{\theta}{n+1}$$

$$M x_{min}^2 = \int_0^\theta x^2 \frac{n}{\theta}\left(1-\frac{x}{\theta}\right)^{n-1} dx = \left\{ \begin{aligned} 1-\frac{x}{\theta} &= t \\ dx &= -\theta dt \\ x &= \theta(1-t) \end{aligned} \right\} =$$

$$= \frac{n}{\theta}\int_1^0 \theta^2(1-t)^2 t^{n-1}(-\theta dt) = n\theta^2 \int_0^1 \left(t^{n-1} - 2t^n + t^{n+1}\right) dt =$$

$$= n\theta^2\left(\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}\right) = n\theta^2\left(\frac{n^2+3n+2-2n^2-4n+n^2+n}{n(n+1)(n+2)}\right) =$$

$$= \frac{2\theta^2}{(n+1)(n+2)};$$

$$D x_{min} = \frac{2\theta^2}{(n+1)(n+2)} - \frac{\theta^2}{(n+1)^2} = \frac{2\theta^2(n+1) - \theta^2(n+2)}{(n+1)^2(n+2)} =$$

$$= \frac{n\theta^2}{(n+1)^2(n+2)} \Rightarrow \boxed{D\tilde{\theta}_2 = \frac{n\theta^2}{n+2}}$$

$$D\tilde{\theta}_3 = \left(\frac{n+1}{n}\right)^2 D x_{max}, \quad M x_{max} = \frac{n}{n+1}\theta$$

$$M x_{max}^2 = \int_0^\theta x^2 \frac{n}{\theta}\left(\frac{x}{\theta}\right)^{n-1} dx = \frac{n}{\theta^n}\int_0^\theta x^{n+1} dx =$$

$$= \frac{n}{\theta^n}\frac{\theta^{n+2}}{n+2} = \frac{n\theta^2}{n+2}$$

$$D x_{max} = \frac{n\theta^2}{n+2} - \frac{n^2\theta^2}{(n+1)^2} = \theta^2\left(\frac{(n^2+2n+1)n - n^2(n+2)}{(n+1)^2(n+2)}\right) =$$

$$= \frac{n\theta^2}{(n+1)^2(n+2)} \Rightarrow D\tilde{\theta}_3 = \frac{(n+1)^2}{n^2}\frac{n\theta^2}{(n+1)^2(n+2)} = \boxed{\frac{\theta^2}{n(n+2)}}$$

$D\tilde{\tilde{\theta}}_2 \vee D\tilde{\tilde{\theta}}_3$

$\frac{n\theta^2}{n+2} \vee \frac{\theta^2}{n(n+2)}$

$n \vee \frac{1}{n}$; $D\tilde{\tilde{\theta}}_2 \geq D\tilde{\tilde{\theta}}_3 \quad \forall \theta > 0$

и при $n > 1$ $D\tilde{\tilde{\theta}}_2 > D\tilde{\tilde{\theta}}_3$ $\left(\exists \theta > 0 \right.$ по факту любое $\left.\right)$

$\Rightarrow \tilde{\tilde{\theta}}_3$ эффективнее $\tilde{\tilde{\theta}}_2$

$\Big(\forall \theta > 0 \quad D\tilde{\tilde{\theta}}_3 \leq D\tilde{\tilde{\theta}}_2$

$\exists \theta > 0$ (при $n > 1$) $D\tilde{\tilde{\theta}}_3 < D\tilde{\tilde{\theta}}_2 \rightarrow \tilde{\tilde{\theta}}_3$ эфф. $\tilde{\tilde{\theta}}_2 \Big)$
