Exercise 1: VC Dimension

Consider a binary classification learning problem with feature space $\mathcal{X} = \mathbb{R}^p$ and label space $\mathcal{Y} = \{-1, 1\}$.

(a) Assume that $p = 1$, i.e., $\mathcal{X} = \mathbb{R}$. Let

$$\mathcal{H} = \{h_r : \mathcal{X} \to \mathcal{Y} \,|\, r \in \mathbb{R}\}$$

be the hypothesis space of left-open interval classifiers on the reals, where $h_r(x) = 1$ for $x \in (-\infty, r]$ and $= -1$ otherwise. What is $VC_p(\mathcal{H})$?

(b) Let

$$\tilde{\mathcal{H}} = \{\tilde{h}_l : \mathcal{X} \to \mathcal{Y} \,|\, l \in \mathbb{R}\}$$

be the hypothesis space of right-open interval classifiers on the reals, where $\tilde{h}_l(x) = 1$ for $x \in [l, \infty)$ and $= -1$ otherwise. What is $VC_p(\mathcal{H} \cup \tilde{\mathcal{H}})$?

(c) Consider now the feature space $\mathcal{X} = \{0, 1\}^p$ for some $p \in \mathbb{N}$ and let

$$\mathcal{H} = \{h_t : \mathcal{X} \to \mathcal{Y} \,|\, t \in \{0, 1, 2, \ldots, p + 1\}\}$$

be the hypothesis space of threshold classifiers on bitstrings, where $h_t(\mathbf{x}) = 1$ for $\sum_{i=1}^{p} x_i \geq t$ and $= -1$ otherwise. Thus, instances are bitstrings of length p, and h_t classifies an instance as positive if the number of 1s in the bitstring is at least t, e.g., $h_3(0, 1, 1, 0, 0) = -1$ and $h_3(1, 1, 1, 0, 1) = +1$. What is $VC_p(\mathcal{H})$?

(d) Let the feature space be $\mathcal{X} = \mathbb{R}^p$ and let $\mathcal{H}$ be a finite hypothesis space, i.e., $|\mathcal{H}| < \infty$. Show that $VC_p(\mathcal{H}) \leq \log_2(|\mathcal{H}|)$ holds.

Hint: Consider a set of points of size $\log_2(|\mathcal{H}|) + 1$.