UQ: Stochastic Galerkin on an ODE

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October 17, 2023

1 Problem statement

We are given the ODE:

$$\frac{du}{dt} = -\frac{1}{2}u^2 + \alpha u \tag{1}$$

with the initial condition $u(t=0, Z) = \frac{1}{10}$. The only random parameter is α which is given by $\alpha(Z) = 1 + Z$ and Z is random variable that is normally distributed: $Z \sim \mathcal{N}(0,1)$. Since Z is normally distributed, we will use Hermite basis polynomials.

We have the gPC approximation of u:

$$v_N(t, Z) = \sum_{i=0}^{N} \hat{v}_i(t) H_i(Z) \tag{2}$$

and the gPC approximation of α:

$$\alpha(Z) = \sum_{i=0}^{N} \hat{\alpha}_i(t) H_i(Z) \tag{3}$$

where H_i are the Hermite basis polynomials. Since we know $\alpha(Z) = 1 + Z$ we can simplify the gPC approximation of α (3). We know that $H_0(Z) = 1$ and $H_1(Z) = Z$, so we can deduce that $\hat{\alpha}_0 = 1$ and $\hat{\alpha}_1 = 1$ and $\hat{\alpha}_m = 0$ for all $m > 1$. We have found the coefficients for $alpha$:

$$\alpha_i = \begin{cases} 1, & \text{if } i = 0 \vee i = 1 \\ 0, & \text{else.} \end{cases} \tag{4}$$

Furthermore, it is important to note that when we define e_{ijk} as follows:

$$e_{ijk} = \mathbb{E}[H_i(Z)H_j(Z)H_k(Z)]$$

where H are the Hermite polynomials, we can find an analytical expression of these constants:

$$e_{ijk} = \frac{i!j!k!}{(s-i)!(s-j)!(s-k)!} \text{ for } s \geq i, j, k, \text{ and } 2s = i + j + k \text{ is even.} \tag{5}$$

For Hermite polynomials, we also know that $\gamma_k = k!$ as long as $k \geq 0$.

We want to apply the Stochastic Galerkin method to the above defined ODE (equation 1) using all of the above statements.

2 Stochastic Galerkin

From the approximations of u we get two residuals, one for the ODE and one for the initial condition:

$$r_1 = \frac{dv_N}{dt} + \frac{1}{2}v_N^2 - \alpha v_N \tag{6}$$

$$r_2 = v_N(t=0, Z) - \frac{1}{10} \tag{7}$$

To reduce the error of the approximation, we want these residuals to be orthogonal to the orthogonal basis functions. Thus, we want: $r_1 \perp H_k$ (A) and $r_2 \perp H_k$ (B) $\forall k \leq N$. We start with working out (A):

$$\begin{aligned}
&\langle r_1, H_k \rangle = 0 \\
&\mathbb{E}[r_1 H_k] = 0 \\
&\mathbb{E}\left[\left(\frac{dv_N}{dt} + \frac{1}{2}v_N^2 - \alpha v_N\right) H_k\right] = 0 \\
&\mathbb{E}\left[\frac{dv_N}{dt} H_k\right] + \mathbb{E}\left[\frac{1}{2}v_N^2 H_k\right] - \mathbb{E}[\alpha v_N H_k] = 0
\end{aligned}$$

We will evaluate these different inner products separately.

$$\begin{aligned}
\mathbb{E}\left[\frac{dv_N}{dt} H_k\right] &= \mathbb{E}\left[\sum_{i=0}^{N} \frac{d\hat{v}_k}{dt} H_i(Z) H_k(Z)\right] \\
&= \sum_{i=0}^{N} \frac{d\hat{v}_k}{dt} \mathbb{E}[H_i(Z) H_k(Z)] \\
&= \frac{d\hat{v}_k}{dt} \gamma_k
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{2}v_N^2 H_k\right] &= \frac{1}{2}\sum_{i,j=0}^{N} \hat{v}_i \hat{v}_j \mathbb{E}[H_i(Z) H_j(Z) H_k(Z)] \\
&= \frac{1}{2}\sum_{i,j=0}^{N} \hat{v}_i \hat{v}_j e_{ijk} \\
&= \frac{1}{2}\sum_{i,j=0}^{N} \hat{v}_i \hat{v}_j \frac{i!j!k!}{(s-i)!(s-j)!(s-k)!}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[\alpha v_N H_k] &= \sum_{i,j=0}^{N} \hat{\alpha}_i \hat{v}_j \mathbb{E}[H_i(Z) H_j(Z) H_k(Z)] \\
&= \sum_{i,j=0}^{N} \hat{\alpha}_i \hat{v}_j e_{ijk}
\end{aligned}$$

When we combine these equations together we get:

$$\frac{d\hat{v}_i}{dt} = -\frac{1}{2\gamma_i} \sum_{j,k=0}^{N} \hat{v}_j \hat{v}_k e_{ijk} + \hat{\alpha}_j \hat{v}_k e_{ijk} \tag{8}$$

which is again an ODE to solve.

Now, we also want to solve (B), the initial condition:

$$
\begin{aligned}
&\langle r_2, H_k \rangle = 0 \\
&\mathbb{E}[r_2 H_k] = 0 \\
&\mathbb{E}\left[(v_N(t=0,Z) - \frac{1}{10})H_k\right] = 0 \\
&\mathbb{E}[v_N(t=0,Z)H_k] = \mathbb{E}\left[\frac{1}{10}H_k\right] \\
&\sum_{i=0}^{N} \hat{v}_i(t=0)\mathbb{E}[H_i(Z)H_k(Z)] = \frac{1}{10}\mathbb{E}[H_k(Z)H_0(Z)] \\
&\hat{v}_k(t=0)\gamma_k = \frac{1}{10}\gamma_0\delta_{k0} \\
&\hat{v}_k(t=0) = \begin{cases} \frac{1}{10}, & \text{if } k=0 \\ 0 & \text{everywhere else} \end{cases}
\end{aligned}
$$

Thus, applying stochastic Galerkin to the original ODE (1) leads to the following system of ODEs that need to be solved:

$$
\frac{d\hat{v}_k}{dt} = -\frac{1}{2\gamma_k}\sum_{i,j=0}^{N} \hat{v}_i\hat{v}_j e_{ijk} + \hat{\alpha}_i\hat{v}_j e_{ijk} \tag{9}
$$

$$
\hat{v}_k(t=0) = \begin{cases} \frac{1}{10}, & \text{if } k=0 \\ 0 & \text{everywhere else} \end{cases} \tag{10}
$$