Uncertainty Quantification: Homework 2 - Part 2

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We prove the eigenfunctions and eigenvalues, respectively, of the Brownian bridge below

$$\psi_i(t) = \sin(\pi i t) \tag{1}$$

$$\lambda_i = \frac{1}{(i\pi)^2} \tag{2}$$

on $t \in [0,1]$. The Brownian bridge covariance function is known to be

$$C(t,s) = \min(s,t) - st. \tag{3}$$

We show that $\psi_i(t)$ and λ_i satisfy the eigenfunction problem

$$\forall t \in I \ : \ \int_I C(t,s)\psi_i(s)ds = \lambda_i \psi_i(t) \tag{4}$$

For simplicity, assume that $e(s)$ is an eigenfunction. Substituting into (4),

$$\int_0^1 [\min(s,t) - st]e(s)ds = \lambda e(t) \tag{5}$$

where $\min(s,t)$ is defined as

$$\min(s,t) = \begin{cases} t, \text{if } 0 \le t \le s \\ s, \text{if } s \le t \le T \end{cases} \tag{6}$$

and $T = 1$, so

$$\int_0^T \min(s,t) = \int_0^s tdt + \int_s^T sdt \tag{7}$$

then (5) becomes

$$\int_0^t (s - st)e(s)ds + \int_t^1 (t - st)e(s)ds = \lambda e(t) \tag{8}$$

$$\int_0^t (s - st)e(s)ds + t\int_t^1 (1-s)e(s)ds = \lambda e(t) \tag{9}$$

Since we wish to find $e(s)$, we seek to transform (9) to a differential equation for which solutions are known. Therefore, we differentiate (9) once with respect to t

$$-\int_0^t se(s)ds + \int_t^1 (1-s)e(s)ds = \lambda e'(t) \tag{10}$$

Then we differentiate (10) again with respect to t, applying the fundamental theorem of calculus

$$\begin{aligned} -[te(t) - 0] + [0 - (1-t)e(t)] &= \lambda e''(t) \\ (-t - 1 + t)e(t) &= \lambda e''(t) \\ -e(t) &= \lambda e''(t) \end{aligned} \tag{11}$$

The general solution to (10) and (11) is

$$e(t) = a\sin(\frac{t}{\sqrt{\lambda}}) + b\cos(\frac{t}{\sqrt{\lambda}}). \tag{12}$$

We observe that for $t = 0$, $e(0) = b$, and substituting $t = 0$ into (5),

$$\begin{aligned} e(0) &= 0 \\ a\sin(\frac{0}{\sqrt{\lambda}}) + b\cos(\frac{0}{\sqrt{\lambda}}) &= 0 \\ 0a + 1b &= 0 \\ b &= 0 \end{aligned} \tag{13}$$

and therefore $b = 0$ and

$$e(t) = a\sin(\frac{t}{\sqrt{\lambda}}) \tag{14}$$

Similarly, the Brownian bridge process requires $e(1) = 0$. Hence,

$$\begin{aligned} a\sin(\frac{1}{\sqrt{\lambda}}) &= 0 \\ \sin(\frac{1}{\sqrt{\lambda}}) &= \sin(i\pi), i \in \mathbb{N} \\ \frac{1}{\sqrt{\lambda}} &= i\pi \\ \lambda_i &= \frac{1}{(i\pi)^2} \end{aligned} \tag{15}$$

gives the eigenvalues to the eigenfunctions $\psi_i(t) = e_i(t) = a\sin(\pi i t)$. a is a normalization constant and can be calculated as follows

$$
\begin{aligned}
\int_0^1 e_i^2(t)dt &= 1 \\
\frac{a^2}{2}(t + \cos(t)\sin(t))\Big|_0^1 &= 1 \\
\frac{a^2}{2} &= 1 \\
a &= \sqrt{2}
\end{aligned} \tag{16}
$$

Therefore, we have proven that the eigenfunctions

$$\psi_i(t) = e_i(t) = \sqrt{2}\sin(\frac{t}{\sqrt{\lambda}}) \tag{17}$$

and the eigenvalues

$$\lambda_i = \frac{1}{(i\pi)^2} \tag{18}$$

are the solutions to the eigenfunction problem with the covariance function of the Brownian Bridge function,

$$C(t, s) = \min(s, t) - st. \tag{19}$$

And, that the KL expansion of the Brownian bridge process is $B_t^d = \sqrt{2}Y_t^d$, where Y_t^d is the truncated KL expansion with the eigenfunctions $\psi_i(t) = \sin(\pi i t)$ and eigenvalues $\lambda_i = \frac{1}{(i\pi)^2}$.