

### Solution 1: Logistic Regression, Softmax, Cross-Entropy

- (a) The logistic function is a special case of the softmax for two classes. We have

$$\pi_1(x) = \frac{\exp(\theta_1^\top x)}{\exp(\theta_1^\top x) + \exp(\theta_2^\top x)}$$

and

$$\pi_2(x) = \frac{\exp(\theta_2^\top x)}{\exp(\theta_1^\top x) + \exp(\theta_2^\top x)}.$$

We get:

$$\pi_1(x) = \frac{1}{(\exp(\theta_1^\top x) + \exp(\theta_2^\top x)) / \exp(\theta_1^\top x)} = \frac{1}{\exp((\theta_1 - \theta_2)^\top x) + 1} = \frac{1}{1 + \exp(\theta^\top x)}$$

where  $\theta = \theta_2 - \theta_1$  and  $\pi_2(x) = 1 - \pi_1(x)$ .

- (b) For  $g$  classes and  $n = 1$  trials (actually we are dealing with a multinoulli or categorical distribution), the likelihood  $l(\boldsymbol{\pi})$  of a single observation  $y$  is given by

$$l(\boldsymbol{\pi}) = \prod_{k=1}^g \pi_k^{\mathbb{1}_{\{y=k\}}}.$$

Now let's look at the logarithmic loss in softmax regression:

$$\text{MC logloss} = - \sum_{k=1}^g \mathbb{1}_{\{y=k\}} \log \pi_k.$$

This is in fact just the negative logarithm of our likelihood:  $-\log l(\boldsymbol{\pi}) = - \sum_{k=1}^g \mathbb{1}_{\{y=k\}} \log \pi_k$ .