In[1]:=

```
(*
Title: Homework 1 Simplification and Checking
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Course: Uncertainty Quantification 2023
*)

Clear["Global`*"]
On[Assert]
```

In[40]:=

```
(*Exercise 1*)

(*Return the jacobi polynomial given α, β, and the polynomial order n

This function is from the wikipedia on Jacobi Polynomials*)
JacobiPolynomial[α_, β_, n_] := (
    Sum[(Binomial[n+α, n-s]*Binomial[n+β, s]*((x-1)/2)^s*((x+1)/2)^(n-s)), {s, 0, n}]
)

Assert[
    Simplify[Equal[(α + 1) + (α+β+2)*(x-1)/2, JacobiPolynomial[α, β, 1]]],
    "n = 1 jacobi polynomial matches wiki solution"
]

(*Parameters of weight function*)
α = 6;
β = 2;

(*Special cases of n = 0, 1, 2*)
P0 = JacobiPolynomial[α, β, 0];
P1 = JacobiPolynomial[α, β, 1];
P2 = JacobiPolynomial[α, β, 2];
myP = 28*((x+1)/2)^2+32*(x-1)/2*(x+1)/2+6((x-1)/2)^2; (*Hand computed*)
w = (1-x)^α*(1+x)^β; (*Weight function of beta distribution*)

Print["Simplification of P_2\n"]
Simplify[P2]
Simplify[myP]
```

$$\sum_{s=0}^{n}\left(\text{Binomial}[n+\alpha,\ n-s]*\text{Binomial}[n+\beta,\ s]*\left(\frac{x-1}{2}\right)^{s}*\left(\frac{x+1}{2}\right)^{n-s}\right)$$

$$\text{myP} = 28*\left(\frac{(x+1)}{2}\right)^2+32*\frac{x-1}{2}*\frac{x+1}{2}+6\left(\frac{(x-1)}{2}\right)^2$$

```
(*How the Xiu ch. 3 shows you to compute the normalization constant γ*)
Print["Inner product w.r.t weight function w for n in [0..2]"]
P0InnerProduct = ∫_{-1}^{1} P0^2*w dx
P1InnerProduct = ∫_{-1}^{1} P1^2*w dx
P2InnerProduct = ∫_{-1}^{1} P2^2*w dx

(*Return γ_nm , the normalization constant for the inner product of jacobi polynomials

This function is from the wikipedia on Jacobi Polynomials*)
JacobiNormalizationConstantGamma[α_, β_, n_] := (
    (2^(α+β+1))/(2*n+α+β+1) * (Gamma[n+α+1]*Gamma[n+β+1])/((Gamma[n+α+β+1])*n!)
)

Assert[
    JacobiNormalizationConstantGamma[α, β, 0] == P0InnerProduct,
    "closed form solution for n=0 jacobi normalization const matches
    integrals solution"
]
Assert[
    JacobiNormalizationConstantGamma[α, β, 1] == P1InnerProduct,
    "closed form solution for n=1 jacobi normalization const matches
    integrals solution"
]
Assert[
    JacobiNormalizationConstantGamma[α, β, 2] == P2InnerProduct,
    "closed form solution for n=2 jacobi normalization const matches
    integrals solution"
]
```

Simplification of P_2

Out[50]= $\frac{1}{2}\left(1+22\,x+33\,x^2\right)$

Out[51]= $\frac{1}{2}\left(1+22\,x+33\,x^2\right)$

```
Inner product w.r.t weight function w for n in [0..2]
```

Out[53]= $\frac{128}{63}$

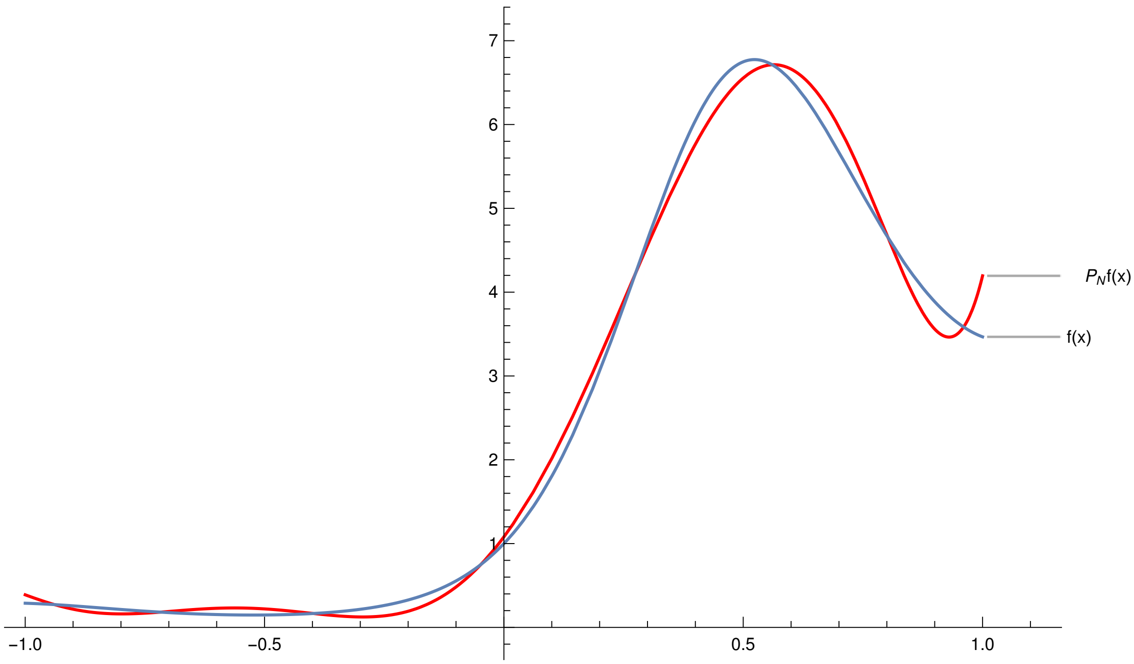
Out[54]= $\frac{128}{33}$

Out[55]= $\frac{1024}{195}$

In[23]:=

```
(*Exercise 2: Polynomial Projection *)

(*Xiu 3.3.1 p. 31*)
InnerProductLP[u_, v_, I_] := NIntegrate[u * v, I]
InnerProductNorm[u_, I_] := Sqrt[NIntegrate[u^2, I]]
FHatK[ϕ_, f_, I_] := (1/(InnerProductNorm[ϕ, I])^2)*InnerProductLP[f, ϕ, I]
OrthogonalProjectionLP[N_, I_] := (
    Sum_{k=0}^{N}(
    (*Evaluate the integrals for f̂_k in order to get the fourier coefficient*)
      f̂_k = FHatK[LegendreP[k, t], Exp[2*t + Sin[4*t]], I];
      ϕ_k = LegendreP[k, x]; (*expression with x to be evaluated later*)
      f̂_k * ϕ_k
    )
)

projectionOnF = OrthogonalProjectionLP[7, {t, -1, 1}];

(*Plots of approximation function and truth function f*)
p1 = Plot[
    projectionOnF,
    {x, -1, 1},
    PlotStyle→Red,
    PlotRange→All,
    PlotLabels→"P_N f(x)",
    ImageSize→Full];

p2 = Plot[
  Exp[2*t + Sin[4*t]],
  {t, -1, 1},
  PlotRange→All,
  ImageSize→Full,
  PlotLabels→"f(x)"];

Show[p1, p2]
```

Out[30]=

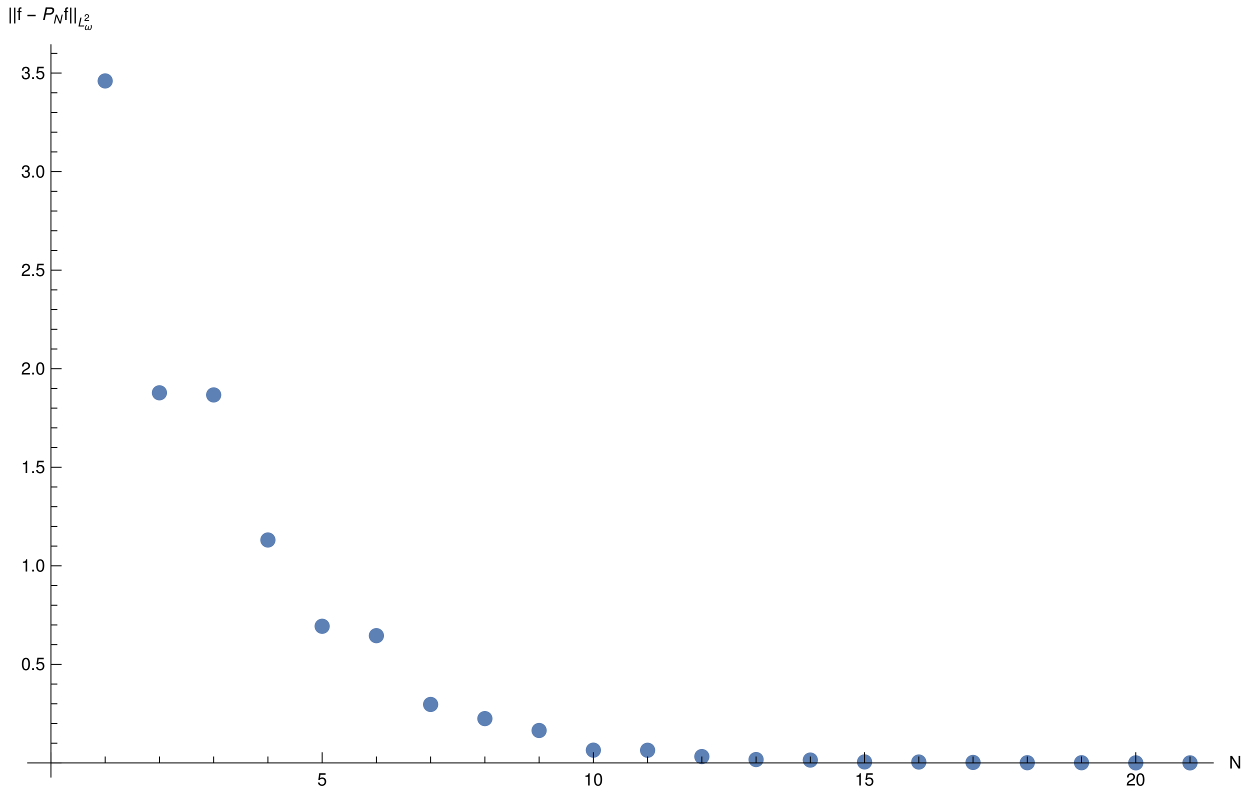


In[31]:=

```
(*Computes approximatino error of projection*)
ApproximationError[projectionOnF_, I_] := Sqrt[
  NIntegrate[Abs[Exp[2*x + Sin[4*x]] - projectionOnF]^2, I]]

(*Compute approximation errors for desired order polynomial*)
errors = {};
For[k = 0, k ≤ 20, k++,
    errors = Append[
      errors,
      ApproximationError[
        OrthogonalProjectionLP[k, {t, -1, 1}],
        {x, -1, 1}
      ]
    ]
]
```

In[34]:=
```
ListPlot[errors, AxesLabel→{"N", "||f - P_N f||_{L^2_ω}"}, ImageSize→Full]
```

Out[34]=




```
(*Exercise 2: Polynomial Interpolation
TODO: How to use the below expression as a function*)
Lagrange[x_, i_, N_] := ∏_{j=0}^{N} (If[i ≠ j, (z - x[[j]])/(x[[i]] - x[[j]]), 1]) (*Careful with indices... this is pro
InterpolatingLagrangePolynomial[x_, y_, N_] := ∑_{i=0}^{N} (y[[i]]*Lagrange[x, i, N])
InterpolatingLagrangePolynomial[x, y, 3];

(*Generate some points using f(x_i) = y_i for {x ...} and {y...}
this will lead to a general formula that can be used to estimate on
the points*)
xs = Range[-1, 1, 0.1];
ys = Exp[2*xs + Sin[4*xs]];
```

Part: Part specification x[[1]] is longer than depth of object.

Part: Part specification x[[1]] is longer than depth of object.

Part: Part specification x[[2]] is longer than depth of object.

General: Further output of Part::partd will be suppressed during this calculation.