

# Roche Lobes in the Second Post-Newtonian Approximation

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## ABSTRACT

Close binary systems of compact stars, due to the emission of gravitational radiation, may evolve into a phase in which the less massive star transfers mass to its companion. We describe mass transfer by using the model of Roche lobe overflow, in which mass is transferred through the first, or innermost, Lagrange point. Under conditions in which gravity is strong, the shapes of the equipotential surfaces and the Roche lobes are modified compared to the Newtonian case. We present calculations of the Roche lobe utilizing the second order post-Newtonian (2PN) approximation in the Arnowitt-Deser-Misner gauge. Heretofore, calculations of the Roche lobe geometry beyond the Newtonian case have not been available. Beginning from the general N-body Lagrangian derived by Damour and Schäffer, we develop the Lagrangian for a test particle in the vicinity of two massive compact objects. As an exact result for the transverse-traceless part of the Lagrangian is not available, we devise an approximation that is valid for regions close to the less massive star. We calculate the Roche lobe volumes, and provide a simple fitting formula for the effective Roche lobe radius analogous to that for the Newtonian case furnished by Eggleton. In contrast to the Newtonian case, in which the

effective Roche radius depends only upon the mass ratio  $q = m_1/m_2$ , in the 2PN case the effective Roche lobe radius also depends on the ratio  $z = 2(m_1 + m_2)/a$  of the total mass and the orbital separation.

*Subject headings:* relativity — binaries: close — stars: mass loss

## 1. Introduction

During the evolution of a close binary system involving compact stars, the stellar separation shrinks due to the emission of gravitational waves. In the event that the stars are not of equal mass, and the less massive star has a larger radius than its companion, mass transfer may ultimately occur. Gravity wave emission generally causes the mutual orbit to circularize (Peters 1964). For circular orbits, conservative mass transfer can be modelled as Roche lobe overflow under the assumption that the star is not significantly disrupted due to tidal interactions. The Roche lobe is the innermost gravitational plus centrifugal equipotential surface encompassing both stars.

In the model Roche lobe overflow, the radius of the less massive star is compared to the effective radius of its Roche lobe. Once the two radii become equal, because the Roche lobe radius decreases due to orbital decay, the star fills its Roche lobe and mass transfer occurs through the first, or innermost, Lagrange point  $L_1$ . Lying on the Roche lobe,  $L_1$  is located between the two stars on the axis connecting their centers and is also a saddle point of the gravitational plus centrifugal potential between the two stars. Due to its saddle point nature, the first Lagrange point acts as a gravitational funnel through which mass transfer occurs.

Values of the Roche lobe radii as a function of orbital separation and mass ratio  $q = m_1/m_2$ , where  $m_1$  refers to the lighter star, have been tabulated by Kopal (1959) for the Newtonian case. Paczyński (1971) and Eggleton (1982) have given analytical fits. We use Eggleton’s functional form, which has the advantage of being a continuous function of  $q$ , as a template in our work.

In this work, we carry out calculations of Roche lobes beyond the Newtonian case. We employ the Arnowitt-Deser-Misner (ADM) form of post-Newtonian expansion and use the corresponding Lagrangian at the second order (2PN) level wherein terms up to  $(M/r)^2$ , where  $M = m_1 + m_2$  and  $r$  is the distance, are retained. The same procedure as used in the Newtonian case for finding the Roche lobes is utilized. Our strategy is to (i) construct the effective potential for the point particle in the vicinity of two stars (the 3-body problem) in the co-rotating frame; (ii) evaluate equipotential surfaces and calculate the corresponding effective Roche volume and radius for this potential; and (iii) provide new fitting formulae as Eggleton did for applications involving mass transfer.

The organization of this work is as follows. In §2, we calculate the effective potential for three bodies at the 2PN level. We establish the Lagrangian in §2.1. The transverse-traceless part of the Lagrangian is evaluated explicitly in §2.2 through the introduction of an approximation valid

for regions near  $m_1$  for test particles. In §3, we evaluate the Roche lobes and their effective radii as a function of  $q$  and a relativity parameter for this potential, and provide a simple analytical fit. In this section, we also show the impact of post-Newtonian corrections on the positions of the Lagrange points and on the position of the center of mass. Our conclusions are contained in §4.

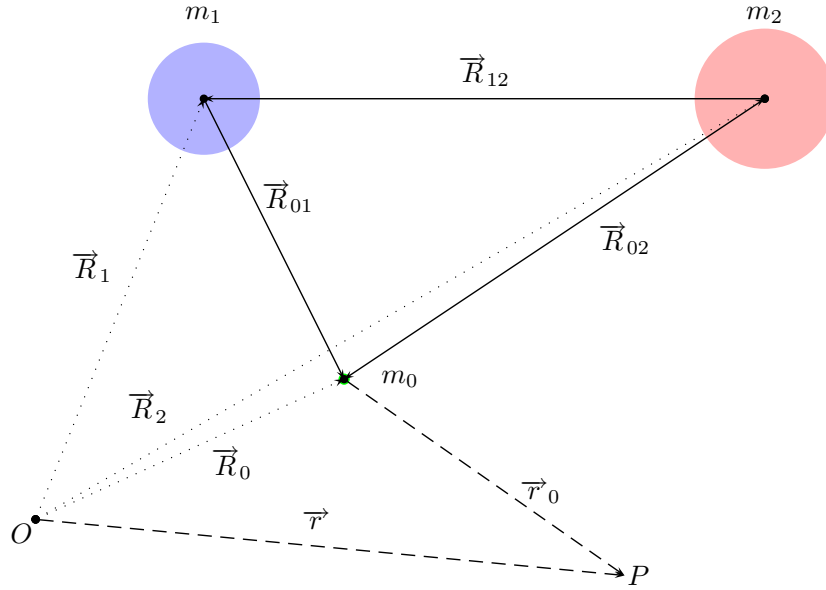


Fig. 1.— The notation used in the evaluation of the Roche lobes in the 2PN approximation. Stellar masses are denoted by  $m_1$  and  $m_2$  and the point-particle mass is taken to be  $m_0$ . Vectors  $\mathbf{R}_A$  ( $A = 0, 1, 2$ ) denote positions of the three bodies with respect to the origin  $O$ ,  $\mathbf{r}$  is the position of a generic point  $P$ , and  $\mathbf{r}_A$  is the position of this point with respect to the mass  $m_A$  (we show only  $\mathbf{r}_0$ ). The vectors  $\mathbf{R}_{AB}$  indicate positions of the three bodies with respect to each other.

## 2. The 2PN potential for 3 bodies

Roche lobes are defined through the acceleration that a point-like particle feels in the frame that is co-rotating with the two massive objects. While velocities of all three bodies disappear in this frame, accelerations are vanishing only for the two massive objects with masses  $m_1$  and  $m_2$ . The effective potential causes acceleration on the third object, the point-particle with mass  $m_0$ .

Starting from the N-body Lagrangian in ADM coordinates derived by Damour & Schäfer (1985), we obtain the 3-body Lagrangian for the situation depicted in Figure 1. We adopt the convention in which we denote masses with uppercase Latin indices ( $A, B, \dots$ ) and coordinates with lowercase Latin indices ( $i, j, \dots$ ). Also, we use units such that  $G = 1$  and  $c = 1$ . We express all inertial-frame velocities in terms of the rotating-frame velocities and the remaining rotationally induced part:

$$\mathbf{v}_A = \mathbf{v}_A^{rot} + \omega \times \mathbf{r}_A, \quad (1)$$

where  $\omega$  is the angular frequency of the rotating frame. Setting  $\mathbf{v}_A^{rot} = 0$  for  $A = \{0, 1, 2\}$ , and  $\dot{\mathbf{v}}_A^{rot} = 0$  for  $A = \{1, 2\}$ , but keeping  $\dot{\mathbf{v}}_0^{rot} \neq 0$  enables us to find the acceleration on the point-like particle (body 0) from

$$m_0 \ddot{\xi}_i = \left. \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\xi}_i} \right) \right|_{v_A^{c,r}=0}, \quad (2)$$

where we have denoted the coordinate of the point-like body in the co-rotating frame by  $\xi$ . We have

$$\xi_i = 0, \quad \dot{\xi} = 0, \quad \text{and} \quad \ddot{\xi}_i \neq 0, \quad (3)$$

since  $\dot{\mathbf{v}}_0^{c,r} = \ddot{\xi}$ . We can find the effective potential for the particle 0 by separating out the “kinetic” part of the Lagrangian that contains terms that are quadratic in  $\dot{\xi}$  and by treating the remaining part of the Lagrangian as the effective potential that we have to determine. It is straightforward to verify that this approach yields the Euler-Lagrange equations for  $\xi_i$ . After setting  $\dot{\xi} = 0$ , we use the resulting potential in order to trace the equipotential surfaces that correspond to the Roche lobes.

## 2.1. The 2PN Lagrangian

The computation of the effective Roche radii requires the effective potential that acts on a point particle in the vicinity of two massive bodies. In order to improve upon the existing Newtonian results, we utilize results that were obtained by using the post-Newtonian approximation of general relativity.

The Roche problem requires a three-body Lagrangian in the case in which one of the bodies is a point-like particle of infinitesimal mass. Such results were derived for the more general N-body case by Damour & Schäfer (1985) who retained terms up to the second order (2PN) in  $M/r$  in the Arnowitt–Deser–Misner (ADM) coordinate gauge.

For completeness, we list the main results of Damour & Schäfer (1985) here. The Newtonian, or zeroth order, result is familiar:

$$L_N = \frac{1}{2} \sum_A m_A v_A^2 + \frac{1}{2} \sum_{A,B \neq A} \frac{m_A m_B}{r_{AB}}, \quad (4)$$

where  $r_{AB} \equiv |\mathbf{R}_{AB}|$ . The first order post-Newtonian correction is

$$\begin{aligned}
 L_2 = & \frac{1}{8} \sum_A m_A v_A^4 \\
 & + \frac{1}{4} \sum_{A,B \neq A} \frac{m_A m_B}{r_{AB}} \left\{ 6v_A^2 - 7(\mathbf{v}_A \cdot \mathbf{v}_B) - (\mathbf{n}_{AB} \cdot \mathbf{v}_A)(\mathbf{n}_{AB} \cdot \mathbf{v}_B) \right\} \\
 & - \frac{1}{2} \sum_{A \neq B \neq C} \frac{m_A m_B m_C}{r_{AB} r_{AC}}, \tag{5}
 \end{aligned}$$

whereas the second order post-Newtonian contribution is

$$\begin{aligned}
 L_4 = & \frac{1}{16} \sum_A m_A v_A^6 + \frac{3}{8} \sum_{\substack{A,B \neq A, \\ C \neq B, D \neq C}} \frac{m_A m_B m_C m_D}{r_{AB} r_{BC} r_{CD}} + \frac{1}{4} \sum_{\substack{A,B \neq A, \\ C \neq A, D \neq A}} \frac{m_A m_B m_C m_D}{r_{AB} r_{AC} r_{AD}} - U_{TT} \\
 & + \frac{1}{4} \sum_{\substack{A,B \neq A, \\ C \neq A}} \frac{m_A m_B m_C}{r_{AB} r_{AC}} \left\{ 9v_A^2 - 7v_B^2 - 17(\mathbf{v}_A \cdot \mathbf{v}_B) + (\mathbf{n}_{AB} \cdot \mathbf{v}_A)(\mathbf{n}_{AB} \cdot \mathbf{v}_B) \right. \\
 & \quad \left. + (\mathbf{n}_{AB} \cdot \mathbf{v}_B)^2 + 16(\mathbf{v}_B \cdot \mathbf{v}_C) \right\} \\
 & + \frac{1}{8} \sum_{\substack{A,B \neq A, \\ C \neq A}} \frac{m_A m_B m_C}{r_{AB}^2} \left\{ -5(\mathbf{n}_{AB} \cdot \mathbf{n}_{AC})v_C^2 + (\mathbf{n}_{AB} \cdot \mathbf{n}_{AC})(\mathbf{n}_{AC} \cdot \mathbf{v}_C)^2 \right. \\
 & \quad \left. - 2(\mathbf{n}_{AB} \cdot \mathbf{v}_A)(\mathbf{n}_{AC} \cdot \mathbf{v}_C) - 2(\mathbf{n}_{AB} \cdot \mathbf{v}_B)(\mathbf{n}_{AC} \cdot \mathbf{v}_C) + 14(\mathbf{n}_{AB} \cdot \mathbf{v}_C)(\mathbf{n}_{AC} \cdot \mathbf{v}_C) \right\} \\
 & - \frac{1}{2} \sum_{\substack{A,B \neq A, \\ C \neq A, B}} \frac{m_A m_B m_C}{(r_{AB} + r_{BC} + r_{CA})^2} \left\{ 3(\mathbf{n}_{AB} + \mathbf{n}_{AC}) \cdot \mathbf{v}_A (\mathbf{n}_{AB} - \mathbf{n}_{BC}) \cdot \mathbf{v}_B \right. \\
 & \quad + (\mathbf{n}_{AB} + \mathbf{n}_{AC}) \cdot \mathbf{v}_A (\mathbf{n}_{AB} - \mathbf{n}_{BC}) \cdot \mathbf{v}_A + 8(\mathbf{n}_{AB} + \mathbf{n}_{AC}) \cdot \mathbf{v}_A (\mathbf{n}_{AB} - \mathbf{n}_{BC}) \cdot \mathbf{v}_C \\
 & \quad \left. - 16(\mathbf{n}_{AB} + \mathbf{n}_{AC}) \cdot \mathbf{v}_C (\mathbf{n}_{AB} - \mathbf{n}_{BC}) \cdot \mathbf{v}_A + 4(\mathbf{n}_{AB} + \mathbf{n}_{AC}) \cdot \mathbf{v}_C (\mathbf{n}_{AB} - \mathbf{n}_{BC}) \cdot \mathbf{v}_C \right\} \\
 & + \frac{1}{2} \sum_{\substack{A,B \neq A, \\ C \neq A, B}} \frac{m_A m_B m_C}{(r_{AB} + r_{BC} + r_{CA}) r_{AB}} \left\{ 3 \left[ (\mathbf{v}_A \cdot \mathbf{v}_B) - (\mathbf{n}_{AB} \cdot \mathbf{v}_A)(\mathbf{n}_{AB} \cdot \mathbf{v}_B) \right] \right. \\
 & \quad + \left[ v_A^2 - (\mathbf{n}_{AB} \cdot \mathbf{v}_A)^2 \right] - 8 \left[ (\mathbf{v}_A \cdot \mathbf{v}_C) - (\mathbf{n}_{AB} \cdot \mathbf{v}_A)(\mathbf{n}_{AB} \cdot \mathbf{v}_C) \right] \\
 & \quad \left. + 4 \left[ v_C^2 - (\mathbf{n}_{AB} \cdot \mathbf{v}_C)^2 \right] \right\} \\
 & - \frac{1}{4} \sum_{A,B \neq A} \frac{m_A m_B^2}{r_{AB}^2} \left\{ v_A^2 + v_B^2 - 2(\mathbf{v}_A \cdot \mathbf{v}_B) \right\} \\
 & + \frac{1}{16} \sum_{A,B \neq A} \frac{m_A m_B}{r_{AB}} \left\{ 14v_A^4 - 28v_A^2(\mathbf{v}_A \cdot \mathbf{v}_B) - 4v_A^2(\mathbf{n}_{AB} \cdot \mathbf{v}_A)(\mathbf{n}_{AB} \cdot \mathbf{v}_B) + 11v_A^2 v_B^2 \right. \\
 & \quad + 2(\mathbf{v}_A \cdot \mathbf{v}_A)^2 - 10v_A^2(\mathbf{n}_{AB} \cdot \mathbf{v}_B)^2 + 12(\mathbf{v}_A \cdot \mathbf{v}_B)(\mathbf{n}_{AB} \cdot \mathbf{v}_A)(\mathbf{n}_{AB} \cdot \mathbf{v}_B) \\
 & \quad \left. + 3(\mathbf{n}_{AB} \cdot \mathbf{v}_A)^2(\mathbf{n}_{AB} \cdot \mathbf{v}_B)^2 \right\}. \tag{6}
 \end{aligned}$$

The term  $U_{TT}$  refers to the transverse-traceless part of the Lagrangian potential which requires special treatment and is explicitly evaluated in §2.2.

We now specialize to the case of three bodies with masses  $m_0$ ,  $m_1$ , and  $m_2$ . For the Newtonian and the first order post-Newtonian correction, we obtain

$$L_N = \frac{1}{2} \left\{ m_0 v_0^2 + m_1 v_1^2 + m_2 v_2^2 \right\} + \frac{m_0 m_1}{r_{01}} + \frac{m_0 m_2}{r_{02}} + \frac{m_1 m_2}{r_{12}}, \quad (7)$$

and

$$\begin{aligned} L_2 = & \frac{1}{8} \left\{ m_0 v_0^4 + m_1 v_1^4 + m_2 v_2^4 \right\} \\ & + \frac{1}{4} \left\{ \frac{m_0 m_1}{r_{01}} \left[ 6v_0^2 + 6v_1^2 - 14(\mathbf{v}_0 \cdot \mathbf{v}_1) - 2(\mathbf{n}_{01} \cdot \mathbf{v}_0)(\mathbf{n}_{01} \cdot \mathbf{v}_1) \right] \right. \\ & \quad + \frac{m_0 m_2}{r_{02}} \left[ 6v_0^2 + 6v_2^2 - 14(\mathbf{v}_0 \cdot \mathbf{v}_2) - 2(\mathbf{n}_{02} \cdot \mathbf{v}_0)(\mathbf{n}_{02} \cdot \mathbf{v}_2) \right] \\ & \quad \left. + \frac{m_1 m_2}{r_{12}} \left[ 6v_1^2 + 6v_2^2 - 14(\mathbf{v}_1 \cdot \mathbf{v}_2) - 2(\mathbf{n}_{12} \cdot \mathbf{v}_1)(\mathbf{n}_{12} \cdot \mathbf{v}_2) \right] \right\} \\ & - \frac{m_0 m_1 m_2}{r_{01} r_{02} r_{12}} \left\{ r_{01} + r_{02} + r_{12} \right\}. \end{aligned} \quad (8)$$

Because the second order post-Newtonian correction  $L_4$  is small, and we are interested in computing the equipotential surfaces for a test particle, we assume

$$m_0 \ll m_1, m_2. \quad (9)$$

We decompose the second order correction in order to facilitate its presentation:

$$L_4 = \sum_{a=i}^{xi} L_4^{(a)} + \mathcal{O}(m_0^2) \quad (10)$$

and we drop terms of  $\mathcal{O}(m_0^2)$ . The decomposition is evident by comparing equation (6) with the following:

$$L_4^{(i)} = \frac{1}{16} \left\{ m_0 v_0^6 + m_1 v_1^6 + m_2 v_2^6 \right\}, \quad (11)$$

$$\begin{aligned} L_4^{(ii)} = & \frac{3}{4} \left\{ \frac{m_0 m_1^2 m_2}{r_{01} r_{12}^2} + \frac{m_0 m_1 m_2^2}{r_{02} r_{12}^2} + \frac{m_0 m_1^2 m_2}{r_{01}^2 r_{12}} + \frac{m_0 m_1 m_2^2}{r_{02}^2 r_{12}} + \frac{m_0 m_1^2 m_2}{r_{01} r_{12} r_{02}} + \frac{m_0 m_1 m_2^2}{r_{01} r_{12} r_{02}} \right. \\ & \left. + \frac{m_1^2 m_2^2}{r_{12}^3} + \frac{m_0 m_1^2 m_2}{r_{01}^2 r_{02}} + \frac{m_0 m_1 m_2^2}{r_{01} r_{02}^2} + \frac{m_0 m_1^2 m_2}{r_{02} r_{12}^2} + \frac{m_0 m_1 m_2^2}{r_{01} r_{12}^2} \right\}, \end{aligned} \quad (12)$$

$$L_4^{(iii)} = \frac{1}{4} \left\{ \frac{m_0 m_1^3}{r_{01}^3} + \frac{m_0 m_2^3}{r_{02}^3} + \frac{m_1 m_2^3}{r_{12}^3} + \frac{m_1^3 m_2}{r_{12}^3} \right\}, \quad (13)$$

$$L_4^{(iv)} = -U_{TT}, \quad (14)$$

$$\begin{aligned} L_4^{(v)} = & \frac{m_0 m_1^2}{r_{01}^2} \left\{ \frac{9}{4} v_0^2 + \frac{13}{8} v_1^2 - \frac{17}{4} (\mathbf{v}_0 \cdot \mathbf{v}_1) + \frac{15}{8} (\mathbf{n}_{01} \cdot \mathbf{v}_1)^2 \right\} \\ & + \frac{m_0 m_2^2}{r_{02}^2} \left\{ \frac{9}{4} v_0^2 + \frac{13}{8} v_2^2 - \frac{17}{4} (\mathbf{v}_0 \cdot \mathbf{v}_2) + \frac{15}{8} (\mathbf{n}_{02} \cdot \mathbf{v}_2)^2 \right\} \\ & + \frac{m_1 m_2^2}{r_{12}^2} \left\{ \frac{9}{4} v_1^2 + \frac{13}{8} v_2^2 - \frac{17}{4} (\mathbf{v}_1 \cdot \mathbf{v}_2) + \frac{15}{8} (\mathbf{n}_{12} \cdot \mathbf{v}_2)^2 \right\} \\ & + \frac{m_1^2 m_2}{r_{12}^2} \left\{ \frac{9}{4} v_2^2 + \frac{13}{8} v_1^2 - \frac{17}{4} (\mathbf{v}_1 \cdot \mathbf{v}_2) + \frac{15}{8} (\mathbf{n}_{12} \cdot \mathbf{v}_1)^2 \right\}, \end{aligned} \quad (15)$$

$$\begin{aligned} L_4^{(vi)} = & \frac{m_0 m_1 m_2}{4 r_{01} r_{12}} \left\{ 18 v_1^2 - 7 v_0^2 - 17 (\mathbf{v}_0 \cdot \mathbf{v}_1) + (\mathbf{n}_{01} \cdot \mathbf{v}_0) (\mathbf{n}_{01} \cdot \mathbf{v}_1) + (\mathbf{n}_{01} \cdot \mathbf{v}_0)^2 \right. \\ & \left. + 32 (\mathbf{v}_0 \cdot \mathbf{v}_2) - 7 v_2^2 - 17 (\mathbf{v}_1 \cdot \mathbf{v}_2) + (\mathbf{n}_{12} \cdot \mathbf{v}_1) (\mathbf{n}_{12} \cdot \mathbf{v}_2) + (\mathbf{n}_{12} \cdot \mathbf{v}_2)^2 \right\} \\ & + \frac{m_0 m_1 m_2}{4 r_{01} r_{02}} \left\{ 18 v_0^2 - 7 v_1^2 - 17 (\mathbf{v}_0 \cdot \mathbf{v}_1) + (\mathbf{n}_{01} \cdot \mathbf{v}_0) (\mathbf{n}_{01} \cdot \mathbf{v}_1) + (\mathbf{n}_{01} \cdot \mathbf{v}_1)^2 \right. \\ & \left. + 32 (\mathbf{v}_1 \cdot \mathbf{v}_2) - 7 v_2^2 - 17 (\mathbf{v}_0 \cdot \mathbf{v}_2) + (\mathbf{n}_{02} \cdot \mathbf{v}_0) (\mathbf{n}_{02} \cdot \mathbf{v}_2) + (\mathbf{n}_{02} \cdot \mathbf{v}_2)^2 \right\} \\ & + \frac{m_0 m_1 m_2}{4 r_{02} r_{12}} \left\{ 18 v_2^2 - 7 v_0^2 - 17 (\mathbf{v}_0 \cdot \mathbf{v}_2) + (\mathbf{n}_{02} \cdot \mathbf{v}_0) (\mathbf{n}_{02} \cdot \mathbf{v}_2) + (\mathbf{n}_{02} \cdot \mathbf{v}_0)^2 \right. \\ & \left. + 32 (\mathbf{v}_0 \cdot \mathbf{v}_1) - 7 v_1^2 - 17 (\mathbf{v}_1 \cdot \mathbf{v}_2) + (\mathbf{n}_{12} \cdot \mathbf{v}_1) (\mathbf{n}_{12} \cdot \mathbf{v}_2) + (\mathbf{n}_{12} \cdot \mathbf{v}_1)^2 \right\}, \end{aligned} \quad (16)$$

$$\begin{aligned} L_4^{(vii)} = & \frac{m_0 m_1 m_2}{8 r_{01}^2} \left\{ -5 (\mathbf{n}_{01} \cdot \mathbf{n}_{02}) v_2^2 + (\mathbf{n}_{01} \cdot \mathbf{n}_{02}) (\mathbf{n}_{02} \cdot \mathbf{v}_2)^2 - 2 (\mathbf{n}_{01} \cdot \mathbf{v}_0) (\mathbf{n}_{02} \cdot \mathbf{v}_2) \right. \\ & - 2 (\mathbf{n}_{01} \cdot \mathbf{v}_1) (\mathbf{n}_{02} \cdot \mathbf{v}_2) + 14 (\mathbf{n}_{01} \cdot \mathbf{v}_2) (\mathbf{n}_{02} \cdot \mathbf{v}_2) + 5 (\mathbf{n}_{01} \cdot \mathbf{n}_{12}) v_2^2 \\ & - (\mathbf{n}_{01} \cdot \mathbf{n}_{12}) (\mathbf{n}_{12} \cdot \mathbf{v}_2)^2 + 2 (\mathbf{n}_{01} \cdot \mathbf{v}_1) (\mathbf{n}_{12} \cdot \mathbf{v}_2) + 2 (\mathbf{n}_{01} \cdot \mathbf{v}_0) (\mathbf{n}_{12} \cdot \mathbf{v}_2) \\ & \left. - 14 (\mathbf{n}_{01} \cdot \mathbf{v}_2) (\mathbf{n}_{12} \cdot \mathbf{v}_2) \right\} \\ & + \frac{m_0 m_1 m_2}{8 r_{02}^2} \left\{ -5 (\mathbf{n}_{01} \cdot \mathbf{n}_{02}) v_1^2 + (\mathbf{n}_{01} \cdot \mathbf{n}_{02}) (\mathbf{n}_{01} \cdot \mathbf{v}_1)^2 - 2 (\mathbf{n}_{01} \cdot \mathbf{v}_1) (\mathbf{n}_{02} \cdot \mathbf{v}_0) \right. \\ & - 2 (\mathbf{n}_{02} \cdot \mathbf{v}_2) (\mathbf{n}_{01} \cdot \mathbf{v}_1) + 14 (\mathbf{n}_{01} \cdot \mathbf{v}_1) (\mathbf{n}_{02} \cdot \mathbf{v}_1) - 5 (\mathbf{n}_{02} \cdot \mathbf{n}_{12}) v_1^2 \\ & + (\mathbf{n}_{02} \cdot \mathbf{n}_{12}) (\mathbf{n}_{12} \cdot \mathbf{v}_1)^2 - 2 (\mathbf{n}_{02} \cdot \mathbf{v}_2) (\mathbf{n}_{12} \cdot \mathbf{v}_1) - 2 (\mathbf{n}_{02} \cdot \mathbf{v}_0) (\mathbf{n}_{12} \cdot \mathbf{v}_1) \\ & \left. + 14 (\mathbf{n}_{02} \cdot \mathbf{v}_1) (\mathbf{n}_{12} \cdot \mathbf{v}_1) \right\} \\ & + \frac{m_0 m_1 m_2}{8 r_{12}^2} \left\{ 5 (\mathbf{n}_{01} \cdot \mathbf{n}_{12}) v_0^2 - (\mathbf{n}_{01} \cdot \mathbf{n}_{12}) (\mathbf{n}_{01} \cdot \mathbf{v}_0)^2 + 2 (\mathbf{n}_{12} \cdot \mathbf{v}_1) (\mathbf{n}_{01} \cdot \mathbf{v}_0) \right. \\ & + 2 (\mathbf{n}_{12} \cdot \mathbf{v}_2) (\mathbf{n}_{01} \cdot \mathbf{v}_0) - 14 (\mathbf{n}_{12} \cdot \mathbf{v}_0) (\mathbf{n}_{01} \cdot \mathbf{v}_0) - 5 (\mathbf{n}_{12} \cdot \mathbf{n}_{02}) v_0^2 \\ & + (\mathbf{n}_{12} \cdot \mathbf{n}_{02}) (\mathbf{n}_{02} \cdot \mathbf{v}_0)^2 - 2 (\mathbf{n}_{12} \cdot \mathbf{v}_2) (\mathbf{n}_{02} \cdot \mathbf{v}_0) - 2 (\mathbf{n}_{12} \cdot \mathbf{v}_1) (\mathbf{n}_{02} \cdot \mathbf{v}_0) \\ & \left. + 14 (\mathbf{n}_{12} \cdot \mathbf{v}_0) (\mathbf{n}_{02} \cdot \mathbf{v}_0) \right\}, \end{aligned} \quad (17)$$

$$\begin{aligned}
L_4^{(viii)} = & -\frac{1}{2} \frac{m_0 m_1 m_2}{(r_{01} + r_{12} + r_{02})^2} \left\{ (\mathbf{n}_{01} \cdot \mathbf{v}_0)^2 + (\mathbf{n}_{01} \cdot \mathbf{v}_1)^2 + (\mathbf{n}_{12} \cdot \mathbf{v}_1)^2 + (\mathbf{n}_{12} \cdot \mathbf{v}_2)^2 \right. \\
& + (\mathbf{n}_{02} \cdot \mathbf{v}_0)^2 + (\mathbf{n}_{02} \cdot \mathbf{v}_2)^2 + 8 \left[ (\mathbf{n}_{01} \cdot \mathbf{v}_2)^2 + (\mathbf{n}_{02} \cdot \mathbf{v}_1)^2 + (\mathbf{n}_{12} \cdot \mathbf{v}_0)^2 \right] \\
& + 32 \left[ (\mathbf{n}_{01} \cdot \mathbf{v}_2) (\mathbf{n}_{12} \cdot \mathbf{v}_0) - (\mathbf{n}_{02} \cdot \mathbf{v}_1) (\mathbf{n}_{12} \cdot \mathbf{v}_0) - (\mathbf{n}_{01} \cdot \mathbf{v}_2) (\mathbf{n}_{02} \cdot \mathbf{v}_1) \right] \\
& + 10 \left[ (\mathbf{n}_{02} \cdot \mathbf{v}_0) (\mathbf{n}_{12} \cdot \mathbf{v}_1) - (\mathbf{n}_{01} \cdot \mathbf{v}_0) (\mathbf{n}_{12} \cdot \mathbf{v}_2) - (\mathbf{n}_{01} \cdot \mathbf{v}_1) (\mathbf{n}_{02} \cdot \mathbf{v}_2) \right] \\
& + 6 \left[ (\mathbf{n}_{01} \cdot \mathbf{v}_0) (\mathbf{n}_{01} \cdot \mathbf{v}_1) + (\mathbf{n}_{02} \cdot \mathbf{v}_0) (\mathbf{n}_{02} \cdot \mathbf{v}_2) + (\mathbf{n}_{12} \cdot \mathbf{v}_1) (\mathbf{n}_{12} \cdot \mathbf{v}_2) \right. \\
& \left. - (\mathbf{n}_{01} \cdot \mathbf{v}_0) (\mathbf{n}_{02} \cdot \mathbf{v}_0) + (\mathbf{n}_{01} \cdot \mathbf{v}_1) (\mathbf{n}_{12} \cdot \mathbf{v}_1) - (\mathbf{n}_{12} \cdot \mathbf{v}_2) (\mathbf{n}_{02} \cdot \mathbf{v}_2) \right] \\
& + 18 \left[ (\mathbf{n}_{01} \cdot \mathbf{v}_0) (\mathbf{n}_{12} \cdot \mathbf{v}_1) - (\mathbf{n}_{02} \cdot \mathbf{v}_0) (\mathbf{n}_{01} \cdot \mathbf{v}_1) - (\mathbf{n}_{02} \cdot \mathbf{v}_0) (\mathbf{n}_{12} \cdot \mathbf{v}_2) \right. \\
& \left. - (\mathbf{n}_{02} \cdot \mathbf{v}_2) (\mathbf{n}_{01} \cdot \mathbf{v}_0) + (\mathbf{n}_{12} \cdot \mathbf{v}_2) (\mathbf{n}_{01} \cdot \mathbf{v}_1) - (\mathbf{n}_{12} \cdot \mathbf{v}_1) (\mathbf{n}_{02} \cdot \mathbf{v}_2) \right] \\
& + 8 \left[ - (\mathbf{n}_{01} \cdot \mathbf{v}_0) (\mathbf{n}_{12} \cdot \mathbf{v}_0) + (\mathbf{n}_{02} \cdot \mathbf{v}_0) (\mathbf{n}_{12} \cdot \mathbf{v}_0) - (\mathbf{n}_{01} \cdot \mathbf{v}_2) (\mathbf{n}_{12} \cdot \mathbf{v}_2) \right. \\
& \left. + (\mathbf{n}_{02} \cdot \mathbf{v}_2) (\mathbf{n}_{01} \cdot \mathbf{v}_2) + (\mathbf{n}_{02} \cdot \mathbf{v}_1) (\mathbf{n}_{12} \cdot \mathbf{v}_1) + (\mathbf{n}_{01} \cdot \mathbf{v}_1) (\mathbf{n}_{02} \cdot \mathbf{v}_1) \right] \\
& + 8 \left[ - (\mathbf{n}_{01} \cdot \mathbf{v}_2) (\mathbf{n}_{01} \cdot \mathbf{v}_0) - (\mathbf{n}_{01} \cdot \mathbf{v}_2) (\mathbf{n}_{01} \cdot \mathbf{v}_1) - (\mathbf{n}_{02} \cdot \mathbf{v}_1) (\mathbf{n}_{02} \cdot \mathbf{v}_0) \right. \\
& \left. - (\mathbf{n}_{02} \cdot \mathbf{v}_1) (\mathbf{n}_{02} \cdot \mathbf{v}_2) - (\mathbf{n}_{12} \cdot \mathbf{v}_0) (\mathbf{n}_{12} \cdot \mathbf{v}_1) - (\mathbf{n}_{12} \cdot \mathbf{v}_0) (\mathbf{n}_{12} \cdot \mathbf{v}_2) \right] \\
& + 24 \left[ (\mathbf{n}_{01} \cdot \mathbf{v}_2) (\mathbf{n}_{02} \cdot \mathbf{v}_0) + (\mathbf{n}_{01} \cdot \mathbf{v}_0) (\mathbf{n}_{02} \cdot \mathbf{v}_1) + (\mathbf{n}_{02} \cdot \mathbf{v}_2) (\mathbf{n}_{12} \cdot \mathbf{v}_0) \right. \\
& \left. + (\mathbf{n}_{02} \cdot \mathbf{v}_1) (\mathbf{n}_{12} \cdot \mathbf{v}_2) - (\mathbf{n}_{01} \cdot \mathbf{v}_2) (\mathbf{n}_{12} \cdot \mathbf{v}_1) - (\mathbf{n}_{01} \cdot \mathbf{v}_1) (\mathbf{n}_{12} \cdot \mathbf{v}_0) \right] \left. \right\}, \quad (18)
\end{aligned}$$

$$\begin{aligned}
L_4^{(ix)} = & \frac{1}{2} \frac{m_0 m_1 m_2}{(r_{01} + r_{12} + r_{02})} \left\{ \frac{1}{r_{01}} \left[ 6 (\mathbf{v}_0 \cdot \mathbf{v}_1) - 6 (\mathbf{n}_{01} \cdot \mathbf{v}_0) (\mathbf{n}_{01} \cdot \mathbf{v}_1) + v_0^2 - (\mathbf{n}_{01} \cdot \mathbf{v}_0)^2 \right. \right. \\
& - 8 (\mathbf{v}_0 \cdot \mathbf{v}_2) + 8 (\mathbf{n}_{01} \cdot \mathbf{v}_0) (\mathbf{n}_{01} \cdot \mathbf{v}_2) + 8 v_2^2 - 8 (\mathbf{n}_{01} \cdot \mathbf{v}_2)^2 + v_1^2 - (\mathbf{n}_{01} \cdot \mathbf{v}_1)^2 \\
& \left. - 8 (\mathbf{v}_1 \cdot \mathbf{v}_2) + 8 (\mathbf{n}_{01} \cdot \mathbf{v}_1) (\mathbf{n}_{01} \cdot \mathbf{v}_2) \right] \\
& + \frac{1}{r_{12}} \left[ 6 (\mathbf{v}_1 \cdot \mathbf{v}_2) - 6 (\mathbf{n}_{12} \cdot \mathbf{v}_1) (\mathbf{n}_{12} \cdot \mathbf{v}_2) + v_1^2 - (\mathbf{n}_{12} \cdot \mathbf{v}_1)^2 - 8 (\mathbf{v}_0 \cdot \mathbf{v}_1) \right. \\
& + 8 (\mathbf{n}_{12} \cdot \mathbf{v}_1) (\mathbf{n}_{12} \cdot \mathbf{v}_0) + 8 v_0^2 - 8 (\mathbf{n}_{12} \cdot \mathbf{v}_0)^2 + v_2^2 - (\mathbf{n}_{12} \cdot \mathbf{v}_2)^2 \\
& \left. - 8 (\mathbf{v}_0 \cdot \mathbf{v}_2) + 8 (\mathbf{n}_{12} \cdot \mathbf{v}_2) (\mathbf{n}_{12} \cdot \mathbf{v}_0) \right] \\
& + \frac{1}{r_{02}} \left[ 6 (\mathbf{v}_0 \cdot \mathbf{v}_2) - 6 (\mathbf{n}_{02} \cdot \mathbf{v}_0) (\mathbf{n}_{02} \cdot \mathbf{v}_2) + v_0^2 - (\mathbf{n}_{02} \cdot \mathbf{v}_0)^2 - 8 (\mathbf{v}_0 \cdot \mathbf{v}_1) \right. \\
& + 8 (\mathbf{n}_{02} \cdot \mathbf{v}_0) (\mathbf{n}_{01} \cdot \mathbf{v}_1) + 8 v_1^2 - 8 (\mathbf{n}_{02} \cdot \mathbf{v}_1)^2 + v_2^2 - (\mathbf{n}_{02} \cdot \mathbf{v}_2)^2 \\
& \left. - 8 (\mathbf{v}_1 \cdot \mathbf{v}_2) + 8 (\mathbf{n}_{02} \cdot \mathbf{v}_1) (\mathbf{n}_{02} \cdot \mathbf{v}_2) \right] \left. \right\}, \quad (19)
\end{aligned}$$



$$\begin{aligned}
L_4^{(x)} = & -\frac{m_0 m_1 (m_0 + m_1)}{4r_{01}^2} (v_0^2 + v_1^2 - 2(\mathbf{v}_0 \cdot \mathbf{v}_1)) - \frac{m_0 m_2 (m_0 + m_2)}{4r_{02}^2} (v_0^2 + v_2^2 - 2(\mathbf{v}_0 \cdot \mathbf{v}_2)) \\
& - \frac{m_1 m_2 (m_1 + m_2)}{4r_{12}^2} (v_1^2 + v_2^2 - 2(\mathbf{v}_1 \cdot \mathbf{v}_2)) ,
\end{aligned} \tag{20}$$

$$\begin{aligned}
L_4^{(xi)} = & \frac{m_0 m_1}{16r_{01}^2} \left\{ 14(v_0^4 + v_1^4) - 28(\mathbf{v}_0 \cdot \mathbf{v}_1)(v_0^2 + v_1^2) - 4(\mathbf{n}_{01} \cdot \mathbf{v}_0)(\mathbf{n}_{01} \cdot \mathbf{v}_1)(v_0^2 + v_1^2) \right. \\
& + 22v_0^2 v_1^2 + 4(\mathbf{v}_0 \cdot \mathbf{v}_1)^2 - 10v_0^2(\mathbf{n}_{01} \cdot \mathbf{v}_1)^2 - 10v_1^2(\mathbf{n}_{01} \cdot \mathbf{v}_0)^2 \\
& \left. + 24(\mathbf{v}_0 \cdot \mathbf{v}_1)(\mathbf{n}_{01} \cdot \mathbf{v}_0)(\mathbf{n}_{01} \cdot \mathbf{v}_1) + 6(\mathbf{n}_{01} \cdot \mathbf{v}_0)^2(\mathbf{n}_{01} \cdot \mathbf{v}_1)^2 \right\} \\
& + \frac{m_0 m_2}{16r_{02}^2} \left\{ 14(v_0^4 + v_2^4) - 28(\mathbf{v}_0 \cdot \mathbf{v}_2)(v_0^2 + v_2^2) - 4(\mathbf{n}_{02} \cdot \mathbf{v}_0)(\mathbf{n}_{02} \cdot \mathbf{v}_2)(v_0^2 + v_2^2) \right. \\
& + 22v_0^2 v_2^2 + 4(\mathbf{v}_0 \cdot \mathbf{v}_2)^2 - 10v_0^2(\mathbf{n}_{02} \cdot \mathbf{v}_2)^2 - 10v_2^2(\mathbf{n}_{02} \cdot \mathbf{v}_0)^2 \\
& \left. + 24(\mathbf{v}_0 \cdot \mathbf{v}_2)(\mathbf{n}_{02} \cdot \mathbf{v}_0)(\mathbf{n}_{02} \cdot \mathbf{v}_2) + 6(\mathbf{n}_{02} \cdot \mathbf{v}_0)^2(\mathbf{n}_{02} \cdot \mathbf{v}_2)^2 \right\} \\
& + \frac{m_1 m_2}{16r_{12}^2} \left\{ 14(v_1^4 + v_2^4) - 28(\mathbf{v}_1 \cdot \mathbf{v}_2)(v_1^2 + v_2^2) - 4(\mathbf{n}_{12} \cdot \mathbf{v}_1)(\mathbf{n}_{12} \cdot \mathbf{v}_2)(v_1^2 + v_2^2) \right. \\
& + 22v_1^2 v_2^2 + 4(\mathbf{v}_1 \cdot \mathbf{v}_2)^2 - 10v_1^2(\mathbf{n}_{12} \cdot \mathbf{v}_2)^2 - 10v_2^2(\mathbf{n}_{12} \cdot \mathbf{v}_1)^2 \\
& \left. + 24(\mathbf{v}_1 \cdot \mathbf{v}_2)(\mathbf{n}_{12} \cdot \mathbf{v}_1)(\mathbf{n}_{12} \cdot \mathbf{v}_2) + 6(\mathbf{n}_{12} \cdot \mathbf{v}_1)^2(\mathbf{n}_{12} \cdot \mathbf{v}_2)^2 \right\} .
\end{aligned} \tag{21}$$

## 2.2. The transverse–traceless part of the Lagrangian $U_{TT}$

The calculation of the 3-body ADM Lagrangian from Damour & Schäfer (1985) is somewhat lengthy, but straightforward, except for the transverse-traceless part of the interaction potential  $U_{TT}$  in equations (6) and (14). It is not known in general how to evaluate this term explicitly, except for the two-body case (see Damour & Schäfer (1985); Ohta et al. (1973, 1974)). In order to circumvent this problem, and since we are interested only in the vicinity of the star with mass  $m_1$ , we assume

$$r_{01} \ll r_{12} , \quad \text{and} \quad r_{01} \ll r_{02} . \tag{22}$$

and expand equations (5) and (6) in terms of  $r_{01}/r_{12}$ . As in the rest of the Lagrangian, we assume the body 0 to be a point-particle and therefore drop all terms that are of quadratic (or higher) power in  $m_0$ .

Utilizing these two physically motivated assumptions, we expand  $U_{TT}$  as

$$U_{TT} = U_{TT}^{(12,12)} + U_{TT}^{(10,12)} + U_{TT}^{(12,02)} + \mathcal{O}(m_0^2) , \tag{23}$$

with

$$U_{TT}^{(AB,CD)} = -\frac{1}{4\pi} \int d^3x [f_{ij}^{TT}(A, B)]_{,k} [f_{ij}^{TT}(C, D)]_{,k}$$

$$= \frac{1}{2\pi} \int d^3x \left( \frac{m_A}{r_A} \right)_{,i} \left( \frac{m_B}{r_B} \right)_{,j} f_{ij}^{TT}(C, D), \quad (24)$$

where  $A, B, C, D = 0, 1, 2$  and  $g_{,i} \equiv \partial g(x)/\partial x^i$  for any function  $g(x)$ .

The integrand  $f_{ij}^{TT}(A, B)$  is given by

$$\begin{aligned} f_{ij}^{TT}(A, B) = & \left( \frac{\partial}{\partial R_A^i} \frac{\partial}{\partial R_B^j} + \frac{\partial}{\partial R_A^j} \frac{\partial}{\partial R_B^i} \right) \ln(r_A + r_B + r_{AB}) \\ & - \frac{\delta_{ij}}{8} \left( \frac{2}{r_A r_B} - \frac{2}{r_A r_{AB}} - \frac{2}{r_B r_{AB}} + 2 \frac{\mathbf{n}_A \cdot \mathbf{n}_{AB}}{r_{AB}^2} - 2 \frac{\mathbf{n}_B \cdot \mathbf{n}_{AB}}{r_{AB}^2} \right) \\ & - \frac{\delta_{ij}^2}{8} \left( 2 \ln(r_A + r_B + r_{AB}) - \frac{r_A + r_B}{r_{AB}} + \frac{r_A^2 (\mathbf{n}_A \cdot \mathbf{n}_{AB})}{2 r_{AB}^2} - \frac{r_B^2 (\mathbf{n}_B \cdot \mathbf{n}_{AB})}{2 r_{AB}^2} \right) \\ & + \frac{1}{2 r_{AB}^2} \left( n_A^i n_{AB}^j + n_A^j n_{AB}^i - n_B^i n_{AB}^j - n_B^j n_{AB}^i \right), \end{aligned} \quad (25)$$

where  $i = 1, 2, 3$  denotes spatial components. Here, as in Figure 1,

$$\mathbf{r}_A = \mathbf{r} - \mathbf{R}_A, \quad \mathbf{R}_A = \mathbf{R}_B + \mathbf{R}_{AB}, \quad \text{and} \quad \mathbf{r}_A = \mathbf{r}_B + \mathbf{R}_{AB}. \quad (26)$$

In the preceding equations, we denote the position of the integration point with  $\mathbf{r}$ , the position (i.e. “trajectory”) of the body  $A$  with  $\mathbf{R}_A$ , the vector between the integration point and the body  $A$  with  $\mathbf{r}_A$ , and the vector defined by two bodies  $A$  and  $B$  with  $\mathbf{R}_{AB}$ .

Two out of three terms in equation (23) are already available or easily derived from previous calculations of the two-body Lagrangian. The result for  $U_{TT}^{(12,12)}$  is given in Damour & Schäfer (1985); Ohta et al. (1973, 1974):

$$U_{TT}^{(12,12)} = -\frac{1}{2} \frac{m_1^2 m_2^2}{r_{12}^3}. \quad (27)$$

It is straightforward to obtain an approximate result for  $U_{TT}^{(12,02)}$  by using an expansion in terms of  $r_{01}/r_{12}$  and keeping only terms with nonpositive powers of  $r_{01}/r_{12}$ . This approximation has a straightforward physical interpretation that  $r_{02} \approx r_{12}$  around  $\mathbf{R}_1$ . A quick calculation yields

$$U_{TT}^{(12,02)} \approx -\frac{1}{2} \frac{m_0 m_1 m_2^2}{r_{12}^3} + \mathcal{O}\left(\frac{r_{01}}{r_{12}}\right). \quad (28)$$

However, the remaining term turns out to require a much more laborious calculation. We start from the expression

$$\begin{aligned} U_{TT}^{(10,12)} &= 2 \int \frac{d^3x}{4\pi} \left( \frac{m_2}{r_2} \right)_{,i} \left( \frac{m_1}{r_1} \right)_{,j} f_{ij}^{TT}(1, 0) \\ &= 2 m_0 m_1^2 m_2 \int \frac{d^3x}{4\pi} \frac{x^i - R_2^i}{r_2^3} \frac{x^j - R_1^j}{r_1^3} \end{aligned}$$

$$\begin{aligned}
& \times \left\{ \left( \frac{\partial}{\partial R_1^i} \frac{\partial}{\partial R_0^j} + \frac{\partial}{\partial R_1^j} \frac{\partial}{\partial R_0^i} \right) \ln(r_1 + r_0 + r_{01}) \right. \\
& - \frac{\delta_{ij}}{8} \left( \frac{2}{r_1 r_0} - \frac{2}{r_1 r_{01}} - \frac{2}{r_0 r_{01}} + 2 \frac{\mathbf{n}_0 \cdot \mathbf{n}_{01}}{r_{01}^2} - 2 \frac{\mathbf{n}_1 \cdot \mathbf{n}_{01}}{r_{01}^2} \right) \\
& - \frac{\partial_{ij}^2}{8} \left( 2 \ln(r_1 + r_0 + r_{01}) - \frac{r_1 + r_0}{r_{01}} + \frac{r_0^2 (\mathbf{n}_0 \cdot \mathbf{n}_{01})}{2 r_{01}^2} - \frac{r_1^2 (\mathbf{n}_1 \cdot \mathbf{n}_{01})}{2 r_{01}^2} \right) \\
& \left. + \frac{1}{2 r_{01}^2} \left( n_0^i n_{01}^j + n_0^j n_{01}^i - n_1^i n_{01}^j - n_1^j n_{01}^i \right) \right\}. \tag{29}
\end{aligned}$$

Instead of performing the entire calculation, we present all the necessary ingredients and techniques for an (alas, still long!) example. As an illustration, we show details of the calculation for the logarithmic term in equation (29). The procedures for the remaining terms are identical and are omitted here for brevity.

After differentiating the logarithmic term with respect to body positions  $\mathbf{R}_A$ , expanding the result in terms of  $r_{01}/r_{12}$ , and grouping terms according to their composition in terms of  $n_{AB}^i$ , we obtain

$$\begin{aligned}
& 2 \int \frac{d^3 x}{4\pi} \frac{n_1^i}{r_1^2} \frac{n_2^j}{r_2^2} \left( \frac{\partial}{\partial R_1^i} \frac{\partial}{\partial R_0^j} + \frac{\partial}{\partial R_1^j} \frac{\partial}{\partial R_0^i} \right) \ln(r_1 + r_0 + r_{01}) \\
& = 2 \int \frac{d^3 x}{4\pi} \frac{n_1^i}{r_1^2} \frac{n_2^j}{r_2^2} \left[ \frac{(n_1^i + n_{01}^i)(-n_0^j + n_{01}^j)}{(r_1 + r_0 + r_{01})^2} + \frac{1}{r_1 + r_0 + r_{01}} \frac{n_{01}^i n_{01}^j - \delta^{ij}}{r_{01}} + (i \leftrightarrow j) \right] \\
& \approx 2 \int \frac{d^3 x}{4\pi} \frac{n_1^i}{r_1^2} \frac{n_2^j}{r_2^2} \left[ -\frac{1}{4} \frac{(n_1^i + n_{01}^i)(n_1^j - n_{01}^j)}{r_1^2} + \frac{1}{2} \frac{n_{01}^i n_{01}^j - \delta^{ij}}{r_{01} r_1} + (i \leftrightarrow j) \right] \\
& = \frac{2}{r_{01}} (n_{01}^i n_{01}^j - \delta^{ij}) \int \frac{d^3 x}{4\pi} \frac{n_1^i}{r_1^3} \frac{n_2^j}{r_2^2} + (n_{01}^i n_{01}^j - \delta^{ij}) n_{01}^k \int \frac{d^3 x}{4\pi} \frac{n_1^i n_1^k}{r_1^4} \frac{n_2^j}{r_2^2}, \tag{30}
\end{aligned}$$

where terms  $\mathcal{O}(r_{01}/r_{12})$  have been dropped.

Various combinations of unit vectors can be expressed through derivatives with respect to particles' distances about the integration point  $P$

$$\frac{n_A^i}{r_A^N} = \frac{1}{N-1} \tilde{\partial}_i \left( \frac{1}{r_A^{N-1}} \right), \tag{31}$$

where  $\tilde{\partial}_i \equiv \partial/\partial R_A^i$ . For combinations of several  $n_A^i$ 's, analogous relations can be derived to be:

$$\begin{aligned}
\frac{n_A^i n_A^j}{r_A^N} &= \frac{1}{N(N-2)} \tilde{\partial}_{ij}^2 \left( \frac{1}{r_A^{N-2}} \right) + \frac{1}{N} \frac{\delta^{ij}}{r_A^N}, \\
\frac{n_A^i n_A^j n_A^k}{r_A^N} &= \frac{1}{(N-3)(N-1)(N+1)} \tilde{\partial}_{ijk}^3 \left( \frac{1}{r_A^{N-3}} \right)
\end{aligned} \tag{32}$$

$$+\frac{1}{(N-1)(N+1)}\left(\delta^{ij}\tilde{\partial}_k+\delta^{jk}\tilde{\partial}_i+\delta^{ki}\tilde{\partial}_j\right)\left(\frac{1}{r_A^{N-1}}\right), \quad (33)$$

$$\begin{aligned} \frac{n_A^i n_A^j n_A^k n_A^l}{r_A^N} &= \frac{1}{(N-4)(N-2)N(N+2)}\tilde{\partial}_{ijkl}^4\left(\frac{1}{r_A^{N-4}}\right) \\ &+ \frac{1}{(N-2)N(N+2)}\left(\delta^{ij}\tilde{\partial}_{kl}^2+\delta^{li}\tilde{\partial}_{jk}^2+\delta^{kl}\tilde{\partial}_{ij}^2\right. \\ &\quad \left.+\delta^{jk}\tilde{\partial}_{li}^2+\delta^{ik}\tilde{\partial}_{jl}^2+\delta^{jl}\tilde{\partial}_{ik}^2\right)\left(\frac{1}{r_A^{N-2}}\right) \\ &+ \frac{1}{N(N-2)}\frac{1}{r_A^N}\left(\delta^{ij}\delta^{kl}+\delta^{ik}\delta^{jl}+\delta^{il}\delta^{kj}\right), \end{aligned} \quad (34)$$

where in equations (32) through (34) we have abbreviated multiple derivatives as

$$\tilde{\partial}_{ij\dots k}^N \equiv \underbrace{\frac{\partial}{\partial R_A^i}\frac{\partial}{\partial R_A^j}\cdots\frac{\partial}{\partial R_A^k}}_{N \text{ derivatives}}. \quad (35)$$

After expressing the unit vectors in equation (30) as derivatives and taking derivatives in front of the integral sign, all integrations are performed over integrands that are combinations of powers of  $r_1$  and  $r_2$  only. Such integrals can be evaluated by the use of the formula (see Damour & Schäfer (1985))

$$I(\alpha, \beta) = \int \frac{d^3x}{4\pi} r_A^\alpha r_B^\beta = \frac{\sqrt{\pi}}{4} \frac{\Gamma\left(\frac{\alpha+3}{2}\right)\Gamma\left(\frac{\beta+3}{2}\right)\Gamma\left(-\frac{\alpha+\beta+3}{2}\right)}{\Gamma\left(-\frac{\alpha}{2}\right)\Gamma\left(-\frac{\beta}{2}\right)\Gamma\left(\frac{\alpha+\beta+6}{2}\right)} r_{AB}^{\alpha+\beta+3}. \quad (36)$$

After performing integrations, differentiations with respect to body trajectories  $R_A$  have to be performed. Integrations yield results that depend on  $R_A$  as powers of  $r_{AB}^N = |\mathbf{R}_A - \mathbf{R}_B|^N$  and in order to calculate  $U_{TT}^{(10,12)}$ , we need to perform up to five consecutive differentiations with respect to  $R_A$ :

$$\frac{\partial}{\partial R_A^i} \frac{1}{r_{AB}^N} = -N \frac{n_{AB}^i}{r_{AB}^{N+1}}, \quad (37)$$

$$\frac{\partial}{\partial R_A^i} \frac{\partial}{\partial R_A^j} \frac{1}{r_{AB}^N} = \frac{N}{r_{AB}^{N+2}} \left( (N+2)n_{AB}^i n_{AB}^j - \delta^{ij} \right), \quad (38)$$

$$\begin{aligned} \frac{\partial}{\partial R_A^i} \frac{\partial}{\partial R_A^j} \frac{\partial}{\partial R_A^k} \frac{1}{r_{AB}^N} &= \frac{N(N+2)}{r_{AB}^{N+3}} \left( \delta^{ij} n_{AB}^k + \delta^{ki} n_{AB}^j + \delta^{jk} n_{AB}^i \right. \\ &\quad \left. - (N+4) n_{AB}^i n_{AB}^j n_{AB}^k \right), \end{aligned} \quad (39)$$

$$\frac{\partial}{\partial R_A^i} \frac{\partial}{\partial R_A^j} \frac{\partial}{\partial R_A^k} \frac{\partial}{\partial R_A^l} \frac{1}{r_{AB}^N} = \frac{N(N+2)}{r_{AB}^{N+4}} \left[ \delta^{ij} \delta^{kl} + \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk} + (N+4)(N+6) n_{AB}^{ijkl} \right]$$

$$-(N+4) \left( \delta^{ij} n_{AB}^{kl} + \delta^{ik} n_{AB}^{jl} + \delta^{il} n_{AB}^{jk} + \delta^{jk} n_{AB}^{il} + \delta^{jl} n_{AB}^{ik} + \delta^{kl} n_{AB}^{ij} \right) \Big], \quad (40)$$

$$\begin{aligned} \frac{\partial}{\partial R_A^i} \frac{\partial}{\partial R_A^j} \frac{\partial}{\partial R_A^k} \frac{\partial}{\partial R_A^l} \frac{\partial}{\partial R_A^m} \frac{1}{r_{AB}^N} &= -\frac{N(N+2)(N+4)}{r_{AB}^{N+5}} \left\{ \left( \delta^{ij} \delta^{kl} + \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk} \right) n_{AB}^m \right. \\ &\quad + \left( \delta^{mj} \delta^{kl} + \delta^{mk} \delta^{jl} + \delta^{ml} \delta^{jk} \right) n_{AB}^i \\ &\quad + \left( \delta^{im} \delta^{kl} + \delta^{ik} \delta^{ml} + \delta^{il} \delta^{mk} \right) n_{AB}^j \\ &\quad + \left( \delta^{ij} \delta^{ml} + \delta^{im} \delta^{jl} + \delta^{il} \delta^{jm} \right) n_{AB}^k \\ &\quad + \left( \delta^{ij} \delta^{km} + \delta^{ik} \delta^{jm} + \delta^{im} \delta^{jk} \right) n_{AB}^l \\ &\quad - (N+6) \left[ \delta^{ij} n_{AB}^{klm} + \delta^{ik} n_{AB}^{jlm} + \delta^{il} n_{AB}^{jkm} + \delta^{im} n_{AB}^{jkl} \right. \\ &\quad + \delta^{jk} n_{AB}^{ilm} + \delta^{jl} n_{AB}^{ikm} + \delta^{jm} n_{AB}^{ikl} \\ &\quad + \delta^{kl} n_{AB}^{ijm} + \delta^{km} n_{AB}^{ijl} + \delta^{lm} n_{AB}^{ijk} \Big] \\ &\quad \left. + (N+6)(N+8) n_{AB}^{ijklm} \right\}, \quad (41) \end{aligned}$$

where  $n_{AB}^{ij\dots m} \equiv n_{AB}^i n_{AB}^j \dots n_{AB}^m$ . We also note that differentiations with respect to  $R_1^i$  and  $R_2^i$  can be related through the identity

$$\frac{\partial}{\partial R_B^i} \frac{1}{r_{AB}^N} = -\frac{\partial}{\partial R_A^i} \frac{1}{r_{AB}^N}. \quad (42)$$

We now assemble these results. Firstly, after rewriting the unit vectors as derivatives through equations (31) and (32), we perform the necessary integrations by employing equation (36):

$$\begin{aligned} \frac{2}{r_{01}} (n_{01}^i n_{01}^j - \delta^{ij}) \int \frac{d^3 x}{4\pi} \frac{n_1^i}{r_1^3} \frac{n_2^j}{r_2^2} = \\ \lim_{\alpha, \beta \rightarrow 0} \left[ \frac{2}{r_{01}} (n_{01}^i n_{01}^j - \delta^{ij}) \frac{1}{2-\alpha} \frac{1}{1-\beta} \frac{\partial}{\partial R_1^i} \frac{\partial}{\partial R_2^j} I(\alpha-2, \beta-1) \right], \quad (43) \end{aligned}$$

$$\begin{aligned} (n_{01}^i n_{01}^j - \delta^{ij}) n_{01}^k \int \frac{d^3 x}{4\pi} \frac{n_1^i n_1^k}{r_1^4} \frac{n_2^j}{r_2^2} = \\ \lim_{\alpha, \beta \rightarrow 0} \left[ (n_{01}^i n_{01}^j - \delta^{ij}) n_{01}^k \left( \frac{1}{4-\alpha} \frac{1}{2-\alpha} \frac{1}{1-\beta} \frac{\partial}{\partial R_1^i} \frac{\partial}{\partial R_2^j} \frac{\partial}{\partial R_1^k} I(\alpha-2, \beta-1) \right. \right. \\ \left. \left. + \frac{1}{4-\alpha} \delta^{ik} \frac{\partial}{\partial R_2^j} I(\alpha-4, \beta-1) \right) \right]. \quad (44) \end{aligned}$$

Finally, the application of differentiations in equations (37) through (41) yields the result

$$2 \int \frac{d^3x}{4\pi} \frac{n_1^i}{r_1^2} \frac{n_2^j}{r_2^2} \left( \frac{\partial}{\partial R_1^i} \frac{\partial}{\partial R_0^j} + \frac{\partial}{\partial R_1^j} \frac{\partial}{\partial R_0^i} \right) \ln(r_1 + r_0 + r_{01}) \approx$$

$$-2 \frac{(\mathbf{n}_{01} \cdot \mathbf{n}_{12})^2}{r_{01} r_{12}^2} - \frac{1}{2} \frac{(\mathbf{n}_{01} \cdot \mathbf{n}_{12})}{r_{12}^3} + \frac{(\mathbf{n}_{01} \cdot \mathbf{n}_{12})^3}{r_{12}^3}. \quad (45)$$

In order to perform the full calculation for  $U_{TT}^{(10,12)}$ , we need to compute the remaining terms in equation (29). They can be calculated by applying the technique described above on the following expressions:

$$\partial_i \partial_j \ln(r_0 + r_1 + r_{01}) \approx -2 \frac{n_1^i n_1^j}{r_1^2} + \frac{\delta^{ij}}{r_1^2}, \quad (46)$$

$$\partial_i \partial_j \left( -\frac{r_0 + r_1}{r_{01}} \right) \approx 2 \frac{n_1^i n_1^j - \delta^{ij}}{r_1 r_{01}} + \frac{3n_1^i n_1^j - \delta^{ij}}{r_1^2} (\mathbf{n}_1 \cdot \mathbf{n}_{01})$$

$$- \frac{n_1^i n_{01}^j}{r_1^2} - \frac{n_1^j n_{01}^i}{r_1^2}, \quad (47)$$

$$\partial_i \partial_j \left( \frac{r_1 (\mathbf{r}_1 \cdot \mathbf{r}_{01})}{2r_{01}^3} - \frac{r_0 (\mathbf{r}_0 \cdot \mathbf{r}_{01})}{2r_{01}^3} \right) \approx \frac{1}{2r_1 r_{01}} \left\{ -2 (n_1^i n_{01}^j + n_1^j n_{01}^i) (\mathbf{n}_1 \cdot \mathbf{n}_{01}) \right.$$

$$+ \delta^{ij} (1 - (\mathbf{n}_1 \cdot \mathbf{n}_{01})^2)$$

$$+ 2n_{01}^i n_{01}^j + n_1^i n_1^j (3(\mathbf{n}_1 \cdot \mathbf{n}_{01})^2 - 1) \left. \right\}$$

$$+ \frac{1}{2r_1^2} \left\{ \frac{3}{2} (n_1^i n_{01}^j + n_1^j n_{01}^i) (1 - 3(\mathbf{n}_1 \cdot \mathbf{n}_{01})^2) \right.$$

$$+ 3n_{01}^i n_{01}^j (\mathbf{n}_1 \cdot \mathbf{n}_{01})$$

$$+ \frac{3}{2} \delta^{ij} (\mathbf{n}_1 \cdot \mathbf{n}_{01}) (1 - (\mathbf{n}_1 \cdot \mathbf{n}_{01})^2)$$

$$+ \frac{3}{2} n_1^i n_1^j (\mathbf{n}_1 \cdot \mathbf{n}_{01}) (5(\mathbf{n}_1 \cdot \mathbf{n}_{01})^2 - 3) \left. \right\} \quad (48)$$

$$\frac{n_0^i n_{01}^j}{r_{01}^2} \approx \frac{n_1^i n_{01}^j}{r_{01}^2} + \frac{n_1^i n_{01}^j}{r_1 r_{01}} (\mathbf{n}_1 \cdot \mathbf{n}_{01}) + \frac{n_1^i n_{01}^j}{2r_1^2} (3(\mathbf{n}_1 \cdot \mathbf{n}_{01})^2 - 1)$$

$$- \frac{n_{01}^i n_{01}^j}{r_{01} r_1} - \frac{n_{01}^i n_{01}^j}{r_1^2} (\mathbf{n}_1 \cdot \mathbf{n}_{01}). \quad (49)$$

A straightforward, but somewhat lengthy, calculation yields the following approximate result

$$2 \int \frac{d^3x}{4\pi} \frac{n_1^i}{r_1^2} \frac{n_2^j}{r_2^2} \left\{ -\frac{\delta_{ij}}{8} \left( \frac{2}{r_1 r_0} - \frac{2}{r_1 r_{01}} - \frac{2}{r_0 r_{01}} + 2 \frac{\mathbf{n}_0 \cdot \mathbf{n}_{01}}{r_{01}^2} - 2 \frac{\mathbf{n}_1 \cdot \mathbf{n}_{01}}{r_{01}^2} \right) \right.$$

$$\begin{aligned}
& -\frac{\partial_{ij}^2}{8} \left( 2 \ln(r_1 + r_0 + r_{01}) - \frac{r_1 + r_0}{r_{01}} + \frac{r_0^2(\mathbf{n}_0 \cdot \mathbf{n}_{01})}{2r_{01}^2} - \frac{r_1^2(\mathbf{n}_1 \cdot \mathbf{n}_{01})}{2r_{01}^2} \right) \\
& + \frac{1}{2r_{01}^2} \left( n_0^i n_{01}^j + n_0^j n_{01}^i - n_1^i n_{01}^j - n_1^j n_{01}^i \right) \} \\
& \approx \frac{11}{16} \frac{1}{r_{01}r_{12}^2} - \frac{(\mathbf{n}_{01} \cdot \mathbf{n}_{12})^2}{r_{01}r_{12}^2} - \frac{9}{32} \frac{(\mathbf{n}_{01} \cdot \mathbf{n}_{12})}{r_{12}^3} - \frac{7}{48} \frac{(\mathbf{n}_{01} \cdot \mathbf{n}_{12})^3}{r_{12}^3}. \tag{50}
\end{aligned}$$

The final result for the transverse–traceless term is

$$\begin{aligned}
U_{TT} & \approx U_{TT}^{(12,12)} + U_{TT}^{(10,12)} + U_{TT}^{(12,02)} \\
& \approx -\frac{1}{2} \frac{m_1^2 m_2^2}{r_{12}^3} - \frac{1}{2} \frac{m_0 m_1 m_2^2}{r_{12}^3} \\
& + m_0 m_1^2 m_2 \left[ \frac{11}{16} \frac{1}{r_{01}r_{12}^2} - 3 \frac{(\mathbf{n}_{01} \cdot \mathbf{n}_{12})^2}{r_{01}r_{12}^2} - \frac{25}{32} \frac{(\mathbf{n}_{01} \cdot \mathbf{n}_{12})}{r_{12}^3} + \frac{41}{48} \frac{(\mathbf{n}_{01} \cdot \mathbf{n}_{12})^3}{r_{12}^3} \right]. \tag{51}
\end{aligned}$$

### 3. Roche Lobes for 2PN

We are now ready to find the equipotential surfaces that correspond to the Roche lobes and to compute volumes that are contained inside the lobes. We employ equations (11) through (21), supplemented with equation (51). After setting the three bodies on quasicircular orbits around the center of mass, we obtain the potential that gives rise to forces that act on the point–particle of mass  $m_0$ . In Figure 2, we show the potential for  $y = 0$  (we position the three bodies on the  $x$ – $y$  plane so  $z = 0$  automatically) and the corresponding Roche lobes for mass ratios  $q = 0.1, 0.2, 0.5$ , and  $1.0$ , respectively. Cusps on equipotential lines correspond to the first Lagrange points  $L_1$ . All distances are scaled by the stellar radial separation  $a \equiv r_{12}$ . Note that we only show the lobe around the first star, since our approximation ( $r_{01} \ll r_{12}$ ) is valid only in this region.

As in the Newtonian case, the volume within the equipotential surface (Roche volume) grows with  $q$  for fixed  $a$ . However, the potential and the equipotential surfaces acquire an additional dependence. Unlike for the Newtonian case, the total mass modifies the result: for low  $q$ , the Roche volumes become smaller as the total mass increases, whereas for  $q$  greater than about  $0.7$  the volumes increase. As for coordinate positions, where we have eliminated the separation  $a$ , we can introduce a new dimensionless parameter

$$z \equiv \frac{2M}{a}, \tag{52}$$

involving the ratio of the total mass  $M$  and the separation  $a$ . This parameter also corresponds to the ratio of Schwarzschild radius for  $M$  and the separation distance  $a$ .

Integration of volumes enclosed by equipotential surfaces is straightforward; we utilize a Newton–Cotes type algorithm to find the enclosed volumes. Our results are shown in Figure 3 for  $q \in [0, 1]$ , and  $z = 0$ ,  $z = 0.2$ , and  $z = 0.4$ , respectively.

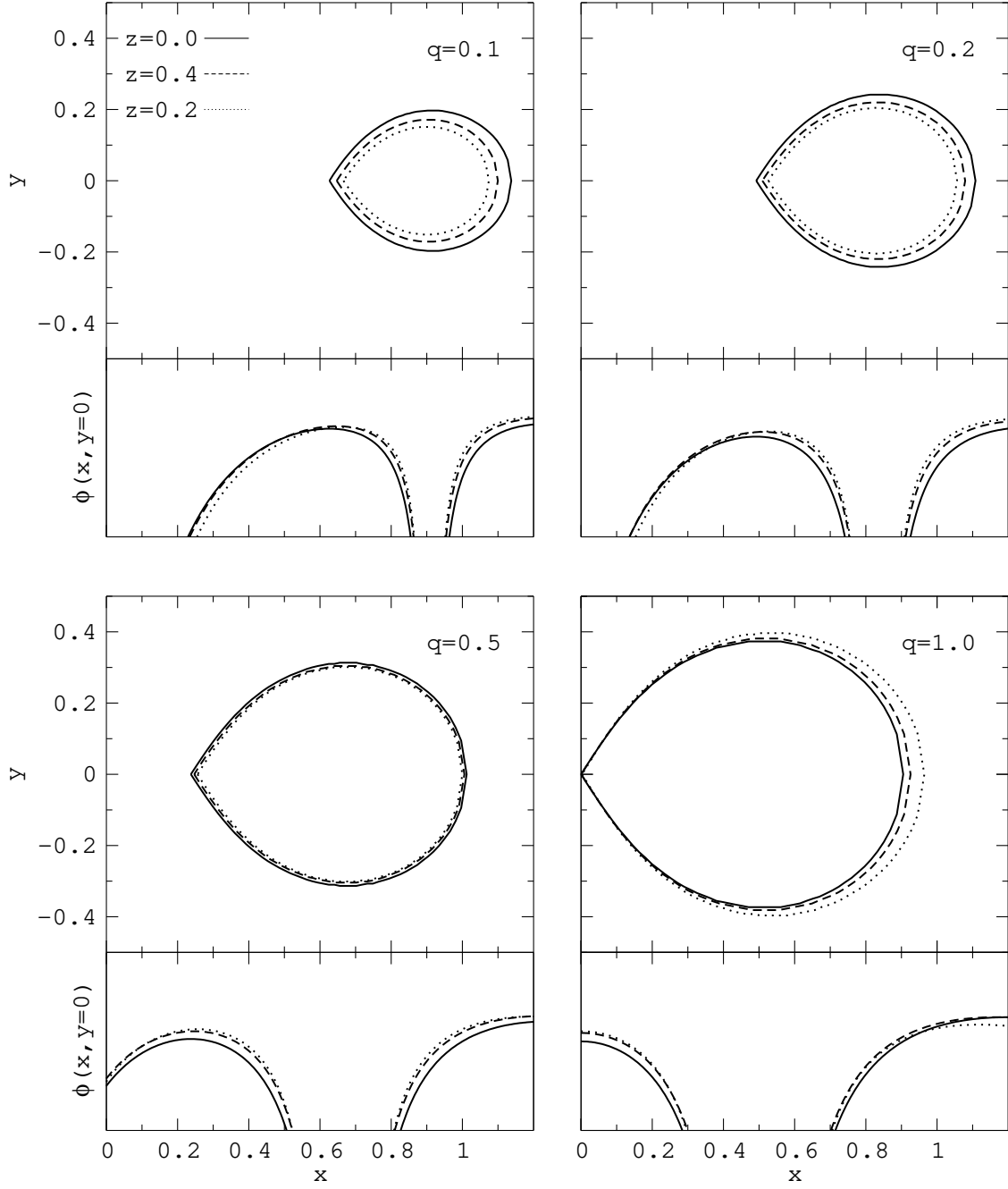


Fig. 2.— Roche lobes and the corresponding potentials for  $y = 0$ . Coordinates  $x$  and  $y$  are scaled by the stellar separation  $a$  and are shown for  $q = 0.1, 0.2, 0.5$ , and  $1.0$ , respectively. Results shown are for values of  $z = 0$ ,  $z = 0.2$ , and  $z = 0.4$ , respectively.



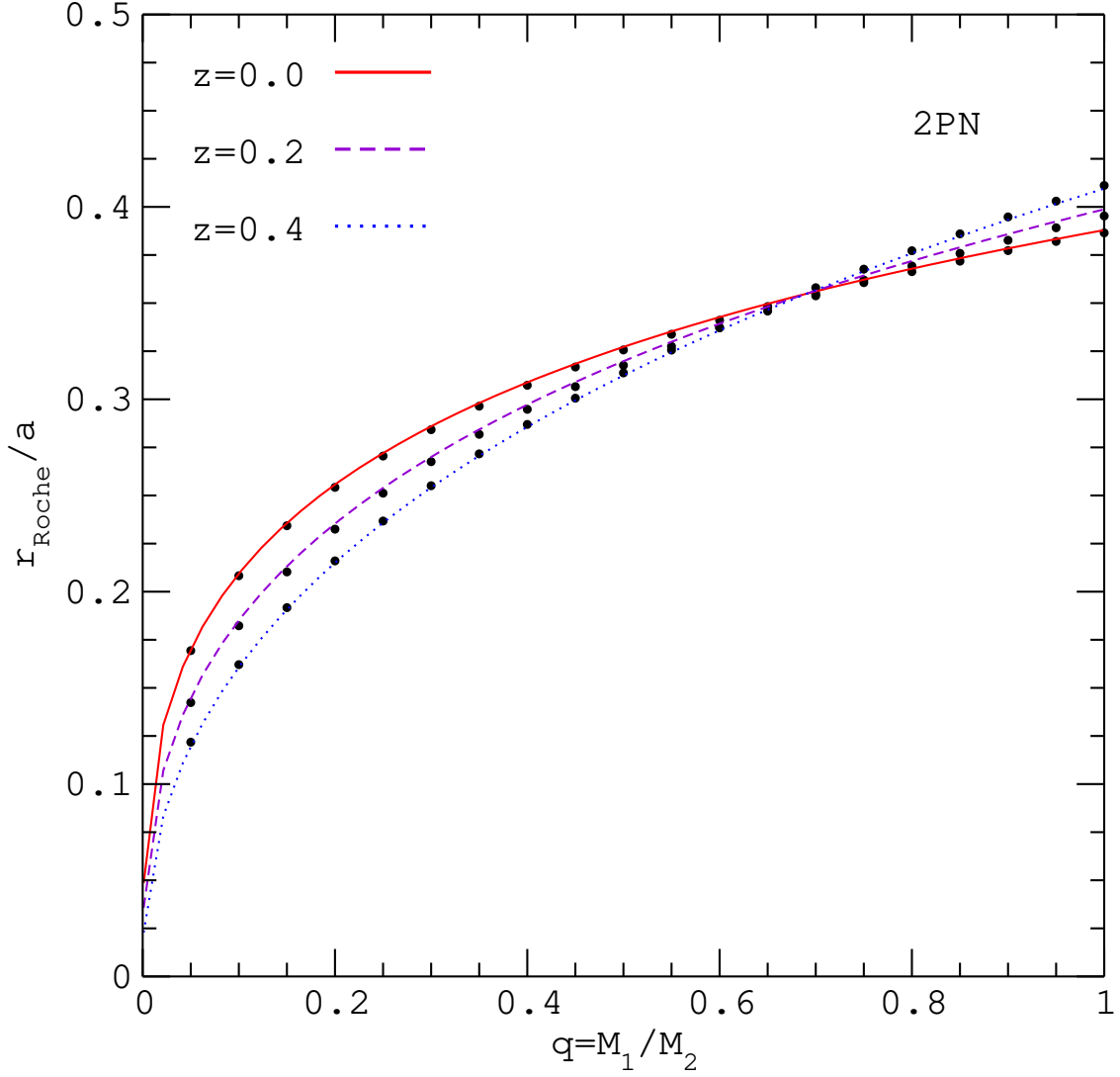


Fig. 3.— Effective Roche lobe radii  $r_{Roche}$  scaled by the stellar separation  $a$  versus  $q$  for  $z = 0$ ,  $z = 0.2$ , and  $z = 0.4$ , respectively.

In Figure 3, we also show results for the fitting function. Following Eggleton (1982), we choose the parametrization in which the scaled effective Roche radius is

$$r_{Roche}/a = Q(q) C(q, z), \quad (53)$$

where

$$Q(q) = \frac{\alpha_Q q^{2/3}}{\beta_Q q^{2/3} + \ln(1 + q^{1/3})} \quad (54)$$

is the fitting function previously given by Eggleton (1982) and

$$C(q, z) = 1 + z(\alpha_C q^{1/5} - \beta_C) \quad (55)$$

is the correction function that stems from post-Newtonian effects. Fitting parameters of  $Q(q)$  are identical to the ones obtained by Eggleton (1982) ( $\alpha_Q = 0.49$  and  $\beta_Q = 0.57$ ) and values of

$$\alpha_C = 1.951, \quad \text{and} \quad \beta_C = 1.812 \quad (56)$$

refer to the fitting function for  $C(q, z)$ . This functional form describes extremely well the dependence of  $q$  for  $q < 1.0$ . We note that the crossover of reduced versus enlarged Roche lobes with respect to the Newtonian ( $z = 0$ ) case occurs at  $q = (\beta_C/\alpha_C)^5 \approx 0.69$ , in which case the Roche radius is virtually  $z$ -independent.

### 3.1. Lagrange points and the center of mass

While performing the computation of the effective Roche volumes and radii, it is necessary to find the first Lagrange point  $L_1$ . In addition, there are two more extrema of the potential along the  $x$ -axis that correspond to the second and third Lagrange points,  $L_2$  and  $L_3$ , respectively. Moreover, it is necessary to find the position of the center of mass of the system while setting the particles onto quasicircular orbits. We briefly outline below how the positions of these points depend on the mass ratio and how the results get modified compared to the Newtonian case. We consider only the first three out of five Lagrange points ( $L_1$ ,  $L_2$ , and  $L_3$ ) as  $L_4$  and  $L_5$  become local extrema only for nonvanishing velocity  $\mathbf{v}_0^{rot}$ .

#### 3.1.1. Lagrange points

In the Newtonian case, the positions of the Lagrange points can be fully described in terms of the mass ratio  $q = M_1/M_2$ . As the mass ratio drops towards 0, the position of the first Lagrange point quickly shifts away from the center of mass and in the direction of the lighter star. The positions of Lagrange points  $L_2$  and  $L_3$  show a much weaker  $q$ -dependence as can be seen in Figure 4, where the center of mass is positioned at  $x = 0$ . The second point  $L_2$  shifts towards the lighter mass for  $q < 0.2$  after it passes through the maximum of its distance from the center of mass at  $q \approx 0.22$ . The third Lagrange point is slowly, but steadily, pulled toward the center of mass as  $q \rightarrow 0$ .

As we might expect from the Roche lobe analysis, effects of post-Newtonian corrections give rise to an additional  $(M_1 + M_2)/r_{12} = M/a$  dependence. In our model, this can be parametrized with the quantity  $z = 2M/a$ . In Figure 4, we show two additional sets of lines for  $z = 0.2$  and  $0.4$  ( $z = 0$  corresponds to the Newtonian case). For all values of  $q$  given in Figure 4, the positions of  $L_1$  and  $L_2$  are slightly pulled towards the lighter mass compared to the Newtonian case. The position of  $L_3$  is, however, pulled towards the center of mass for  $q \gtrsim 0.45$  and away from it for  $q \lesssim 0.4$ .

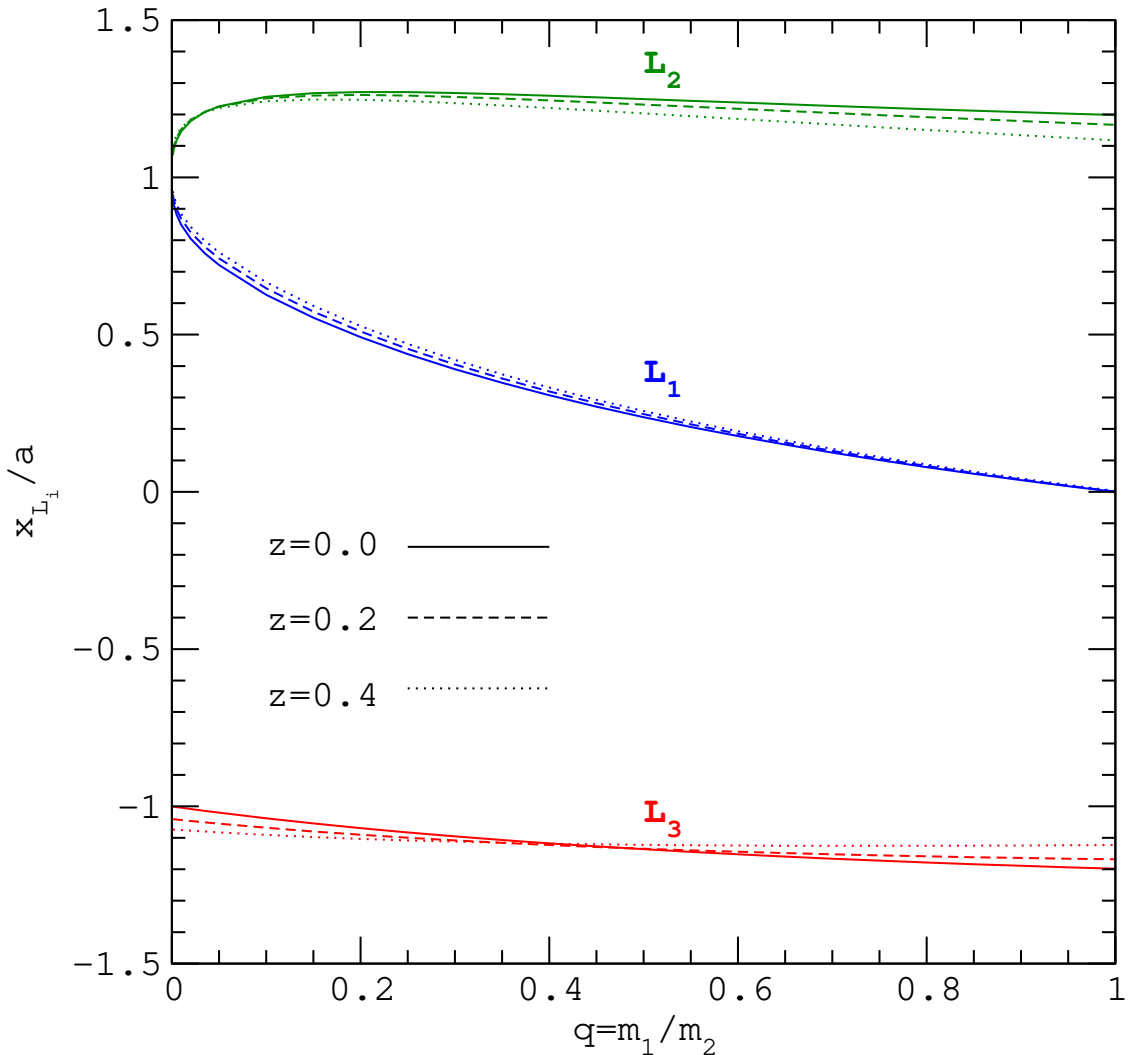


Fig. 4.— The positions of the Lagrange points  $L_1$ ,  $L_2$ , and  $L_3$  as a function of the mass ratio  $q = M_1/M_2$ . Results are for  $z = 0, 0.2$ , and  $0.4$ , respectively.

### 3.1.2. Position of the center of mass

In order to find the Roche lobes, we set the two stars on quasicircular orbits and set the center of mass at the origin. The post-Newtonian approximation modifies the position of the center of mass compared to the Newtonian result since the conservation of total momentum results in a slight shift of the position from the Newtonian case. In this section, we show how important these corrections are and parametrize them in terms of  $z = 2M/a$ .

We denote the ratio of the distance between  $m_1$  and the center of mass by  $a_1$ . The separation

between the two stars is  $a$ , and their ratio is

$$\frac{a_1}{a} = \beta, \quad \text{and} \quad \frac{a_2}{a} = 1 - \beta. \quad (57)$$

Setting the center of mass at the origin yields the condition (de Andrade et al. 2001)

$$\begin{aligned} & \beta m_1 \left\{ 1 + \frac{p_1^2}{2m_1^2} - \frac{m_2}{2a} - \frac{p_1^4}{8m_1^4} + \frac{m_2}{4a} \left( -5 \frac{p_1^2}{m_1^2} - \frac{p_2^2}{m_2^2} + 7 \frac{\mathbf{p}_1 \cdot \mathbf{p}_2}{m_1 m_2} \right) + \frac{m_2 (m_1 + m_2)}{a^2} \right\} = \\ & (1 - \beta) m_2 \left\{ 1 + \frac{p_2^2}{2m_2^2} - \frac{m_1}{2a} - \frac{p_2^4}{8m_2^4} + \frac{m_1}{4a} \left( -5 \frac{p_2^2}{m_2^2} - \frac{p_1^2}{m_1^2} + 7 \frac{\mathbf{p}_1 \cdot \mathbf{p}_2}{m_1 m_2} \right) + \frac{m_1 (m_1 + m_2)}{a^2} \right\}, \end{aligned} \quad (58)$$

which contains an implicit dependence on  $\beta$ , since for quasicircular motion we can write the velocities of the two stars as

$$v_1^2 = \beta^2 a^2 w^2, \quad \text{and} \quad v_2^2 = (1 - \beta)^2 a^2 w^2, \quad (59)$$

and their product as

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = -\beta(1 - \beta) a^2 w^2, \quad (60)$$

where at the 2PN level (Blanchet & Iyer 2003)

$$\omega^2 = \frac{M}{a^3} \left( 1 + \frac{M}{a} (\nu - 3) + \left( \frac{M}{a} \right)^2 \left( \frac{21}{4} - \frac{5}{8} \nu + \nu^2 \right) \right), \quad (61)$$

with  $\nu = q/(1 + q)^2$ . Squares of the two momenta are

$$p_1^2 = m_1^2 \beta^4 a^4 \omega^4 + \frac{1}{2} m_1^2 a \omega^2 (2a + 5m_2) \beta^2 + \frac{7}{2} m_1^2 a \omega^2 m_2 \beta, \quad (62)$$

$$\begin{aligned} p_2^2 &= m_2^2 a^4 \omega^4 \beta^4 - 4m_2^2 a^4 \omega^4 \beta^3 + \left( 3m_2^2 a^4 \omega^4 + \frac{1}{2} m_2^2 a \omega^2 (2a + 6a^3 \omega^2 + 5m_1) \right) \beta^2 \\ &+ \left( -\frac{1}{2} m_2^2 a \omega^2 (2a + 6a^3 \omega^2 + 5m_1) + \frac{1}{2} m_2^2 a \omega^2 (-2a - 2a^3 \omega^2 - 12m_1) \right) \beta \\ &- \frac{1}{2} m_2^2 a \omega^2 (-2a - 2a^3 \omega^2 - 12m_1). \end{aligned} \quad (63)$$

The product of the two momenta can be computed to be

$$\begin{aligned} \mathbf{p}_1 \cdot \mathbf{p}_2 &= m_1 m_2 a^4 \omega^4 \beta^4 - 2m_1 m_2 \beta^3 a^4 \omega^4 + \frac{1}{4} m_1 m_2 a \omega^2 (6a^3 \omega^2 + 4a + 5m_1 + 5m_2) \beta^2 \\ &+ \frac{1}{4} m_1 m_2 a \omega^2 (-4a - 2a^3 \omega^2 + 2m_2 - 12m_1) \beta - \frac{7}{4} m_1 m_2^2 a \omega^2, \end{aligned} \quad (64)$$

and their fourth powers (up to 2PN order) are

$$p_1^4 = m_1^4 v_1^4, \quad (65)$$

$$p_2^4 = m_2^4 v_2^4. \quad (66)$$

Inserting these results into equation (58), the position of the center of mass can be obtained by solving the 5<sup>th</sup> order polynomial

$$0 = a_0 + a_1\beta + a_2\beta^2 + a_3\beta^3 + a_4\beta^4 + a_5\beta^5, \quad (67)$$

with coefficients  $a_i$  given by

$$\begin{aligned} a_0 = & -16m_1^2m_2 - 16m_2^2m_1 - 8m_2a^4\omega^2 + 49m_1m_2^2a^2\omega^2 - 6m_2a^6\omega^4 - 16m_2a^2 \\ & + 8m_2m_1a - 28m_2a^3\omega^2m_1 + 120m_1^2m_2a^2\omega^2 + 20m_1m_2a^5\omega^4, \end{aligned} \quad (68)$$

$$\begin{aligned} a_1 = & 32m_2^2m_1 + 16m_2a^2 + 16m_1a^2 + 24m_2a^4\omega^2 + 30m_2a^6\omega^4 + 80m_2a^3\omega^2m_1 \\ & - 90m_1m_2a^5\omega^4 + 32m_1^2m_2 - 98m_1m_2^2a^2\omega^2 - 16m_2m_1a - 230m_1^2m_2a^2\omega^2, \end{aligned} \quad (69)$$

$$\begin{aligned} a_2 = & 51m_1^2m_2a^2\omega^2 + 146m_1m_2a^5\omega^4 - 24m_2a^4\omega^2 - 81m_1m_2^2a^2\omega^2 \\ & - 72m_2a^3\omega^2m_1 - 60m_2a^6\omega^4, \end{aligned} \quad (70)$$

$$\begin{aligned} a_3 = & 48m_2a^3\omega^2m_1 + 8a^4\omega^2m_1 + 60m_2a^6\omega^4 + 10m_1m_2^2a^2\omega^2 + 10m_1^2m_2a^2\omega^2 \\ & + 8m_2a^4\omega^2 - 84m_1m_2a^5\omega^4, \end{aligned} \quad (71)$$

$$a_4 = -30m_2a^6\omega^4 - 20m_1m_2a^5\omega^4, \quad (72)$$

$$a_5 = 6m_2a^6\omega^4 + 8m_1m_2a^5\omega^4 + 6a^6\omega^4m_1. \quad (73)$$

The results obtained by numerical root-finding are shown in Figure 5, in which we show the relative change of  $\beta$  compared to the Newtonian case:

$$\frac{\Delta\beta}{\beta} = \frac{\beta - \beta_N}{\beta}, \quad (74)$$

where  $\beta_N = m_2/M$ . The dependence on the total mass  $M$  and on the separation  $a$  is entirely through their ratio and is parametrized in Figure 5 through the parameter  $z$ .

The post-Newtonian analysis can be expected to be reliable up to moderate values of  $z$  ( $z < 0.4 - 0.5$ ). We note that in this region post-Newtonian corrections to  $\beta$  are less than about 4%. Moreover, in all cases of interest ( $q < 1.0$ ), corrections to  $\beta$  are smaller than 0.5%. As expected, deviations from the Newtonian case increase for close or massive configurations, whereas for large separations or low masses the results converge toward the Newtonian case. Whereas for the more massive star ( $q > 1.0$ ) the distance to the center of mass increases with  $z$  compared to the Newtonian case, the shift of the less massive star ( $q < 1.0$ ) does not have a monotonic dependence on  $q$ . As expected from the symmetry of the problem, corrections vanish for  $q = 1.0$ .

## 4. Conclusion

We have utilized the second order post-Newtonian approximation in the Arnowitt–Deser–Misner gauge to calculate Roche lobe volumes. These results are an improvement over the Newto-

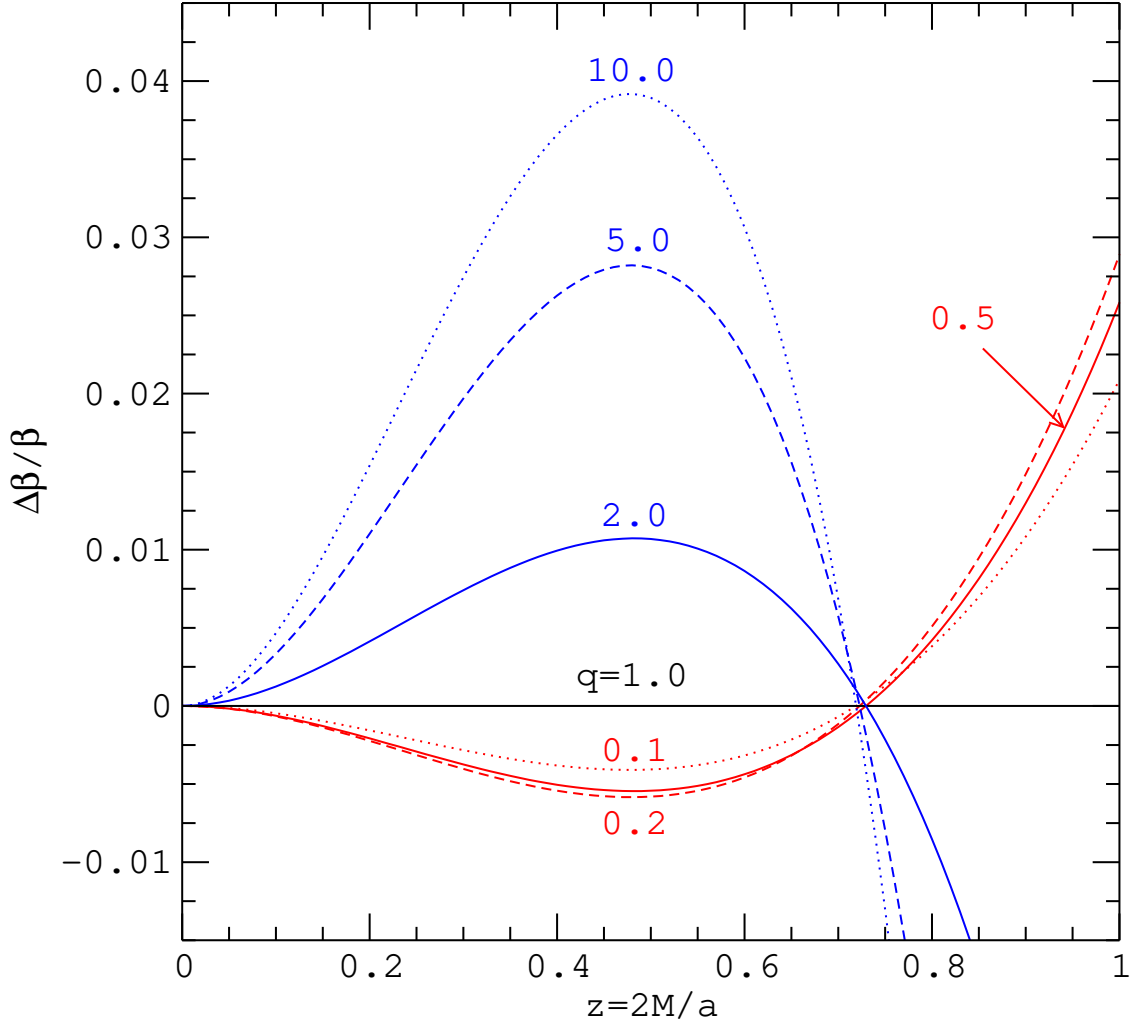


Fig. 5.— The position of the center of mass as a function of the relativity parameter  $z$  shown in the form of the ratio  $\Delta\beta/\beta$  for  $q = 0.1 \dots 10.0$ . We note an increase of  $\beta$  compared to the Newtonian case for  $q > 1.0$ . For  $q > 1.0$ ,  $\beta$  decreases compared to the Newtonian case.

nian case in that post-Newtonian gravity introduces corrections in the case of moderately strong gravitational field.

In the course of our calculations, we have derived an approximate three-body Lagrangian that is valid in the case when one of the bodies is a point particle. This calculation requires an evaluation of the transverse-traceless term  $U_{TT}$  of the Lagrangian for which an exact result is not available. However, as shown in this work, utilization of an approximation valid in the vicinity of the less massive star enables this problem to be circumvented.

Using these results, we calculated Roche lobes in the 2PN effective potential in the co-rotating

frame and computed effective Roche lobe radii that can be used to model mass transfer through Roche lobe overflow. In addition, we computed changes to the positions of the Lagrange points and to the center of mass due to post-Newtonian effects. We find that corrections to Newtonian results for Roche lobe radii can be as significant as 20–30% at low mass ratio  $q \lesssim 0.1$ . Whereas for  $q \gtrsim 0.7$  the Roche lobe radius increases ( $\approx 15\%$  for  $q = 1.0$ ), for low  $q$ 's the Roche lobe is smaller than in the Newtonian case. We have provided our results in the form of a simple fitting formula that depends on two physical parameters: the mass ratio and the ratio of the total mass and the separation.

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