Hull-White Model and Calibration
Perfect Fit of the Term Structure

M. Lobenwein, N. Weber, M. Pommer

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Introduction

- Hull-White model: $dr_t = [\theta(t) - ar_t]dt + \sigma dW_t$
- Since the Hull-White model is inhomogenous we can exactly fit the term structure
- In order to achieve the perfect fit we construct a trinomial tree in two consecutive steps

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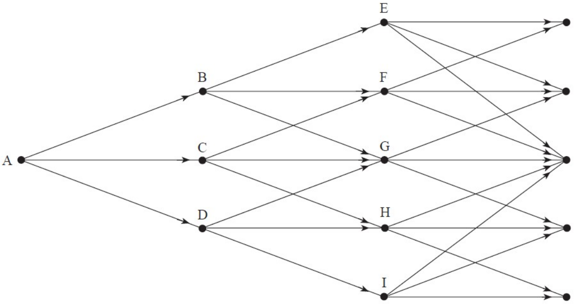
Trinomial tree according to Hull and White step one

Options, Futures and other Derivatives by Hull and White

Figure 30.8 Tree for R^* in Hull–White model (first stage).

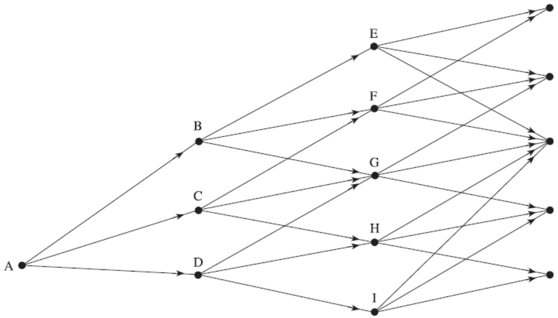


Dynamics: $dR_t^* = \sigma dW_t$, assuming $a = 0$

Trinomial tree according to Hull and White step two

Options, Futures and other Derivatives by Hull

Figure 31.9 Tree for R in Hull–White model (the second stage).



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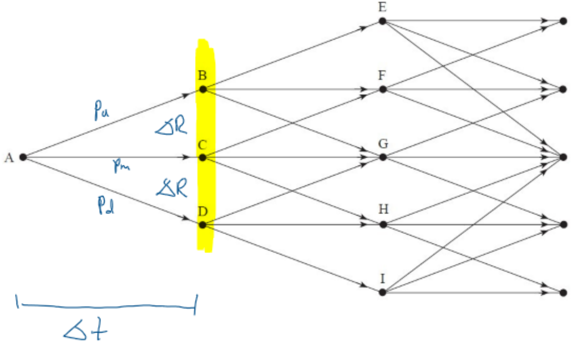

Dynamics: $dR_t^* = \sigma dW_t$, assuming $a = 0$
$\rightarrow$ one step has mean zero and variance $\sigma^2 \Delta t$.

Trinomial tree according to Hull and White step one

- During the first step the branching probabilities for going up, down and remaining constant are calculated
- The difference $R^*(t+\Delta t) - R^*$ is normally distributed with mean $-aR^*\Delta t$ and variance $\sigma^2 \Delta t$
- We have to equate the theoretical mean and variance with the one in the model

$$p_u \Delta R - p_u \Delta R = -aj\Delta R \Delta t$$
$$p_u \Delta R^2 + p_d \Delta R^2 = \sigma^2 \Delta t + a^2 j^2 \Delta R^2 \Delta t^2$$
$$p_u + p_m + p_d = 1$$

Trinomial tree according to Hull and White step one

This leads to the following probabilities:

$$p_u = \frac{1}{6} + \frac{1}{2}(a^2 j^2 \Delta t^2 - aj\Delta t)$$
$$p_m = \frac{2}{3} + a^2 j^2 \Delta t^2$$
$$p_u = \frac{1}{6} + \frac{1}{2}(a^2 j^2 \Delta t^2 + aj\Delta t)$$

Trinomial tree according to Hull and White step two

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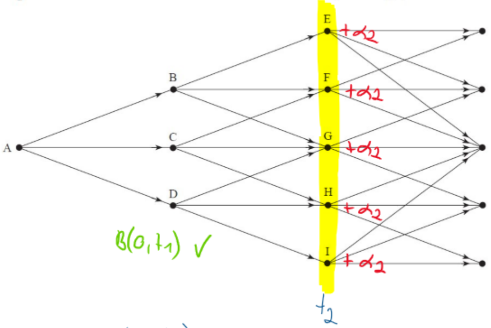

$B(0, t_2) = 0{,}95$

vs. Exp. value of 1 in $t_2 \neq 0{,}95$

$\Rightarrow r_{t_{2,\cdot}} + \alpha_2$

Trinomial tree according to Hull and White step two

We have to shift the nodes in order to fit the term structure

- We define $Q_{i,j}$ as the present value of a security that pays off 1 EUR at node (i,j)
- We initialize $Q_{0,0} = 1$
- We calcuate α_0 such that the right price of the given zero coupond bond at time Δt is perfectly met
- Next we calculate the prices $Q_{1,j}$ at time 0 for all nodes j as the multiplication of the respective probabilities and the zero coupond bonds according to α_0
- We can iteratively calculate all α_i and $Q_{i,j}$ to exactly fit the term structure with the following formulas

$$\alpha_m = \frac{ln \sum_{j=-n_m}^{n_m} Q_{m,j} e^{-j\Delta R \Delta t} - ln P_{m+1}}{\Delta t}$$

$$Q_{m+1,j} = sum_k Q_{m,k} q(k,j) exp[-(\alpha_m + k\Delta R)\Delta t]$$

Trinomial tree according to Hull and White step two

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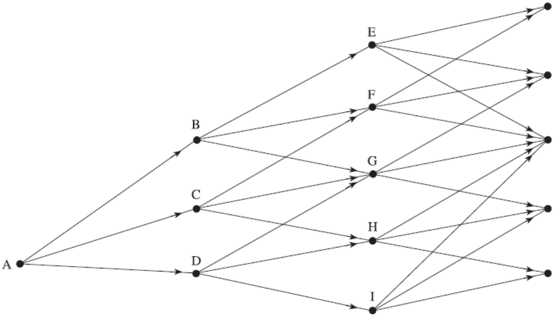

Dynamics: $dR_t = \theta(t)dt + \sigma dW_t$, assuming $a = 0$

$\rightarrow \alpha \approx \theta$

(alpha correction corresponds to introduction of drift for perfect fit)