Solution 1: Risk Minimizers for the L2-Loss

(a) As seen in L2-loss slide 2, the risk minimizer f^* is the conditional mean:

$$f^*(\mathbf{x}) = \mathbb{E}_{y_{|x}}\left[y_{|x}\right]$$

The distribution of y given x is known, and can be plugged-in:

$$f^*(\mathbf{x}) = \mathbb{E}\left[\mathcal{N}(a + bx, 1)\right] = a + bx$$

(b) The resulting risk can be calculated by using the definition:

$$\begin{aligned}
\mathcal{R}_L(f^*) &= \mathbb{E}_{xy}\left[(y - f^*(\mathbf{x}))^2\right] \\
&= \mathbb{E}_{xy}\left[(y - \mathbb{E}_{y_{|x}}\left[y_{|x}\right])^2\right] \\
&= \mathbb{E}_{x}\left[\mathsf{Var}_{y_{|x}}(y_{|x})\right] \\
&= \mathbb{E}_{x}\left[\mathsf{Var}(\mathcal{N}(a + bx, 1))\right] \\
&= \mathbb{E}_{x}\left[1\right] \\
&= 1
\end{aligned}$$

(c) The risk minimizer for the L2 loss is the conditional mean. Considering that the hypothesis space is now restricted to constant models, $f(x)$ is a constant for any x. The optimal constant model in terms of the theoretical risk for the L2 loss is the expected value over y.

$$\bar{f}(x) = \mathbb{E}_{y_{|x}}\left[y_{|x}\right] = \mathbb{E}_y[y]$$

The Law of total expectation can be used in this case:

$$\begin{aligned}
\mathbb{E}_y[y] &= \mathbb{E}_x\left[\mathbb{E}_{y_{|x}}\left[y_{|x}\right]\right] \\
&= \mathbb{E}_x\left[a + bx\right] \\
&= a + b \cdot \mathbb{E}_x[x] \\
&= a + b \cdot 0 \\
&= a
\end{aligned}$$

(d) To obtain the risk of the optimal constant model, the definition can be used:

$$\begin{aligned}
\mathcal{R}_L(\bar{f}) &= \mathbb{E}_{xy}\left[(y - \bar{f}(x))^2\right] \\
&= \mathbb{E}_{xy}\left[(y - a)^2\right] \\
&= \mathbb{E}_y[y^2] - 2a\underbrace{\mathbb{E}_y[y]}_{a} + a^2 \\
&= \mathbb{E}_y[y^2] - a^2
\end{aligned}$$

As $\mathbb{E}_y[y^2]$ is yet unknown, the calculation is detailed below:

$$\begin{aligned}
\mathbb{E}_y[y^2] &= \mathbb{E}_x\left[\mathbb{E}_{y_{|x}}\left[y_{|x}^2\right]\right] \\
&= \mathbb{E}_x\left[\mathsf{Var}_{y_{|x}}[y_{|x}] + \mathbb{E}_{y_{|x}}\left[y_{|x}\right]^2\right] \\
&= \mathbb{E}_x\left[1 + (a+bx)^2\right] \\
&= \mathbb{E}_x[1 + a^2 + 2abx + b^2x^2] \\
&= 1 + a^2 + 2ab\underbrace{\mathbb{E}_x[x]}_{=0} + b^2\underbrace{\mathbb{E}_x[x^2]}_{=\frac{100}{3}} \\
&= 1 + a^2 + b^2\frac{100}{3}
\end{aligned}$$

Using this result, the risk can be calculated:

$$\mathcal{R}_L(\bar{f}) = 1 + b^2\frac{100}{3}$$