Uncertainty Quantification course
Homework assignment 8

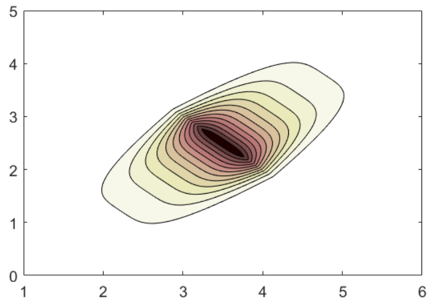
November 9, 2023

We discuss this exercise in the class meeting on 15 November 2023.

Consider the following probability density function:

$$\rho(x, y) \propto \exp(-V(x, y)) \quad \text{with } V(x, y) = (x - y - 1)^4 + |x + y - 6|$$

In the figure below you see a contour plot of this pdf.



Use Markov-Chain Monte Carlo (MCMC) sampling to draw samples from ρ. Use the Metropolis algorithm with a Gaussian proposal distribution q centered at the last value of the Markov Chain:

$$q(\mathbf{w} \mid \mathbf{u}_i) \sim \mathcal{N}(\mathbf{u}_i, K_q).$$

Here, $\mathbf{w} = (w_1, w_2)^T$ are candidate samples to be accepted or rejected, and $\mathbf{u}_i = (u_1^{(i)}, u_2^{(i)})^T$ is the previous state of the Markov chain. The covariance matrix K_q is given by

$$K_q = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}.$$

Manually tune the value of α until you get an acceptance rate (i.e., mean acceptance probability) of approximately 30%, and start the Markov chain at $\mathbf{u}_0 = (0, 0)^T$. Plot the Markov chain in the 2D plane. Does it sample the pdf shown in the figure above? Also, make plots of the running mean of the Markov chain. Comment on the convergence of the MCMC estimate of the mean of the pdf, as you observe it in these plots.

Tip: Instead of directly using the acceptance probability, use the log of the acceptance probability for a better numerical implementation.