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Асимптотическое исследование модели страховой компании с инвестициями и скачками в обе стороны

Asymptotic research of the insurance company model with investments and two-side jumps

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Аннотация

В данной работе исследуется асимптотика вероятности разорения страховой компании, процесс которой описывается классической моделью теории коллективного риска Лундберга - Крэмера. Компания инвестируют в рисковый актив, динамика которого соответствует геометрическому броуновскому движению. Подход к нахождению асимптотики вероятности разорения основан на выводе и анализе интегро-дифференциальных уравнений второго порядка. В случае страхования жизни (скачки процесса риска вверх) или страхования не жизни (скачки процесса риска вниз) можно дифференцировать ИДУ, исключить интегральную часть оператора и использовать теорию асимптотического поведения линейных дифференциальных уравнений с заданным начальным условием. В настоящей работе мы будем изучать асимптотику для случая объединения этих двух моделей с экспоненциально распределенными скачками процесса вверх и вниз. Новизна настоящей работы состоит в том, что мы докажем гладкость функции разорения, как функции начального капитала и покажем, что она является решением дифференциального уравнения 4-го порядка. С помощью методов асимптотического анализа будет доказано, что вероятность разорения страховой компании убывает так же, как и в случае односторонних скачков.

1. Introduction

In the classical Lundberg–Cramér models of collective risk theory insurance companies keep their reserve in cash (or in a bank account, typically, paying zero interest rate). In recent, more realistic models, it is assumed that the capital reserves may vary not only due to the business activity but also due to a stochastic interest rate. In other words, an insurance company may invest all or just a part of its capital reserve into risky assets. These models lead to an important conclusion that the financial risk contributes enormously in the asymptotical behavior of the ruin probabilities: even under the Cramér condition, they are not exponentially decaying when the initial capital grows to infinity and the ruin always happens if the volatility of the risky asset is large with respect to the instantaneous interest rate. Moreover; they allows to quantify which proportion of risky investments may lead to the imminent ruin.

Due to their practical importance, ruin problems with investments became a vast and quickly growing chapter of the collective risk theory studying numerous models with various level of generality. The ruin problem can be treated, at least, in two different ways: using the techniques of integro-differential equations, see, e.g., [6], [7], [9], [2], [3], or results from the implicit renewal theory, see [10] and references wherein. The first approach, allowing not only to obtain asymptotic but also calculate ruin probabilities for given values of the capital reserves, has some interesting mathematical questions.

Our paper is a complement to the papers [7] and [9], that extend, respectively, the Lundberg–Cramér models for non-life insurance and life-insurance to the case where the capitals of insurance companies are invested into a risky asset with the price dynamics given by a geometric Brownian motion. In both, the business activity is given by a compound Poisson process either with negative jumps and positive drift (non-life insurance), or with positive jumps and negative drift (life insurance). Technically speaking, these two models are quite different: in the first case the downcrossing of zero may happen only at an instant of jump (thus, the model can be reduced to a discrete-time one) while in the second case the downcrossing happens in a continuous way and the reduction to a discrete-time model is not possible. Of course, in the classical setting if jumps are exponentially distributed, a model with upward jumps can be transformed into a model with up downward jumps and vice versa. In models with investment the duality arguments does not work and one needs to treat them separately. In the mentioned papers it was shown that in the Lundberg-type model, that is, with exponentially distributed jumps, the ruin probability decreases with the rate $Cu^{-\beta}$, $C > 0$, when $\beta := 2a/\sigma^2 - 1 > 0$ and the ruin happens with probability one when $\beta \geq 1$ (the result for $\beta = 1$ was established in [18] and [19]).

Here we consider a model of a company combining both types of business activities, that is, for which the corresponding compound Poisson process has positive and negative jumps, [21]. We prove, under some minor assumptions on the distributions of jumps, that the ruin probability as a function of the initial value is smooth and satisfies an integro-differential equation (IDE). For a more specific case of exponential distributions we show that the ruin probability is a solution of the 4th order ordinary differential equation. Asymptotic analysis of this ODE leads us to our main result (Theorem 2.2) which looks exactly as those of [7] and [9]. Though the arguments follow the same general line as in [9], they are different in many aspects. Several seemingly new results are obtained. In particular, for the model of Sparre Andersen with investments, where the interarrival times form a renewal process, we derive a sufficient condition for the ruin with probability one (Theorem 3.1) and also a lower asymptotic bound (Proposition 4.1). For the model where the business activity of the company can be represented as the difference of two compound Poisson processes with exponentially distributed jumps we prove a result on smoothness of the ruin probabilities.

2. The model

Let $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ be a stochastic basis, that is a filtered probability space, where we are given a Wiener process $W = (W_t)_{t\geq 0}$ and an independent compound Poisson process $P = (P_t)_{t\geq 0}$ with drift c and successive jump instants T_n. We denote $p_P(dt, dx)$ the jump measure of the latter with its mean measure $\Pi_P(dx)dt$ where $\Pi_P(\mathbf{R}) < \infty$. We assume that $\Pi_P(\mathbf{R}_+) > 0$ and $\Pi_P(\mathbf{R}_-) > 0$ (some comments for the cases where Π_P charges only of the half-axes will be also given).

We consider the process $X = X^u$ which is the solution of non-homogeneous linear stochastic equation

$$X_t = u + \int_0^t X_s dR_s + P_t, \tag{2.1}$$

where $R_t = at + \sigma W_t$ is the relative price process of the risky asset and P is a compound Poisson process with drift $c \in \mathbf{R}$ representing the business activity of the insurance company, $u > 0$ is the initial capital at time zero; we assume that $\sigma^2 > 0$. The process X can be written in "differential" form as $dX_t = X_t dR_t + dP_t$, $X_0 = u$, where

$$dP_t = cdt + \int x p_P(dt, dx), \quad P_t = 0. \tag{2.2}$$

In the actuarial context $X = X^u$ represents the dynamics of the reserve of an insurance company combining life and non-life insurance business and investing into a stock with the price given by a geometric Brownian motion

$$S_t = S_0 e^{(a-\sigma^2/2)t+\sigma W_t}$$

solving the linear stochastic differential equation $dS_t = S_t dR_t$ with initial value S_0. Changing in the need the money unit, we assume with our lost of generality that $S_0 = 1$. In this case we have by the product formula that

$$X_t = S_t\left(u + \int_0^t S_s^{-1} dP_s\right). \tag{2.3}$$

In the classical collective risk theory the process P are usually represented in the form

$$P_t = ct + \sum_{i=1}^{N_t} \xi_i, \tag{2.4}$$

where $N_t := p_P([0,t] \times \mathbf{R})$ is a Poisson process with intensity $\alpha = \Pi_P(\mathbf{R})$ independent on the random variables $\xi_i = \Delta P_{T_n}$ where T_n are successive instants of jumps of N; the random variables ξ_i has the probability distribution $F(dx) = \Pi_P(dx)/\Pi_P(\mathbf{R})$. Alternatively, can represent P using independent compound Poisson processes given by the sums in the following representation.

$$P_t = ct + \sum_{i=1}^{N_t^2} \xi_i^2 - \sum_{i=1}^{N_t^1} \xi_i^1.$$

Here the Poisson processes N^1 and N^2 with intensities $\alpha_1 = \Pi_P(\mathbf{R}_-)$ and $\alpha_2 = \Pi_P(\mathbf{R}_+)$, count, respectively, downward and upward jumps of P and have successive jump instants T_n^1, T_n^2. The corresponding jump sizes ξ_n^1, ξ_n^2 are positive and have the distribution functions $F_1(x) := \Pi_P(]-x,0])/\alpha_1$, $F_2(x) := \Pi_P(]0,x])/\alpha_2$ for $x > 0$.

Let $\tau^u := \inf\{t : X_t^u \le 0\}$ (the instant of ruin), $\Psi(u) := P(\tau^u < \infty)$ (the ruin probability), and $\Phi(u) := 1 - \Psi(u)$ (the survival probability).

Note that the same formula (2.4) N can be an independent renewal process, that is, the counting process in which the lengths of the interarrival intervals $T_n - T_{n-1}$ form an i.i.d. sequence. In the collective risk theory this corresponds to the Sparre Andersen model.

The main aims of the present work:

1) to get a result on smoothness in u of the ruin probability Ψ justifying that Ψ solves IDE in the classical sense;

2) to deduce from the latter, in the spacial case of exponentially distributed jumps, an ODE and use it to obtain the exact asymptotic of the ruin probability as $u \to \infty$.

Of course, in the model with only upward jumps there is no ruin if $c \ge 0$. The case where $c < 0$ is studied in details in [9] and [18]. The paper [7] deals with the model with downward jumps, i.e. with the non-life insurance, but the question of smoothness is not discussed in and, seemingly, needs to be revisited (in [22] the smoothness is established under restrictions on parameters).

Notations. Throughout the paper we shall use the following abbreviations:

$$\beta := 2a/\sigma^2 - 1, \qquad \kappa := a - \sigma^2/2, \qquad \eta_t := \ln S_t = \kappa t + \sigma W_t.$$

The following simple result holds for any Sparre Andersen model with investments.

Lemma 2.1. *Suppose that there is $\beta' \in]0, \beta \wedge 1[$ such that $E(\xi_1^-)^{\beta'} < 1$. Then $\Psi(u) \to 0$ as $u \to \infty$.*

Proof. Let $\tilde{\Psi}(u)$ be the ruin probability for the reserve process $\tilde{X}^u$ corresponding to the process

$$\tilde{P}_t = -|c|t - \sum_{i=1}^{N_t} \xi_i^-.$$

Then $\Psi(u) \le \tilde{\Psi}(u) \le \mathbf{P}(Z_\infty > u)$ where

$$Z_\infty := -\int_0^\infty S_s^{-1} d\tilde{P}_s = |c| \int_0^\infty e^{-\kappa s - \sigma W_s} ds + \sum_{n=1}^\infty e^{-\kappa T_n - \sigma W_{T_n}} \xi_k^-.$$

Then $\kappa s + \sigma W_s = \sigma^2 s(\beta + (2/\sigma)W_s)/s$. By the law of large numbers for almost all ω there is $s_0(\omega)$ such that $\beta + (2/\sigma)W_s/s > \beta/2$ when $s \geq s_0(\omega)$. Thus, the integral in the right-hand side above is a finite random variable. Also

$$e^{-\kappa T_n - \sigma W_{T_n}} = \prod_{k=1}^{n} \zeta_k,$$

where $\zeta_k := e^{-\kappa(T_k - T_{k-1}) - \sigma(W_{T_k} - W_{T_{k-1}})}$ form an i.i.d. sequence. Note that $\mathbf{E}\zeta_1^{\beta} = 1$. Hence, $\mathbf{E}\zeta_1^{\beta'} < 1$ and

$$\mathbf{E}\sum_{n=1}^{\infty}(\xi_n^-)^{\beta'}\prod_{k=1}^{n}\zeta_k^{\beta'} < \infty.$$

That is, $\sum_{n=1}^{\infty}(\xi_n^- \prod_{k=1}^{n}\zeta_k)^{\beta'} < \infty$ (a.s.). But then $\sum_{n=1}^{\infty}\xi_n^- \prod_{k=1}^{n}\zeta_k < \infty$ (a.s.). $\square$

Theorem 2.2. *Let $F_1(x) = 1 - e^{-x/\mu_1}$ and $F_2(x) = 1 - e^{-x/\mu_2}$ for $x > 0$. Assume that $\sigma, c > 0$.*

(i) If $\beta > 0$, then for some $K > 0$

$$\Psi(u) = Ku^{-\beta}(1 + o(1)), \quad u \to \infty. \tag{2.5}$$

(ii) If $\beta \leq 0$, then $\Psi(u) = 1$ for all $u > 0$.

3. Large volatility case: ruin with probability one

The result below gives a sufficient condition on the ruin with probability one for Sparre Andersen models with risky investments. It implies the statement (ii) of Theorem 2.2.

Theorem 3.1. *Suppose that P is a non-increasing compound renewal process with drift (i.e., $c < 0$ or $\mathbf{P}(\xi_1 < 0) > 0$) such that $\mathbf{E}|\,\xi_1|^{\varepsilon} < \infty$ and $\mathbf{E}e^{\varepsilon T_1} < \infty$ for some $\varepsilon > 0$. If $\beta \leq 0$, then $\Psi(u) = 1$ for any u.*

Remark. In this formulation one can replace the assumption $\mathbf{E}|\xi_1|^{\varepsilon} < \infty$ on a formally weaker $\mathbf{E}(\xi_1^+)^{\varepsilon} < \infty$.

As in [9] the arguments are based on the ergodic property of the discrete-time autoregressive process $(\tilde{X}_n^u)_{n\geq 1}$ with random coefficients which is defined recursively by the relations

$$\tilde{X}_n^u = A_n\tilde{X}_{n-1}^u + B_n, \qquad n \geq 1, \quad \tilde{X}_0^u = u, \tag{3.1}$$

where $(A_n, B_n)_{n\geq 1}$ is an i.i.d. sequence in $\mathbb{R}^2$, For the following result see [18], Prop. 7.1.

Lemma 3.2. *Suppose that* $\mathbf{E}|A_1|^\delta < 1$ *and* $\mathbf{E}|B_1|^\delta < \infty$ *for some* $\delta \in]0,1[$. *Then for any* $u \in \mathbb{R}$ *the sequence* $\tilde{X}_n^u$ *converges in* L^δ *(hence, in probability) to the random variable*

$$\tilde{X}_\infty^0 = \sum_{k=1}^{\infty} B_k \prod_{j=1}^{k-1} A_j$$

and for any bounded uniformly continuous function f

$$\frac{1}{n}\sum_{k=1}^{n} f(\tilde{X}_k^u) \to \mathbf{E} f(\tilde{X}_\infty^0) \quad \text{in probability as } n \to \infty. \tag{3.2}$$

Corollary 3.3. *Suppose that* $\mathbf{E}|A_1|^\delta < 1$ *and* $\mathbf{E}|B_1|^\delta < \infty$ *for some* $\delta \in]0,1[$.
(i) *If* $\mathbf{P}(\tilde{X}_\infty^0 < 0) > 0$*, then* $\inf_{k\geq 1} \tilde{X}_k^u < 0$.
(ii) *If* $A_1 > 0$ *and* B_1/A_1 *is unbounded from below, then* $\inf_{k\geq 1} \tilde{X}_k^u < 0$.

Proof. (i) Let $f(x) := -I_{\{x<-1\}} + xI_{\{-1\leq x<0\}}$. Then $\mathbf{E} f(\tilde{X}_\infty^0) < 0$ and (3.2) may hold only if $\inf_{k\geq 1} \tilde{X}_k^u < 0$.

(ii) Put $\tilde{X}_\infty^{0,1} := \sum_{n=2}^{\infty} B_n \prod_{j=2}^{n-1} A_j$. Then

$$\tilde{X}_\infty^0 = B_1 + A_1\tilde{X}_\infty^{0,1} = A_1(\tilde{X}_\infty^{0,1} + B_1/A_1).$$

Since B_1/A_1 and $\tilde{X}_\infty^{0,1}$ are independent random variables and B_1/A_1 is unbounded from below, so is the sum $\tilde{X}_\infty^{0,1} + B_1/A_1$. It follows that the probability $\mathbf{P}(\tilde{X}_\infty^0 < 0) > 0$ and we can use (i). □

Proof of Theorem 3.1. Let $\tilde{X}_n = \tilde{X}_n^u := X_{T_n}^u$ where X^u is the process given by the formulae (2.3) and (2.4) assuming that N is a renewal process. In this case

$$X_{T_n}^u = e^{\eta_{T_n}} u + \sum_{k=1}^{n} e^{\eta_{T_n} - \eta_{T_k}} \left(c \int_{T_{k-1}}^{T_k} e^{\eta_{T_k} - \eta_v} dv + \xi_k \right)$$

and

$$A_k := e^{\eta_{T_k} - \eta_{T_{k-1}}}, \qquad B_k := \xi_k + c\int_{T_{k-1}}^{T_k} e^{\eta_{T_k} - \eta_v} dv. \tag{3.3}$$

Since the Wiener process W and P (a compound renewal process with drift) are independent, $(A_k, B_k)_{k\geq 1}$ is an i.i.d. sequence.

According to the formulae (2.3) and (2.4)

$$\tilde{X}_n = \mathcal{E}_n u + \sum_{k=1}^{n} B_k \frac{\mathcal{E}_n}{\mathcal{E}_k}, \qquad \mathcal{E}_n := \prod_{j=1}^{n} A_j. \tag{3.4}$$

Clearly, $\tilde{X}_n = A_n \tilde{X}_{n-1} + B_n$, that is, $\tilde{X}_n$ satisfies (3.1).

Lemma 3.4. *Suppose that $c < 0$ or $\mathbf{P}(\xi_1 < 0) > 0$. Then the ratio*

$$\frac{B_1}{A_1} = e^{-\eta_{T_1}} \xi_1 + c \int_0^{T_1} e^{-\eta_s} ds$$

is unbounded from below on the set $\{W_{T_1} < 0\}$.

Proof. Recall that the conditional law of the the Wiener process W on $[0, t]$ given $W_t = x$ is the same as the Brownian bridge $[0, t]$ ending at x, i.e. coinciding with the law of the process $(W_s - (s/t)W_t + (s/t)x)_{s \le t}$.

Fix $N > 0$. Let $c < 0$. Then

$$\mathbf{P}(B_1/A_1 \le -N | (T_1, W_{T_1}) = (t, x)) = \mathbf{P}\left(e^{-\kappa t - \sigma x} \xi_1 + c \zeta_t^x \le -N\right) \tag{3.5}$$

where the random variable

$$\zeta_t^x := \int_0^t e^{-\kappa s - \sigma(W_s - (s/t)W_t + (s/t)x)} ds$$

is independent on W. The law of ζ_t^x charges every open interval in $\mathbb{R}_+$. Thus, the right-hand side of (3.5) is strictly positive for every $(t, x) \in]0, \infty[\times \mathbb{R}$.

Let $\mathbf{P}(\xi_1 < 0) > 0$. Take $\varepsilon > 0$ and $T > 0$ such that the probabilities $\mathbf{P}(\xi_1 \le -\varepsilon)$ and $P(T_1 \le T)$ are strictly positive. Then

$$\mathbf{P}(B_1/A_1 \le -N, \xi \le -\varepsilon | (T_1, W_{T_1}) = (t, x)) \ge \mathbf{P}\left(-\varepsilon e^{-\kappa T - \sigma x} + c \zeta_t^x \le -N\right) > 0$$

for all sufficiently small x, namely, satisfying the inequality $\varepsilon e^{-\kappa T - \sigma x} > N$. Since the law of (T_1, W_{T_1}) charges the set $[0, T] \times] - \infty, y]$ whatever is $y \in \mathbb{R}$, the result follows. $\square$

Lemma 3.5. *Let $\beta \le 0$. If $\mathbf{E}|\xi|^\varepsilon < \infty$ and $\mathbf{E}e^{\varepsilon T_1} < \infty$ for some $\varepsilon > 0$, then $\mathbf{E}|B_1|^\delta < \infty$ for some $\delta \in]0, 1[$.*

Proof. For any $\delta \in]0,1[$ we have $\eta_{T_1} - \eta_s \le \sigma(W_{T_1} - W_s)$ and, hence,

$$\mathbf{E}|B_1|^\delta \le \mathbf{E}|\xi_1|^\delta + \mathbf{E}\left(\int_0^{T_1} e^{\eta_{T_1}-\eta_s} ds\right)^\delta \le \mathbf{E}|\xi_1|^\delta + \frac{1}{|\kappa|}\mathbf{E}\sup_{s\le T_1} e^{\delta\sigma W_s}.$$

Let $m(dt)$ denote the distribution of T_1. Substituting the density of distribution of the running maximum of the Wiener process we get that

$$\begin{aligned}\mathbf{E}\sup_{s\le T_1} e^{\delta\sigma W_s} &= \int_0^\infty \int_0^t \sqrt{\frac{2}{\pi t}} e^{\delta\sigma x} e^{-x^2/(2t)} dx m(dt) \\ &= \int_0^\infty e^{(1/2)\delta^2\sigma^2 t} \int_0^{\sqrt{t}} \sqrt{\frac{2}{\pi}} e^{-(y-\delta\sigma\sqrt{t})^2/2} dy m(dt) \le 2\mathbf{E}e^{(1/2)\delta^2\sigma^2 T_1}.\end{aligned}$$

Hence, $\mathbf{E}|B_1|^\delta < \infty$ for all sufficiently small $\delta > 0$. □

In the particular case where $\beta < 0$ we can get the result by direct reference to Corollary 3.3(ii) because

$$\mathbf{E}A_1^{-\beta} = \mathbf{E}e^{-\beta\eta_{T_1}} = \int_0^\infty \mathbf{E}e^{-\beta\sigma W_t - (1/2)(\beta\sigma)^2 t} m(dt) = 1$$

and, therefore, $\mathbf{E}A_1^\delta < 1$ for any $\delta \in]0, -\beta[$.

To cover the general case, we consider a suitably chosen random subsequence of $\tilde{X}_n$ which satisfies a linear difference equation with needed properties.

Let $\widehat{X}_n = \widehat{X}_n^u := \tilde{X}_{\theta^n}$ where $\theta_n := \inf\{k > \theta_{n-1} \colon \mathcal{E}_k < \mathcal{E}_{\theta_{n-1}}\}$. Thus, θ_n are ladder times for the random walk $M_k = \ln \mathbf{E}_k$. If $\beta = 0$, then $M_1 = \sigma W_{T_1}$ and $\mathbf{E}M_1 = 0$, $\mathbf{E}M_1^2 = \mathbf{E}T_1 < \infty$. Therefore, there is a finite constant C such that

$$\mathbf{P}(\theta_1 > n) \le Cn^{-1/2}, \tag{3.6}$$

see Theorem 1a in Ch. XII.7 of Feller's book [5] and the remark before it. In the general case, $M_1 = \beta\sigma^2 T_1 + \sigma W_{T_1}$ and the above bound holds also when $\beta < 0$.

In particular, $\theta_n < \infty$ and the differences $\theta_n - \theta_{n-1}$ form a sequence of finite independent random variables distributed as θ_1. The discrete-time process

$$\widehat{X}_n^u = \mathcal{E}_{\theta_n} u + \sum_{k=1}^{\theta_n} B_k \frac{\mathcal{E}_{\theta_n}}{\mathcal{E}_k}$$

solves the linear equation

$$\widehat{X}_n^u = \widehat{A}_n \widehat{X}_{n-1}^u + \widehat{B}_n, \qquad n \geq 1, \quad \widehat{X}_0^u = u,$$

where

$$\widehat{A}_n := \frac{\mathcal{E}_{\theta_n}}{\mathcal{E}_{\theta_{n-1}}}, \qquad \widehat{B}_n := \sum_{k=\theta_{n-1}+1}^{\theta_n} B_k \frac{\mathcal{E}_{\theta_n}}{\mathcal{E}_k}.$$

By construction, $\widehat{A}_1 < 1$ and

$$|\widehat{B}_1| \leq \sum_{k=\theta_{n-1}+1}^{\theta_n} |B_k| \frac{\mathcal{E}_{\theta_n}}{\mathcal{E}_k} \leq \sum_{j=1}^{\theta_1} |B_j|.$$

According to Lemma 3.5 $\mathbf{E}|B_1|^\delta < \infty$ for some $\delta \in]0,1[$. Taking $r \in]0, \delta/5[$ and defining the sequence $l_n := [n^{4r}]$, we have, using the Chebyshev inequality and (3.6), that

$$\begin{aligned}
\mathbf{E}|\widehat{B}_1|^r &\leq 1 + r \sum_{n\geq 1} n^{r-1} \mathbf{P}\left(\sum_{j=1}^{\theta_1} |B_j| > n \right) \\
&\leq 1 + r \sum_{n\geq 1} n^{r-1}\, \mathbf{P}\left(\sum_{j=1}^{l_n} |B_j| > n \right) + r \sum_{n\geq 1} n^{r-1} \mathbf{P}(\theta_1 > l_n) \\
&\leq 1 + r\mathbf{E}|Q_1|^\delta \sum_{n\geq 1} l_n n^{r-1-\delta} + rC \sum_{n\geq 1} n^{r-1} l_n^{-1/2} < \infty.
\end{aligned}$$

To apply Corollary 3.3(ii) it remains to check that the random variable $\widehat{B}_1/\widehat{A}_1$ is unbounded from below. But Lemma 3.4 asserts that this ratio coinciding with B_1/A_1 on the set $\{W_{T_1} < 0\}$ of strictly positive probability is unbounded from below on this set. □

4. Lower asymptotic bound

The next result we are needed at our asymptotic analysis indicates that the ruin probability decreases at infinity not faster than a certain power function. The proof given below covers also the more general case where P is a compound renewal process with drift given by the representation (2.4) where N is a counting renewal process.

Proposition 4.1. *Suppose that $c < 0$ or the random variable ξ_1 is unbounded from below. Then there exists $\beta_* > 0$ such that*

$$\liminf_{u\to\infty} u^{\beta_*}\, \Psi(u) \;>\; 0\,. \tag{4.1}$$

Proof. Let $\tilde X_n = \tilde X_n^u := X_{T_n}^u$ and let $\theta^u := \inf\{n:\ \tilde X_n^u \le 0\}$. If $c<0$ then the ruin may happen between jump times but in all cases,

$$\Psi(u) := \mathbf{P}(\tau^u < \infty) \ge \mathbf{P}(\theta^u < \infty).$$

Recall that for $(\tilde X_n)$ we have the formulae (3.4) and (3.1) with (A_k, B_k) defined by (3.3).

For reals $\varrho \in]0,1[$ and $b > 1/(\varrho^2(1-\varrho))$ we define the sets

$$\Gamma_k := \{A_k \le \varrho\} \cap \{B_k \le \varrho^{-1}\}\,, \quad D_k := \{A_k \le \varrho^{-1}\} \cap \{B_k \le -b\}. \tag{4.2}$$

Note that $\mathbf{P}(\Gamma_k) = \mathbf{P}(\Gamma_1)$ and $\mathbf{P}(D_k) = \mathbf{P}(D_1)$ for all k.

Lemma 4.2. *If there are ϱ and b such that $\mathbf{P}(\Gamma_1) > 0$ and $\mathbf{P}(D_1) > 0$, then (4.1) holds.*

Proof. Using (3.4) we easily get that on the set $\cap_{k=1}^n \Gamma_k$

$$\tilde X_n \le u\varrho^n + \frac{1}{\varrho(1-\varrho)}.$$

From the representation $\tilde X_{n+1} = A_{n+1}\tilde X_n + A_{n+1}$ and the above bound we infer that on the set $\big(\cap_{k=1}^n\ \Gamma_k\big) \cap D_{n+1}$

$$\tilde X_{n+1} \le u\varrho^{n-1} + \frac{1}{\varrho^2(1-\varrho)} - b = u\varrho^{n-1} - b_1.$$

where $b_1 := b - 1/(\varrho^2(1-\varrho)) > 0$.

Let $u > b^1$ and let $n = n(u) := 2 + [(1/\ln\varrho)\ln(b_1/u)]$ where $[x]$ means the integer part, $x - 1 < [x] \le x$. Then

$$u\varrho^{n-1} \le ue^{(n-1)\ln\varrho} < ue^{\ln(b_1/u)} < b_1$$

and, therefore,

$$\mathbf{P}(\theta^u < \infty) \ge \mathbf{P}\left(\cap_{k=1}^n \Gamma_k \cap D_{n+1}\right) = \left(\mathbf{P}(\Gamma_1)\right)^n \mathbf{P}(D_1).$$

Take $\beta_* := \frac{\ln \mathbf{P}(\Gamma_1)}{\ln \varrho}$. Then

$$u^{\beta_*}\mathbf{P}(\theta^u < \infty) = e^{\frac{\ln \mathbf{P}(\Gamma_1)}{\ln \varrho}\ln u + n \ln \mathbf{P}(\Gamma_1)}\mathbf{P}(D_1) \geq e^{\left(2+\frac{\ln b_1}{\ln \varrho}\right)\ln \mathbf{P}(\Gamma_1)}\mathbf{P}(D_1).$$

The lemma is proven. □

It remains to show that our assumptions ensure that for any $\varrho, b > 0$ the probability $\mathbf{P}(A_1 \leq \varrho,\ B_1 \leq -b)$ is strictly positive. We use the fact that the conditional distribution of the Wiener process $(W_s)_{s\leq t}$ given $W_t = x$ is the (unconditional) distribution of the Brownian bridge $(B^x_s)_{s\leq T}$ which is the same the distribution of the process $(W_s - (s/t)W_t)_{s\leq t}$. Take $v_1, v_2 > 0$ and x_0 such that $\mathbf{P}(T_1 \in [v_1, v_2]) > 0$ and $|\kappa|v_2 + \sigma x_0 \leq \ln \varrho$. Then for any $(t, x) \in \Delta := [v_1, v_2]\times] - \infty, x_0]$ we have that

$$\mathbf{P}(A_1 \leq \varrho,\ B_1 \leq -b \mid (T_1, W_{T_1}) = (t, x)) = \mathbf{P}(\xi_1 + c\Upsilon^x_t \leq -b) \tag{4.3}$$

where

$$\Upsilon^x_t := \int_0^t e^{\kappa(t-s)+\sigma x-\sigma(W_s-(s/t)W_t)}ds.$$

Since the distribution of Wiener process has a full support, i.e. charges any open set in the space $C_0([0, T])$ of the trajectories, the random variable Υ^x_t is unbounded from above (whatever are x and $t > 0$). Any of our assumptions implies that $\xi_1 + c\Upsilon^x_t$ is unbounded from below, that is the probability in the right-hand side of (4.3) is strictly positive. Integrating (4.3) over Δ with respect to the distribution of (T_1, W_{T_1}) charging Δ we get the result. □

5. Regularity of the ruin probability

Following tradition (seemingly, justified by notational convenience) we shall work with the survival probability $\Phi = 1 - \Psi$ which has the same regularity property and satisfies the same equations.

Theorem 5.1. *Suppose that $F_k(dx) = f_k(x)dx$ where the densities f_k are two times differentiable on $]0, \infty[$ and $f'_k, f''_k \in L^1(\mathbb{R}_+)$, $k = 1, 2$. Then Φ is two times continuously differentiable on $]0, \infty[$ and Φ', Φ'' are bounded.*

Proof. Define the continuous process

$$Y^u_t := S_t\left(u + c\int_{[0,t]} S^{-1}_s ds\right) \tag{5.1}$$

coinciding with X^u on $[0, T_1[$ and introduce the stopping time

$$\theta^u := \inf\{t \geq 0 \colon\ Y_t^u \leq 0\}.$$

By virtue of the strong Markov property of X^u

$$\Phi(u) = \mathbf{E}\Phi(X^u_{\theta^u \wedge T_1}) = \mathbf{E}\Phi(Y^u_{\theta^u \wedge T_1} + \Delta X^u_{\theta^u \wedge T_1}). \tag{5.2}$$

Due to independence of W and the Poisson processes N^1, N^2, the values of $\theta^u(\omega)$, $T_1^1(\omega)$ and $T_1^2(\omega)$ are all different for almost all ω. Since $\Phi(X^u_{\theta^u})I_{\{\theta^u < T_1\}} = 0$ we have the representation $\Phi = K^{\downarrow} + K^{\uparrow}$ where

$$K^{\downarrow}(u) := \mathbf{E}I_{\{Y^u_{T_1^1} > 0\}} I_{\{T_1^1 < T_1^2\}} \Phi(Y^u_{T_1^1} - \xi_1^1), \quad K^{\uparrow}(u) := \mathbf{E}I_{\{Y^u_{T_1^2} > 0\}} I_{\{T_1^1 > T_1^2\}} \Phi(Y^u_{T_1^2} + \xi_1^2).$$

The analysis of the smoothness of these two functions is similar, so we consider the first one. It is convenient to represent it as the sum $K^{\downarrow} = K_1^{\downarrow} + K_2^{\downarrow}$ where

$$K_1^{\downarrow}(u) \ := \ \int_{\mathbf{R}^3} I_{\{0 < s < t \wedge 2\}} \mathbf{E}G(Y_s^u, w) m(ds, dt)\, F_1(dw), \tag{5.3}$$

$$K_2^{\downarrow}(u) \ := \ \int_{\mathbf{R}^3} I_{\{2 < s < t\}} \mathbf{E}G(Y_s^u, w) m(ds, dt)\, F_1(dw), \tag{5.4}$$

with $G(y, w) := I_{\{y > 0\}}\Phi\,(y - w) = I_{\{y - w > 0\}}\Phi\,(y - w) = \Phi\,(y - w)$ for $w \geq 0$ and

$$m(ds, dt) := \alpha_1 \alpha_2 e^{-(\alpha_1 s + \alpha_2 t)} ds\, dt.$$

Lemma 5.2. *For any bounded measurable function $G(y, w)$ the function $K_2^{\downarrow}(u)$ defined (5.4) belongs to $C^\infty(]0, \infty[)$ and has bounded derivatives of any order.*

Proof. Using the representation

$$Y_s^u = e^{\eta_s - \eta_1} Y_1^u + \int_{[1,s]} e^{\eta_s - \eta_r} dr, \qquad s \geq 2,$$

and noticing that the random variable Y_1^u is independent of the process $(\eta_s - \eta_1)_{s \geq 1}$, we get that

$$\mathbf{E}(G(Y_s^u, w) | Y_1^u) = G(s, Y_1^u, s, w)$$

where

$$G(s,y,w) := \mathbf{E}G\left(e^{\eta_s-\eta_1}y + \int_{[1,s]} e^{\eta_s-\eta_r}dr, w\right).$$

Substituting the expression for Y_1^u we obtain the formula

$$\mathbf{E}G(Y_s^u,w) = \mathbf{E}G(s,Y_1^u,w) = \mathbf{E}G(s,e^{\kappa+\sigma W_1}(u+cR_1),w)$$

with

$$R_1 := \int_{[0,1]} e^{-\kappa r-\sigma W_r}dr.$$

Recall that the conditional distribution of the process $(W_r)_{r\le 1}$ given $W_1 = x$, i.e. the unconditional distribution of the Brownian bridge B^x with $B_0^x = 0$ and $B_1^x = x$, which is the same as of the process $W_r + r(x-W_1)$, $r \le 1$. It follows that

$$\mathbf{E}G(Y_s^u,w) = \int_{\mathbf{R}} \mathbf{E}G\big(s,e^{\kappa+\sigma x}(u+\zeta^x),w\big)\varphi_{0,1}(x)dx$$

where the random variable

$$\zeta^x := c\int_{[0,1]} e^{-\kappa r-\sigma(W_r+r(x-W_1))}dr.$$

The case $c=0$ is easy. By the change of variable $z = \kappa+\sigma x+\ln u$ we get that

$$\mathbf{E}G(Y_s^u,w) = \frac{1}{\sigma}\int_{\mathbf{R}} G(s,e^z,w)\varphi_{0,1}((z-\kappa-\ln u)/\sigma)dz.$$

The function $u \mapsto \varphi_{0,1}((z-\kappa-\ln u)/\sigma)$ belongs to $C^\infty(]0,\infty[)$ and its derivatives on any interval $[u_1,u_2] \subset]0,\infty[$ are dominated by integrable functions. It follows that the function $u \mapsto \mathbf{E}G(Y_s^u,w)$ belongs to $C^\infty(]0,\infty[)$ and its derivatives are locally bounded.

Let $c \neq 0$. Lemma 5.3 below asserts that for every x the random variable ζ^x has a density $\rho(x,.)$ so that

$$\mathbf{E}G(Y_s^u,w) = \int_{\mathbf{R}^2} G\big(s,e^{\kappa+\sigma x}z,w\big)\rho(x,z-u)\varphi_{0,1}(x)dxdz.$$

Since this density belongs to C^∞ and its derivatives are of sub exponential growth in x the function $y \mapsto \mathbf{E}G(Y_s^u,w)$ also belongs to C^∞ and has bounded derivatives. So, the same property has the function $u \mapsto K_2^{\downarrow}(u)$ and the lemma is proven.

Lemma 5.3 (see [9], Lemma 5.2)**.** *The random variable ζ^x has a density $\rho(x,.) \in C^\infty$ such that for any $n \geq 1$*

$$\sup_{y\geq 0}\left|\frac{\partial^n}{\partial y^n}\rho(x,y)\right| \leq C_n e^{C_n|x|} \tag{5.5}$$

with some constant C_n and $(\partial^n/\partial y^n)\rho(x,0) = 0$.

The needed smoothness property of $K_1^{\downarrow}$ follows from the following lemma.

Lemma 5.4. *Let $\xi > 0$ be a random variable with a density f which is two times differentiable on $]0,\infty[$ and $f', f'' \in L^1(\mathbb{R}_+)$. Let $G : \mathbf{R} \to [0,1]$ be a measurable function vanishing on $]-\infty,0]$ and let $h(y) := \mathbf{E}G(y-\xi)$. Then the function $(s,u) \mapsto \mathbf{E}h(Y_s^u)$ has two continuous derivatives in u bounded on $[0,t]\times]0,\infty]$.*

Proof. First, we observe that

$$h(y) = \int_{\mathbf{R}} G(y-x)f(x)dx = \int_{\mathbf{R}} G(z)f(y-z)dz$$

and

$$h'(y) = \int_{\mathbf{R}} G(z)f'(y-z)dz.$$

It follows that $|h'(y)| \leq ||f'||_{L^1}$ and $|h''(y)| \leq ||f''||_{L^1}$.

Using the representation

$$Y_s^u = e^{\eta_s}u + \int_{[0,s]} e^{\eta_s-\eta_r}ds$$

and arguing in the same spirit as above but conditioning this time on the random variable $W_s \sim \mathcal{N}(0,s)$ and considering the Brownian bridge on $[0,s]$ we obtain that

$$\mathbf{E}G(Y_s^u - \xi) = \frac{1}{\sqrt{s}}\int_{\mathbf{R}} \mathbf{E}h(e^{\kappa s+\sigma x}(u+\zeta^{s,x}))\phi_{0,1}(x/\sqrt{s})dx$$

where

$$\zeta^{s,x} := c\int_{[0,s]} e^{-(\sigma rx/s+\kappa r+\sigma(W_r-(r/s)W_s)}dr.$$

If $c = 0$, then

$$\mathbf{E}G(Y_s^u - \xi) = \int_{\mathbf{R}} h(e^{\kappa s + \sigma\sqrt{s}x} u)\phi_{0,1}(x)dx$$

and the needed property is obvious.

It is easily seen that the random variable $\zeta^{s,x}$ has a C^∞ density (the same as of ζ^x but with the parameters cs, κs, and $\sigma s^{1/2}$). Unfortunately, derivatives of this density have non-integrable singularities at $s = 0$. By this reason arguments used above do not work.

Let $c \neq 0$. Then the smooth function $x \to \zeta^{s,x}$ is strictly decreasing and maps $\mathbb{R}$ onto $\mathbb{R}_+$ (when $c > 0$) or strictly increasing and maps $\mathbb{R}$ onto $\mathbb{R}_-$) (when $c < 0$). Let denote $z(s, .)$ its inverse which is a function decreasing from $+\infty$ to $-\infty$ (when $c > 0$) and increasing from $-\infty$ to $+\infty$ (when $c < 0$). The partial derivative in x is given by the formula

$$z_x(s, x) = -\frac{s}{L(s, z(s, x))}, \tag{5.6}$$

where

$$L(s, x) = c\sigma \int_{[0,s]} r e^{-(\sigma r x/s + \kappa r + \sigma(W_r - (r/s)W_s)} dr. \tag{5.7}$$

In both cases $z_{xx}(s, x) > 0$ for $s > 0$.

Changing the variable, we obtain that $\mathbf{E}h(Y_s^u) = \mathbf{E}H(s, u)$ where

$$H(s, u) := \frac{1}{\sqrt{s}} \int_0^\infty h(e^{\kappa s + \sigma z(s,x)}(u + x))\varphi_{0,1}\left(z(s, x)/\sqrt{s}\right) z_x(s, x)dx, \qquad c > 0, \tag{5.8}$$

For $s \in]0, 2]$

$$\begin{aligned} \frac{1}{\sqrt{s}} \int_0^\infty e^{\sigma z(s,x)} \varphi_{0,1}\left(z(s, x)/\sqrt{s}\right) z_x(s, x)dx &= \int e^{\sigma\sqrt{s}z} \varphi_{0,1}(z)dz \\ &\leq \int e^{\sigma\sqrt{2}z} \varphi_{0,1}(z)dz \end{aligned}$$

where the last integral is a finite constant. This estimate and the boundedness of h' legitimate the differentiation under the sign of the integrals. In the case $c > 0$ we have the formula

$$H_u(s, u) = \frac{1}{\sqrt{s}} \int_0^\infty e^{\kappa s + \sigma z(s,x)} h'(e^{\kappa s + \sigma z(s,x)}(u + x))\varphi_{0,1}\left(z(s, x)/\sqrt{s}\right) z_x(s, x)dx.$$

and the bound $|H_u(s,u)| \le C$ for all $s \in]0,2]$.

Repeating the arguments we get that

$$H_{uu}(s,u) = \frac{1}{\sqrt{s}} \int_0^\infty e^{2\kappa s + 2\sigma z(s,x)} h'(e^{\kappa s+\sigma z(s,x)}(u+x))\varphi_{0,1}\left(z(s,x)/\sqrt{s}\right) z_x(s,x)dx.$$

$|H_u(s,u)| \le C$ for all $s \in]0,2]$.

Similar arguments are applied in the case $c < 0$. $\square$

Theorem 5.1 is proven. $\square$

Remark. Let $V : \mathbb{R} \to [0,1]$ be a measurable function, $V(x) = 0$ for $x \le 0$. Put

$$\Psi_V(u) := EV(X^u_{\tau^u}) I_{\{\tau^u < \infty\}}).$$

Then the statement of Theorem 5.1 holds for Ψ_V with the same proof. Indeed, the strong Markov property for Ψ_V has the same form as for Φ in (5.2) that is $\Psi_V(u) = \mathbf{E}\Phi_V(X^u_{\theta^u \wedge T_1})$. Also Proposition 6.1 below holds for Ψ_V. In the particular case where $V(x) = 1$ for all $x < 0$, the function Ψ_V coincides with Ψ on $]0,\infty[$ but they are different on $\mathbb{R}_-$.

6. The integro-differential equation for the survival probability

Proposition 6.1. *Suppose that* $\Psi \in C^2$. *Then the function* Φ *on* $]0,\infty[$ *satisfies the following equation:*

$$\frac{1}{2}\sigma^2 u^2 \Phi''(u) + (au+c)\Phi'(u) + \int (\Phi(u+y) - \Phi(u))\Pi_P(dy) = 0. \qquad (6.1)$$

Proof. For $h > 0$ and $\epsilon > 0$ small enough to ensure that $u \in]\epsilon, \epsilon^{-1}[$ we put

$$\tau_h^\epsilon := \inf\left\{t \ge 0\colon\ X^u_t \notin [\epsilon\,, \epsilon^{-1}]\right\} \wedge h \wedge T_1.$$

Let $\mathcal{L}^0\Phi(u) := (1/2)\sigma^2 u^2 \Phi''(u) + (au+c)\Phi'(u)$. By the Itô formula

$$\begin{aligned}\Phi(X^u_{\tau_h^\epsilon}) \;&=\; \Phi(u) + \sigma \int_0^{\tau_h^\epsilon} X^u_s \Phi'(X^u_s)\, dW_s + \int_0^{\tau_h^\epsilon} \mathcal{L}\Phi(X^u_s) ds \\ &\quad + \int_0^{\tau_h^\epsilon} \int (\Phi(X^u_{s-} + x) - \Phi(X^u_{s-})) p_P(ds,dx).\end{aligned}$$

Due to the strong Markov property $\Phi(u) = \mathbf{E}\,\Phi(X^u_{\tau_h^\epsilon})$ (since $\Phi(u) = 0$ for $u \le 0$). For every $\epsilon > 0$ the integrands above are bounded by constants and, hence, the expectation of the stochastic integral with respect to the Wiener process is zero. The expectation of the integral with respect to then integer-valued measure $p_P(ds,dx)$ is equal to the integral with respect to the compensator of the latter, that is to

$$\mathbf{E}\int_0^{\tau_h^\epsilon}\int(\Phi(X^u_{s-}+x)-\Phi(X^u_{s-}))ds\Pi_P(dx).$$

Moreover, $\tau_h^\epsilon = h$ when h is sufficiently small (the threshold below which the equality holds, of course, depends on ω).

It follows that, independently of ϵ,

$$\frac{1}{h}\mathbf{E}\int_0^{\tau_h^\epsilon}\Big(\frac{1}{2}\sigma^2(X^u_s)^2\Phi''(X^u_s)+(aX^u_s+c)\Phi'(X^u_s)\Big)ds\to\mathcal{L}\Phi(u)$$

as $h\to 0$. Finally,

$$\frac{1}{h}\mathbf{E}\int_0^{\tau_h^\epsilon}\int(\Phi(X^u_{s-}+x)-\Phi(X^u_{s-}))ds\Pi_P(dx)\to\int(\Phi(u+y)-\Phi(u))\Pi_P(dy).$$

It follows that Φ satisfies the equation (6.1). □

Remark. The equation (6.1) holds in the viscosity sense, i.e. without additional assumptions on smoothness of Ψ, see [1].

7. Exponentially distributed jumps: from IDE to ODE

In the case of exponentially distributed jumps the integro-differential equation can be written as

$$\mathcal{L}\Phi(u)+\frac{\alpha_1}{\mu_1}\int_0^\infty\Phi(u-y)e^{-y/\mu_1}dy+\frac{\alpha_2}{\mu_2}\int_0^\infty\Phi(u+y)e^{-y/\mu_2}dy=0. \quad (7.1)$$

where

$$\mathcal{L}\Phi(u):=\frac{1}{2}\sigma^2u^2\Phi''(u)+(au+c)\Phi'(u)-(\alpha_1+\alpha_2)\Phi(u).$$

Changing variables in the integrals we get that

$$\mathcal{L}\Phi(u)+\frac{\alpha_1}{\mu_1}I_1(u)+\frac{\alpha_2}{\mu_2}I_2(u)=0 \quad (7.2)$$

where

$$I_1(u) := \int_{-\infty}^{u} \Phi(z)e^{-(u-z)/\mu_1}dz, \qquad I_2(u) := \int_{u}^{\infty} \Phi(z)e^{-(z-u)/\mu_1}dz.$$

Note that $I_1' = \Phi - (1/\mu_1)I_1$ and $I_2' = -\Phi + (1/\mu_2)I_2$.

Put $\mathcal{T}f := \mu_1\mu_2 f'' + (\mu_2 - \mu_1)f' - f$. It is easily seen that

$$\mathcal{T}I_1 = \mu_1\mu_2\Phi' - \mu_1\Phi, \qquad \mathcal{T}I_2 = -\mu_1\mu_2\Phi' - \mu_2\Phi.$$

Applying the operator $\mathcal{T}$ to both sides of the equation (7.2) we get that the survival probability Φ (as well as the ruin probability $\Psi := 1 - \Phi$) solves the differential equation $\mathcal{D}\Phi = 0$ where $\mathcal{D}$ is the differential operator of the 4th order:

$$\begin{aligned}
\mathcal{D}\Phi &:= \mathcal{T}\mathcal{L}\Phi + (\alpha_1\mu_2 - \alpha_2\mu_1)\Phi' - (\alpha_1 + \alpha_2)\Phi \\
&= \mu_1\mu_2(\mathcal{L}\Phi)'' + (\mu_2 - \mu_1)(\mathcal{L}\Phi)' - \mathcal{L}\Phi + (\alpha_1\mu_2 - \alpha_2\mu_1)\Phi' - (\alpha_1 + \alpha_2)\Phi
\end{aligned}$$

After simple calculation we get that that the obtained 4th order equation in fact is the following third order differential equation for $G := \Phi'$:

$$\tilde{g}_3(u)G''' + \tilde{g}_2(u)G'' + \tilde{g}_1(u)G' + \tilde{g}_0(u)G = 0$$

where the coefficients (depending on u) are:

$$\begin{aligned}
\tilde{g}_3(u) &:= \frac{1}{2}\sigma^2\mu^2u^2, \\
\tilde{g}_2(u) &:= \mu^2((a + 2\sigma^2)u + c) + \frac{1}{2}\Delta\mu\sigma^2u^2, \\
\tilde{g}_1(u) &:= \mu^2(2a + \sigma^2 - \alpha_1 - \alpha_2) + \Delta\mu(\sigma^2u + au + c) - \frac{1}{2}\sigma^2u^2, \\
\tilde{g}_0(u) &:= -au - c + \Delta\mu(a - \alpha_1 - \alpha_2) + (\alpha_1\mu_2 - \alpha_2\mu_1)
\end{aligned}$$

with the abbreviations $\Delta\mu := \mu_2 - \mu_1$, $\mu^2 := \mu_1\mu_2$. Since $\mu_1 > 0, \mu_2 > 0$ we get from here the equation with unit coefficient at the third derivative:

$$G''' + q_2(u)\ G'' + q_1(u)\ G' + q_0(u)\ G(u) = 0 \tag{7.3}$$

where

$$\begin{aligned}
q_2(u) &:= \frac{\Delta\mu}{\mu^2} + \frac{2(a + 2\sigma^2)}{\sigma^2}\frac{1}{u} + \frac{2c}{\sigma^2}\frac{1}{u^2}, \\
q_1(u) &:= -\frac{1}{\mu^2} + \frac{2(a + \sigma^2)\Delta\mu}{\sigma^2\mu^2}\frac{1}{u} + \frac{2(\Delta\mu\ c + \mu^2(2a + \sigma^2 - \alpha_1 - \alpha_2))}{\mu^2\sigma^2}\frac{1}{u^2}, \\
q_0(u) &:= -\frac{2a}{\mu^2\sigma^2}\frac{1}{u} + \frac{2(\Delta\mu(a - \alpha_1 - \alpha_2) + (\alpha_1\mu_2 - \alpha_2\mu_1) - c)}{\mu^2\sigma^2}\frac{1}{u^2}.
\end{aligned}$$

Let denote $\mathcal{J}$ the operator in the left-hand side of the basic IDE acting in the space of sufficiently smooth functions. Then $\mathcal{T}\mathcal{J} = \mathcal{D}$. Note that $\dim \operatorname{Ker} \mathcal{D} = 4$. Also, we have the inclusion $\operatorname{Ker} \mathcal{J} \subseteq \operatorname{Ker} \mathcal{D}$. The kernel of the 2nd order differential operator $\mathcal{T}$ is a 2-dimensional linear subspace generated by the functions $h_1(u) := e^{-u/\mu_1}$, $h_2(u) := e^{u/\mu_2}$. Let f_1 and f_2 be solutions of the equations $\mathcal{J} f_j = h_j$. Then f_1 and f_2 are linearly independent and belong to $\operatorname{Ker} \mathcal{D}$. The 4 functions: the identity, the survival probability Φ (assumed not to be a constant), f_1, f_2 form a basis in $\operatorname{Ker} \mathcal{D}$. Indeed, if their linear combination

$$a_1 f_1 + a_2 f_2 + a_3 1 + a_4 \Phi = 0,$$

then $a_1 \mathcal{J} f_1 + a_2 \mathcal{J} f_2 = 0$. That is, $a_1 h_1 + a_2 h_2 = 0$ and $a_1 = a_2 = 0$. But the equality $a_3 + a_4 \Phi = 0$ holds only if $a_3 = a_4 = 0$.

8. Asymptotic analysis of the differential equation for the survival probability

We analyze the behavior of solutions of the equation (7.3) using a result on systems with asymptotically constant coefficients. To this end, we put $y = (y_1, y_2, y_3) := (G, G', G'')$. Using the matrix notation where the vectors are columns we get from (7.3) that $y' = A(u)y$ where

$$A(u) := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -q_0(u) & -q_1(u) & -q_2(u) \end{bmatrix}, \qquad A(\infty) := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1/\mu^2 & -(\Delta\mu)/\mu^2 \end{bmatrix}.$$

Then $A(u) = A(\infty) + V(u)$ where the matrix $V(u) := A(u) - A(\infty)$ is such that its norm $|V'(u)|$ (the Euclidean or any other) is integrable on $[1, \infty[$.

Let $\lambda_j = \lambda_j(u)$, $j = 1, 2, 3$, be the roots of the characteristic equation

$$\lambda^3 + q_2(u)\ \lambda^2 + q_1(u)\ \lambda + q_0(u) = 0. \tag{8.1}$$

Note that

$$\lambda_1 + \lambda_2 + \lambda_3 = -q_2(u), \quad \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 = q_1(u), \quad \lambda_1\lambda_2\lambda_3 = -q_0(u). \tag{8.2}$$

Recall that we are working under the assumptions $a > \sigma^2/2 > 0$ and $\mu_1, \mu_2 > 0$. Then $q_0(u) \to 0$,

$$q_2(u) \to \frac{\Delta\mu}{\mu^2} \neq 0, \quad q_1(u) \to -\frac{1}{\mu^2} < 0, \quad u q_0(u) \to -\frac{2a}{\mu^2\sigma^2} < 0, \quad u \to \infty.$$

The Cardano formulae imply that the roots $\lambda_j(u)$ are continuous functions having finite limits as $u \to \infty$. According to the last equation in (8.2), at least one root, say, $\lambda_3(u)$ tends to zero as $u \to \infty$. Since $q_1(\infty) \neq 0$, two other roots have nonzero limits satisfying the system

$$\lambda_1(\infty) + \lambda_2(\infty) = -q_2(\infty), \qquad \lambda_1(\infty)\lambda_2(\infty) = q_1(\infty),$$

that is, $\lambda_1(\infty) = 1/\mu_2$, $\lambda_2(\infty) = -1/\mu_1$. Thus, we obtain that

$$\lambda_1(u) = \frac{1}{\mu_2} + o(1), \quad \lambda_2(u) = -\frac{1}{\mu_1} + o(1), \quad \lambda_3(u) = -\frac{2a}{\sigma^2}\frac{1}{u} + o(u^{-1}). \tag{8.3}$$

Applying the implicit function theorem we conclude that the function $\lambda_3(u)$ can be expanded in powers of u^{-1} In particular,

$$\lambda_3(u) = -\frac{2a}{\sigma^2}\frac{1}{u} + O(u^{-2}), \quad u \to \infty. \tag{8.4}$$

The numbers $\lambda_1(\infty) = 1/\mu_2$, $\lambda_2(\infty) = -1/\mu_1$, and $\lambda_3(\infty) = 0$ are eigenvalues of the matrix $A(\infty)$.

The conditions of Th. VII-5-3 from [8] are fulfilled and, therefore, the fundamental matrix of the equation $y' = A(u)y$ has the form

$$P_0(u)(I + H(u)) \exp\left\{\int^u \Lambda(s)ds\right\}$$

where the matrix-valued functions $P_0(u)$ and $H(u)$ are continuous, $P_0(u) \to Q$, $H(u) \to 0$ as $u \to \infty$, $\Lambda(s) := \operatorname{diag}(\lambda_1(s), \lambda_2(s), \lambda_3(s))$, the columns of Q are eigenvectors of $A(\infty)$ corresponding to the eigenvalues $\lambda_1(\infty)$, $\lambda_2(\infty)$, $\lambda_3(\infty)$, that is

$$Q = \begin{bmatrix} 1 & 1 & 1 \\ 1/\mu_2 & -1/\mu_1 & 0 \\ 1/\mu_2^2 & 1/\mu_1^2 & 0 \end{bmatrix}.$$

A general solution of solution G of (7.3) is a linear combination of functions

$$\begin{aligned}
h_1(u) &:= (1 + \theta_1(u)) \exp\left\{\int_1^u \left(\frac{1}{\mu_2} + \gamma_1(s)\right) ds\right\}, \\
h_2(u) &:= (1 + \theta_2(u)) \exp\left\{\int_1^u \left(-\frac{1}{\mu_1} + \gamma_2(s)\right) ds\right\}, \\
h_3(u) &:= (1 + \theta_3(u)) \exp\left\{-\frac{2a}{\sigma^2}\ln u\right\} \exp\left\{\int_1^u \gamma_3(s)ds\right\},
\end{aligned}$$

where continuous functions θ_j and γ_j vanish at infinity for $j = 1, 2, 3$ and the function γ_3 is integrable. Thus, the general solution of the 4th order differential equation is a linear combination of a constant function and these three functions above. In particular, the ruin probability Ψ is given by a certain linear combination. Clearly, the latter cannot involve the unbounded function corresponding to the integral of first function. The integrals of others are bounded and, therefore,

$$\Psi(u) = c_0 + c_2 H_2(u) + c_3 H_3(u)$$

where

$$H_j(u) := \int_u^\infty h_j(s)ds, \quad j = 2, 3.$$

Note that $c_0 = 0$, the integral $H_2(u)$ converges to zero exponentially fast, and

$$\frac{H_3(u)}{u^{1-2a/\sigma^2}} \to \frac{1}{2a/\sigma^2 - 1} \exp\left\{\int_1^\infty \gamma_3(s)ds\right\} \neq 0$$

as $u \to \infty$. in virtue of Proposition 4.1 $c_3 \neq 0$ and we easily obtain that $\Psi(u) \sim Cu^{-\beta}$ where $C > 0$ and $\beta = 2a/\sigma^2 - 1$.

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