Solution 1: Entropy

A fair die is rolled at the same time as a fair coin is tossed. Let A be the number on the upper surface of the die and let B describe the outcome of the coin toss, where

$$B = \begin{cases} 1, & \text{head}\,, \\ 0, & \text{tail}\,. \end{cases}$$

Two random variables X and Y are given by $X = A + B$ and $Y = A - B$, respectively.

(a) Calculate the entropies $H(X)$ and $H(Y)$, the conditional entropies $H(Y|X)$ and $H(X|Y)$, the joint entropy $H(X, Y)$ and the mutual information $I(X; Y)$.

Solution:

Let a, b, x, and y denote the realisations of the random variables A, B, X, and Y, respectively. Each event (a, b) is associated with exactly one event (x, y) and the probability for such an event is given by

$$p_{AB}(a,b) = p_{XY}(x,y) = \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}$$

Consequently, we obtain for the joint entropy

$$\begin{aligned} H(X,Y) &= -\sum_{x,y} p_{X,Y}(x,y) \log_2 p_{XY}(x,y) = -12 \cdot \frac{1}{12} \log_2 \frac{1}{12} \\ &= \log_2 12 \\ &= 2 + \log_2 3 \end{aligned}$$

Below we list the possible values of the random variables X and Y, the associated events (a, b), and the probability masses $p_X(x)$ and $p_Y(y)$.

x	events (a,b)	$p_X(x)$
1	$(1,0)$	$1/12$
2	$(2,0),(1,1)$	$1/6$
3	$(3,0),(2,1)$	$1/6$
4	$(4,0),(3,1)$	$1/6$
5	$(5,0),(4,1)$	$1/6$
6	$(6,0),(5,1)$	$1/6$
7	$(6,1)$	$1/12$

y	events (a,b)	$p_Y(y)$
0	$(1,1)$	$1/12$
1	$(1,0),(2,1)$	$1/6$
2	$(2,0),(3,1)$	$1/6$
3	$(3,0),(4,1)$	$1/6$
4	$(4,0),(5,1)$	$1/6$
5	$(5,0),(6,1)$	$1/6$
6	$(6,0)$	$1/12$

The random variable $X = A + B$ can take the values 1 to 7. The probability masses $p_X(x)$ for the values 1 and 7 are equal to $1/12$, since they correspond to exactly one event. The probability masses for the values 2 to 6 are equal to $1/6$, since each of these values corresponds to two events (a, b). An analogue result is obtained for the random variable $Y = A - B$.

The marginal entropies are given by

$$\begin{aligned} H(X) &= -\sum_{x} p_X(x) \log_2 p_X(x) \\ &= -2 \cdot \frac{1}{12} \log_2 \frac{1}{12} - 5 \cdot \frac{1}{6} \log_2 \frac{1}{6} \\ &= \frac{1}{6} \cdot (\log_2 4 + \log_2 3) + \frac{5}{6} \cdot (\log_2 2 + \log_2 3) \\ &= \frac{7}{6} + \log_2 3 \end{aligned}$$

and for Y

$$\begin{aligned}
H(Y) &= -\sum_y p_Y(y)\log_2 p_Y(y) \\
&= -2\cdot\frac{1}{12}\log_2\frac{1}{12} - 5\cdot\frac{1}{6}\log_2\frac{1}{6} \\
&= \frac{1}{6}\cdot(\log_2 4 + \log_2 3) + \frac{5}{6}\cdot(\log_2 2 + \log_2 3) \\
&= \frac{7}{6} + \log_2 3
\end{aligned}$$

We can determine the conditional entropies using

$$H(X|Y) = H(X,Y) - H(Y) = 2 + \log_2 3 - \frac{7}{6} - \log_2 3 = \frac{5}{6}$$

$$H(Y|X) = H(X,Y) - H(X) = 2 + \log_2 3 - \frac{7}{6} - \log_2 3 = \frac{5}{6}$$

The mutual information $I(X;Y)$ can be determined acording to

$$I(X;Y) = H(X) - H(X|Y) = \frac{7}{6} + \log_2 3 - \frac{5}{6} = \frac{1}{3} + \log_2 3$$

or

$$I(X;Y) = H(Y) - H(Y|X) = \frac{7}{6} + \log_2 3 - \frac{5}{6} = \frac{1}{3} + \log_2 3$$

(b) Show that, for independent discrete random variables X and Y,

$$I(X;X+Y) - I(Y;X+Y) = H(X) - H(Y)$$

Solution:

Using the definition of mutual information for discrete random variables, $I(X;Y) = H(Y) - H(Y|X)$, we can write

$$\begin{aligned}
I(X;X+Y) - I(Y;X+Y) &= H(X+Y) - H(X+Y|X) - H(X+Y) + H(X+Y|Y) \\
&= H(X|Y) - H(Y|X) \\
&= H(X) - H(Y)\,.
\end{aligned}$$

The first step follows from the fact that modifying the mean of a pmf doesn't change the entropy. For the second step, we used the fact that the conditional entropy $H(X|Y)$ is equal to the marginal entropy $H(X)$ for independent random variables X and Y.