$$h(x) = \beta^{-1} \log u(x), \quad u(x) = \sum_i e^{\beta x_i}$$

Lemma.

$$\|\nabla^{(m)} h(x)\|_1 \leq \beta^{m-1}, \quad m = \{1, 2, 3\}.$$

Proof. Denote basis elements in $I\!R^m$ as $e_i \otimes e_j \otimes e_k$.

$$\begin{aligned}
\nabla h(x) &= \beta^{-1} \frac{\nabla u}{u} \\
\nabla^2 h(x) &= \beta^{-1} \left(\frac{\nabla \otimes \nabla u}{u} - \frac{\nabla u \otimes \nabla u}{u^2} \right) \\
\nabla^3 h(x) &= \beta^{-1} \left(\frac{\nabla \otimes \nabla \otimes \nabla u}{u} - \frac{\nabla \otimes \nabla u \otimes \nabla u}{u^2} - \frac{\nabla \otimes (\nabla u \otimes \nabla u)}{u^2} + 2 \frac{\nabla u \otimes \nabla u \otimes \nabla u}{u^3} \right).
\end{aligned}$$

Define $p_i = \frac{\nabla u}{u}(i)$, which satisfying condition $\sum_i p_i = 1$. Norm of tensor T with p axes equals to convolution sup with p vectors restricted by norm $\|\gamma\|_\infty = 1$.

$$\begin{aligned}
\alpha^T \nabla^2 h(x) \gamma &= \beta \left(\sum p_i \alpha_i \gamma_i - \sum p_i \alpha_i \sum p_j \gamma_j \right) \\
&= \beta \left(I\!E \alpha \gamma - I\!E \alpha I\!E \beta \gamma \right) = \beta I\!E \overset{o}{\alpha} \overset{o}{\gamma}, \\
\nabla^3_{ijk} h(x) \alpha_i \phi_j \gamma_k &= \beta^2 \left(I\!E \alpha \phi \gamma - I\!E \alpha I\!E \phi \gamma - I\!E \alpha \phi I\!E \gamma - I\!E \alpha \gamma I\!E \phi + 2 I\!E \alpha I\!E \phi I\!E \gamma \right) \\
&= \beta^2 \left(I\!E \overset{o}{\alpha} \overset{o}{\phi} \overset{o}{\gamma} \right).
\end{aligned}$$

Using property $I\!E \overset{o}{\alpha} \overset{o}{\phi} \overset{o}{\gamma} \leq \sqrt{I\!E \alpha^2 I\!E \phi^2 I\!E \gamma^2}$. the restriction for L1 norm follows.

$\square$