FUZZY LOGIC

Introduction

- Fuzziness or vagueness can be found in many areas of daily life.
- This is more frequently seen in which human judgment, evaluation, and decisions are important.
- The word “fuzzy” means vagueness (ambiguity).
- Fuzziness occurs when the boundary of a piece of information is not clear-cut.
- Fuzzy sets - 1965 Lotfi Zadeh as an extension of classical notation set.
- Classical set theory allows the membership of the elements in the set in binary terms.
- Fuzzy set theory permits membership function valued in the interval [0,1]

Example:

Words like young, tall, good or high are fuzzy.

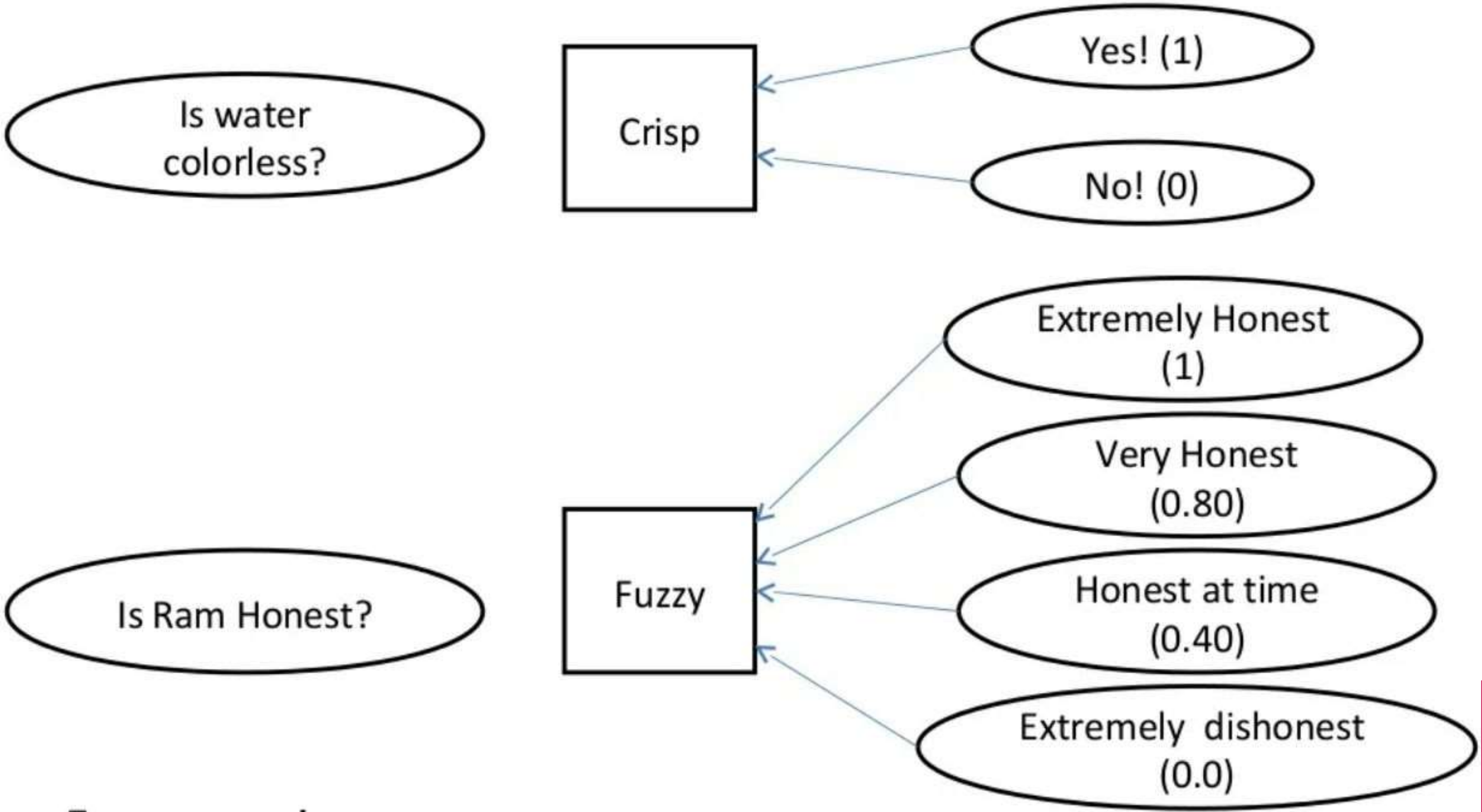
- There is no single quantitative value which defines the term young.
- For some people, age 25 is young, and for others, age 35 is young.
- The concept young has no clean boundary.
- Age 35 has some possibility of being young and usually depends on the context in which it is being considered.

Fuzzy set theory is an extension of classical set theory where elements have degree of membership.

- In real world, there exist much fuzzy knowledge (i.e. vague, uncertain inexact etc).
- Human **thinking** and **reasoning** (analysis, logic, interpretation) frequently involved **fuzzy** information.
- Human can give satisfactory answers, which are probably true.
- Our systems are unable to answer many question because the systems are designed based upon classical set theory (Unreliable and incomplete).
- We want, our system should be able to cope with unreliable and incomplete information.
- Fuzzy system have been provide solution.

Classical set theory	Fuzzy set theory
• Classes of objects with sharp boundaries.	• Classes of objects with un-sharp boundaries.
• A classical set is defined by crisp(exact) boundaries, i.e., there is no uncertainty about the location of the set boundaries.	• A fuzzy set is defined by its ambiguous boundaries, i.e., there exists uncertainty about the location of the set boundaries.
• Widely used in digital system design	• Used in fuzzy controllers.

Example



Fuzzy vs crips

Classical set theory

- A Set is any well defined collection of objects.
- An object in a set is called an element or member of that set.
- Sets are defined by a simple statement,
- Describing whether a particular element having a certain property belongs to that particular set.

$$A = \{a_1, a_2, a_3, \ldots\ldots, a_n\}$$

Classical sets are also called *crisp* (sets).

Lists: A = {apples, oranges, cherries, mangoes}

$A = \{a_1, a_2, a_3\}$

$A = \{2, 4, 6, 8, \ldots\}$

Formulas: $A = \{x \mid x \text{ is an even natural number}\}$

$A = \{x \mid x = 2n, n \text{ is a natural number}\}$

Membership or characteristic function

$$X_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$

Operations on classical set theory

Union: the union of two sets A and B is given as

$$A \cup B = \{ x \mid x \in A \text{ or } x \in B \}$$

Intersection: the intersection of two sets A and B is given as

$$A \cap B = \{ x \mid x \in A \text{ and } x \in B \}$$

Complement: It is denoted by Ã and is defined as

$$\tilde{A} = \{ x \mid x \text{ does not belongs } A \text{ and } x \in X \}$$

Fuzzy Sets

- Fuzzy sets theory is an extension of classical set theory.
- Elements have varying degree of membership. A logic based on two truth values,
- ***True*** and ***False*** is sometimes insufficient when describing human reasoning.
- Fuzzy Logic uses the whole interval between 0 (false) and 1 (true) to describe human reasoning.
- A Fuzzy Set is any set that allows its members to have different degree of membership, called **membership function**, having interval [0,1].

11.5.1 Definition

A **fuzzy set** is a pair (U, m) where U is a set (assumed to be non-empty) and $m : U \to [0, 1]$ a function. The set U is called *universe of discourse*. The function m is called the *membership function*. For each $x \in U$, the value $m(x)$ is called the *grade of membership* of x in (U, m).

The fuzzy set (U, m) is often denoted by a single letter A. In such a case, the membership function is denoted by μ_A.

Types of membership

Let $x \in U$. Then x is said to be

- *not included* in the fuzzy set (U, m) if $m(x) = 0$ (no member),
- *fully included* if $m(x) = 1$ (full member),
- *partially included* if $0 < m(x) < 1$ (fuzzy member).

Support, kernel, α-cut

Let $A = (U, m)$ be a fuzzy set. The support of A is the crisp set

$$\mathrm{Supp}\,(A) = \{x \in U : m(x) > 0\}.$$

The kernel of A is the crisp set

$$\mathrm{Kern}\,(A) = \{x \in : m(x) = 1\}.$$

Notation

Let $A = (U, m)$ be a fuzzy set and let U be a finite set. Let the support of A be Supp $(U) = \{x_1, x_2, \ldots, x_n\}$. Then the fuzzy set A is sometimes written as a set of ordered pairs as follows:

$$A = \{(x_1, m(x_1)), (x_2, m(x_2)), \cdots, (x_n, m(x_n))\}$$

The following notation

$$A = \{m(x_1)/x_1, m(x_2)/x_2, \ldots, m(x_n)/x_n\}$$

as well as the notation

$$A = m(x_1)/x_1 + m(x_2)/x_2 + \ldots + m(x_n)/x_n$$

are also used to denote the fuzzy set A. Still others write

$$A = \left\{ \frac{m(x_1)}{x_1}, \frac{m(x_2)}{x_2}, \ldots, \frac{m(x_n)}{x_n} \right\}.$$

Note that in these representations, the symbols "/" and "+" are just symbols and they do not represent the corresponding arithmetical operations.

Crisp set as a fuzzy set

Let A be a crisp subset of the universe of discourse U. A can be treated as a fuzzy set $A = (U, m)$ where the membership function m is defined as

$$m(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

The membership function of a crisp set is called the characteristic function of the set.

Example 1

Consider the set

$$U = \{5, 10, 15, 20, 30, 35, 40, 45, 60, 70\}$$

and the function $m : U \to [0, 1]$ defined by the following table:

x	5	10	15	20	30	35	40	45	60	70
$m(x)$	1	1	1	1	0.6	0.5	0.4	0.2	0	0

$A = (U, m)$ is a fuzzy set. The support of A is

$$\mathrm{Supp}\,(A) = \{5, 10, 15, 20, 30, 35, 40, 45\}$$

and the kernel of A is

$$\mathrm{Kern}\,(A) = \{5, 10, 15, 20\}.$$

The fuzzy set may be represented as

$$A = \{1/5, 1/10, 1/15, 1/20, 0.6/30, 0.5/35, 0.4/40, 0.2/45\}$$

or as

$$A = 1/5 + 1/10 + 1/15 + 1/20 + 0.6/30 + 0.5/35 + 0.4/40 + 0.2/45.$$

Fuzzy Sets

- **Fuzzy set** is defined as follows:

- If X is an universe of discourse and x is a particular element of X, then a fuzzy set A defined on X and can be written as a collection of ordered pairs

$$A = \{(x, \mu_{\tilde{A}}(x)), x \in X \}$$

Fuzzy Sets (Continue)

Example

- Let X = {g_1, g_2, g_3, g_4, g_5} be the reference set of students.
- Let Ã be the fuzzy set of “smart” students, where “smart” is fuzzy term.

$$\tilde{A} = \{(g_1,0.4)(g_2,0.5)(g_3,1)(g_4,0.9)(g_5,0.8)\}$$

Here Ã indicates that the smartness of g_1 is 0.4 and so on

Fuzzy Sets (Continue)

Membership Function

- The membership function fully defines the fuzzy set
- A membership function provides a measure of *the degree of similarity* of an element to a fuzzy set

Membership functions can

  - either be chosen by the user arbitrarily, based on the user's experience (MF chosen by two users could be different depending upon their experiences, perspectives, etc.)
  - Or be designed using machine learning methods (e.g., artificial neural networks, genetic algorithms, etc.)

Some special membership functions

1. **Triangular membership function**

$$\mu_A(x) = \begin{cases} 0 & x \leq a \\ \dfrac{x-a}{b-a} & a < x \leq b \\ \dfrac{x-b}{c-b} & b < x \leq c \\ 0 & c < x \end{cases}$$

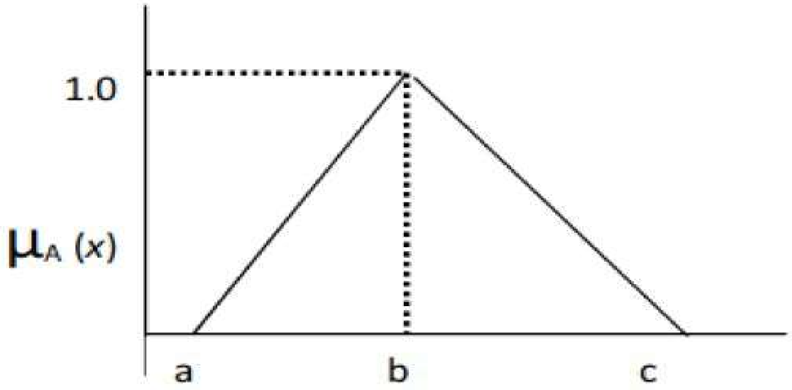


Figure 11.3: Graph of the triangular membership function

2. **Trapezoidal membership function**

$$\mu_A(x) = \begin{cases} 0 & x \leq a \\ \dfrac{x-a}{b-a} & a < x \leq b \\ 1 & b < x \leq c \\ \dfrac{x-c}{d-b} & c < x \leq d \\ 0 & d < x \end{cases}$$

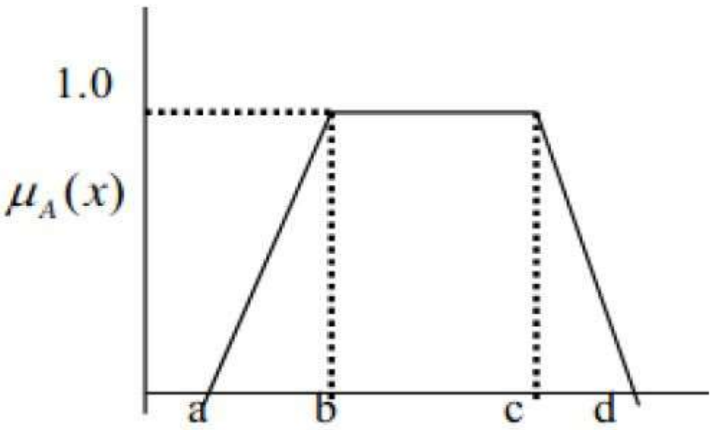


Figure 11.4: Graph of the trapezoidal membership function

3. **Gaussian membership function**

$$\mu_A(x) = \exp\left(-\frac{1}{2}\left|\frac{x-m}{s}\right|^k\right)$$

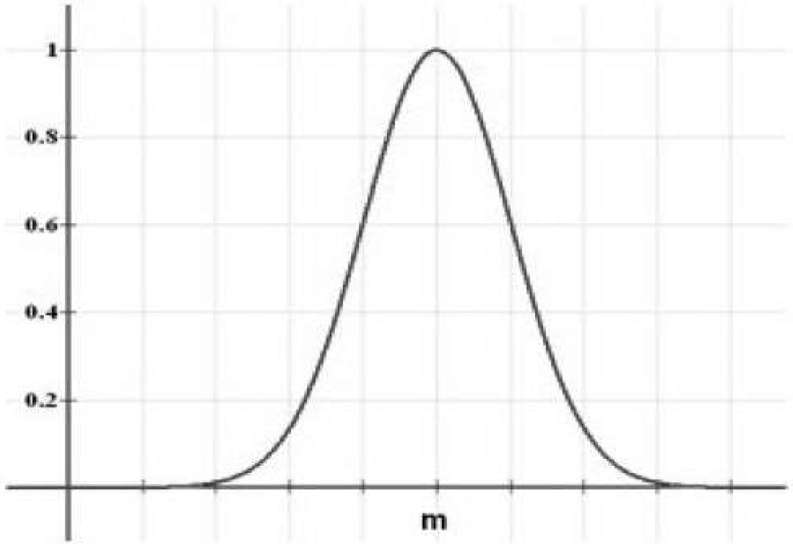


Figure 11.5: Graph of the Gaussian membership function

11.6.1 Definitions

Let A and B be two fuzzy sets in a universe U.

1. **Equality**

   A and B are said to be equal, denoted by $A = B$, if $\mu_A(x) = \mu_B(x)$ for all x in U.

2. **Union**

   The union of A and B, denoted by $A \cup B$, is the fuzzy set C whose membership function is defined by

   $$\mu_C(x) = \max\{\mu_A(x), \mu_B(x)\}, \quad x \in U.$$

3. **Intersection**

   The intersection of A and B, denoted by $A \cap B$, is the fuzzy set D whose membership function is defined by

   $$\mu_D(x) = \min\{\mu_A(x), \mu_B(x)\}, \quad x \in U.$$

4. **Complement**

   The complement of A, denoted by $\overline{A}$, is the fuzzy set E whose membership function is defined by

   $$\mu_E(x) = 1 - \mu_A(x), \quad x \in U.$$

The support of a fuzzy set is a crisp set. We have

$$\mathrm{Supp}\,(A \cup B) = \mathrm{Supp}\,(A) \cup \mathrm{Supp}\,(B)$$
$$\mathrm{Supp}\,(A \cap B) = \mathrm{Supp}\,(A) \cap \mathrm{Supp}\,(B)$$
$$\mathrm{Supp}\,(\overline{A}) = \overline{\mathrm{Supp}\,(A)}$$

Example:

$A = \{(x_1,0.5),(x_2,0.7),(x_3,0)\}$ $B = \{(x_1,0.8),(x_2,0.2),(x_3,1)\}$

Union:

$$A \cup B = \{(x_1,0.8),(x_2,0.7),(x_3,1)\}$$

Because

$$\mu_{A \cup B}(x_1) = \max(\mu_A(x_1), \mu_B(x_1))$$

$$= \max(0.5,0.8)$$

$$= 0.8$$

$$\mu_{A \cup B}(x_2) = 0.7 \text{ and } \mu_{A \cup B}(x_3) = 1$$

Example:

$A = \{(x_1,0.5),(x_2,0.7),(x_3,0)\}$

Complement:

$$A^c = \{(x_1,0.5),(x_2,0.3),(x_3,1)\}$$

Because

$$\mu_A(x_1) = 1 - \mu_A(x_1)$$

$$= 1 - 0.5$$

$$= 0.5$$

$$\mu_A(x_2) = 0.3 \text{ and } \mu_A(x_3) = 1$$

Example:

$A = \{(x_1,0.5),(x_2,0.7),(x_3,0)\}$ $B = \{(x_1,0.8),(x_2,0.2),(x_3,1)\}$

Intersection:

$$A \cap B = \{(x_1,0.5),(x_2,0.2),(x_3,0)\}$$

Because

$$\mu_{A\cap B}(x_1) = \min(\mu_A(x_1), \mu_B(x_1))$$

$$= \max(0.5,0.8)$$

$$= 0.5$$

$$\mu_{A\cap B}(x_2) = 0.2 \text{ and } \mu_{A\cap B}(x_3) = 0$$

EXAMPLE

Given the fuzzy sets

$$A = \{0.3/2, 0.4/3, 0.1/4, 0.8/5, 1.0/6\}$$
$$B = \{0.7/4, 0.5/5, 1.0/6, 0.02/7, 0.75/8\}$$

find $A \cup B$ and $A \cap B$.

11.7 Some more definitions

1. **Fuzzy empty set**

   A fuzzy set A is said to be empty if $\mu_A(x) = 0$ for all $x \in U$, that is, if Supp $(A) = \varnothing$.

2. **Fuzzy subset**

   A fuzzy set A is said to be a subset of a fuzzy set B, denoted by $A \subseteq B$, if $\mu_A(x) \leq \mu_B(x)$ for all $x \in U$.

3. **Disjoint fuzzy sets**

   Two fuzzy sets A and B are said to be disjoint if $\mu_A(x) = 0$ or $\mu_B(x) = 0$ for all $x \in U$, that is, if Supp $(A \cap B) = \varnothing$.

7.3.2 Properties of Fuzzy Sets

Fuzzy sets follow the same properties as crisp sets except for the law of excluded middle and law of contradiction. That is, for fuzzy set $\underset{\sim}{A}$

law of excluded middle

$$\underset{\sim}{A} \cup \overline{\underset{\sim}{A}} \neq U; \quad \underset{\sim}{A} \cap \overline{\underset{\sim}{A}} \neq \phi$$

law of contradiction

Frequently used properties of fuzzy sets are given as follows:

1. Commutativity

$$\underset{\sim}{A} \cup \underset{\sim}{B} = \underset{\sim}{B} \cup \underset{\sim}{A}; \quad \underset{\sim}{A} \cap \underset{\sim}{B} = \underset{\sim}{B} \cap \underset{\sim}{A}$$

2. Associativity

$$\underset{\sim}{A} \cup (\underset{\sim}{B} \cup \underset{\sim}{C}) = (\underset{\sim}{A} \cup \underset{\sim}{B}) \cup \underset{\sim}{C}$$
$$\underset{\sim}{A} \cap (\underset{\sim}{B} \cap \underset{\sim}{C}) = (\underset{\sim}{A} \cap \underset{\sim}{B}) \cap \underset{\sim}{C}$$

3. Distributivity

$$\underset{\sim}{A} \cup (\underset{\sim}{B} \cap \underset{\sim}{C}) = (\underset{\sim}{A} \cup \underset{\sim}{B}) \cap (\underset{\sim}{A} \cup \underset{\sim}{C})$$
$$\underset{\sim}{A} \cap (\underset{\sim}{B} \cup \underset{\sim}{C}) = (\underset{\sim}{A} \cap \underset{\sim}{B}) \cup (\underset{\sim}{A} \cap \underset{\sim}{C})$$

4. Idempotency

$$\underset{\sim}{A} \cup \underset{\sim}{A} = \underset{\sim}{A}; \quad \underset{\sim}{A} \cap \underset{\sim}{A} = \underset{\sim}{A}$$

5. Identity

$$\underset{\sim}{A} \cup \phi = \underset{\sim}{A} \quad \text{and} \quad \underset{\sim}{A} \cup U = U(\text{universal set})$$
$$\underset{\sim}{A} \cap \phi = \phi \quad \text{and} \quad \underset{\sim}{A} \cap U = \underset{\sim}{A}$$

6. Involution (double negation)

$$\overline{\overline{\underset{\sim}{A}}} = \underset{\sim}{A}$$

7. Transitivity

$$\text{If } \underset{\sim}{A} \subseteq \underset{\sim}{B} \subseteq \underset{\sim}{C}, \text{ then } \underset{\sim}{A} \subseteq \underset{\sim}{C}$$

8. De Morgan's law

$$\overline{\underset{\sim}{A} \cup \underset{\sim}{B}} = \overline{\underset{\sim}{A}} \cap \overline{\underset{\sim}{B}}; \overline{\underset{\sim}{A} \cap \underset{\sim}{B}} = \overline{\underset{\sim}{A}} \cup \overline{\underset{\sim}{B}}$$

2. Consider two given fuzzy sets

$$\underset{\sim}{A} = \left\{\frac{1}{2} + \frac{0.3}{4} + \frac{0.5}{6} + \frac{0.2}{8}\right\}$$

$$\underset{\sim}{B} = \left\{\frac{0.5}{2} + \frac{0.4}{4} + \frac{0.1}{6} + \frac{1}{8}\right\}$$

Perform union, intersection, difference and complement over fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$.

Solution: For the given fuzzy sets we have the following

(a) Union

$$\underset{\sim}{A} \cup \underset{\sim}{B} = \max\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\}$$
$$= \left\{\frac{1}{2} + \frac{0.4}{4} + \frac{0.5}{6} + \frac{1}{8}\right\}$$

(b) Intersection

$$\underset{\sim}{A} \cap \underset{\sim}{B} = \min\{\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)\}$$
$$= \left\{\frac{0.5}{2} + \frac{0.3}{4} + \frac{0.1}{6} + \frac{0.2}{8}\right\}$$

(c) Complement

$$\overline{\underset{\sim}{A}} = 1 - \mu_{\underset{\sim}{A}}(x) = \left\{\frac{0}{2} + \frac{0.7}{4} + \frac{0.5}{6} + \frac{0.8}{8}\right\}$$

$$\overline{\underset{\sim}{B}} = 1 - \mu_{\underset{\sim}{B}}(x) = \left\{\frac{0.5}{2} + \frac{0.6}{4} + \frac{0.9}{6} + \frac{0}{8}\right\}$$

(d) Difference

$$\underset{\sim}{A}|\underset{\sim}{B} = \underset{\sim}{A} \cap \overline{\underset{\sim}{B}} = \left\{\frac{0.5}{2} + \frac{0.3}{4} + \frac{0.5}{6} + \frac{0}{8}\right\}$$

$$\underset{\sim}{B}|\underset{\sim}{A} = \underset{\sim}{B} \cap \overline{\underset{\sim}{A}} = \left\{\frac{0}{2} + \frac{0.4}{4} + \frac{0.1}{6} + \frac{0.8}{8}\right\}$$

7.3.1.4 More Operations on Fuzzy Sets

1. *Algebraic sum:* The algebraic sum $(\underset{\sim}{A} + \underset{\sim}{B})$ of fuzzy sets, fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$ is defined as

$$\mu_{\underset{\sim}{A}+\underset{\sim}{B}}(x) = \mu_{\underset{\sim}{A}}(x) + \mu_{\underset{\sim}{B}}(x) - \mu_{\underset{\sim}{A}}(x) \cdot \mu_{\underset{\sim}{B}}(x)$$

2. *Algebraic product:* The algebraic product $(\underset{\sim}{A} \cdot \underset{\sim}{B})$ of two fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$ is defined as

$$\mu_{\underset{\sim}{A}\cdot\underset{\sim}{B}}(x) = \mu_{\underset{\sim}{A}}(x) \cdot \mu_{\underset{\sim}{B}}(x)$$

3. *Bounded sum:* The bounded sum ($\underset{\sim}{A} \oplus \underset{\sim}{B}$) of two fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$ is defined as

$$\mu_{\underset{\sim}{A} \oplus \underset{\sim}{B}}(x) = \min\{1, \mu_{\underset{\sim}{A}}(x) + \mu_{\underset{\sim}{B}}(x)\}$$

4. *Bounded difference:* The bounded difference ($\underset{\sim}{A} \odot \underset{\sim}{B}$) of two fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$ is defined as

$$\mu_{\underset{\sim}{A} \odot \underset{\sim}{B}}(x) = \max\{0, \mu_{\underset{\sim}{A}}(x) - \mu_{\underset{\sim}{B}}(x)\}$$