Linear Algebra

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Contents

Chapter 0

Prerequisites

0.1 Sets

Lecture 01: Sets

Definition (Set). Collection of objects where objects can be almost anything (number, symbol, set, shape...)

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Note. This definition can lead to paradoxes, but its fine. Axiomatic set theories avoid this (e.g., Zermelo-Fraenkel), but usually the subtleties are not necessary.

- A set is described by the objects that **belong** to it (are **in** it)
- Sets are given (usually single, upper-cased, italicized, roman letter) names: A, B, X, P, R, T
- An object in a set is a **member** or **element** of A
- Belonging to (being a member of, being in) a set is denoted by $\in$ (e.g., $2 \in A$)

Definition 0.1.1 (Equal). Two sets, A and B are **equal** (denoted $A = B$) if every member of A is also a member of B and every member of B is also a member of A

Note. There is no order of members in a set (no "first," or "last" member)

0.1.1 Set-builder notation

- To describe a small set, we can list members explicitly with curly braces, separated by commas (e.g., $A = ☺, ☹, ♠$)
- For larger (possibly infinite) sets we describe members using a predicate
  - A is the set of students in Quiz Bowl club

  - $\mathbb{N}$ is the set of Natural numbers

- The set-builder notation, $\{x : \Phi(x)\}$, is a concise expression of this
  - $A = \{x : x \text{ is a student in Quiz Bowl club}\}$
  - $B = \{x : x^2 = 4\}$
  - $C = \{2k : k \in \mathbb{N}\}$

Definition (Predicate). A logical formula that evaluates to True ($\top$) or False ($\bot$)

Definition (Domain of Discourse). Universe of objects that can potentially be in the set if they satisfy the predicate

Usually implied from the context, but can be explicitly defined:

$$E \in \mathbb{N} : (x\%2) = 0\}$$

where $\mathbb{N}$ is the set of natural numbers (counting numbers):

$$\mathbb{N} = 1, 2, 3, \ldots$$

0.1.2 Logic

Definition (Conditional Operator). Denoted $p \Rightarrow q$ ("if p then q" or "p implies q")

$$(p \Rightarrow q) = \begin{cases} \bot & \text{if } p = \top, q = \bot \\ \top & \text{otherwise} \end{cases}$$

Definition (Conjunction Operator). Denoted $p \wedge q$ ("p and q")

$$(p \wedge q) = \begin{cases} \top & \text{if } p = \top, q = \top \\ \bot & \text{otherwise} \end{cases}$$

Definition (Disjunction Operator). Denoted $p \vee q$ ("p or q")

$$(p \vee q) = \begin{cases} \bot & \text{if } p = \bot, q = \bot \\ \top & \text{otherwise} \end{cases}$$

Definition (Negation Operator). Denoted $\neg p$ ("not p")

$$(\neg p) = \begin{cases} \top & p = \bot \\ \bot & p = \top \end{cases}$$

Definition (Biconditional Operator). Denoted $p \Leftrightarrow q$ ("p if and only if q")

$$(p \Leftrightarrow q) = (p \Rightarrow q) \wedge (q \Rightarrow p)$$

0.1.3 Set Notation and Terminology

Let A, B be sets from the same domain of discourse

Definition 0.1.2 (Subset). A is called a **subset** of B, denoted $A \subseteq B$, if

$$(x \in A) \Rightarrow (x \in B)$$

Note.

$$A \subseteq A$$

Definition 0.1.3 (Intersection). The **intersection** of A and B, denoted $A \cap B$, is the set

$$(A \cap B) = \{x : (x \in A) \wedge (x \in B)\}$$

Definition 0.1.4 (Union). The **union** of A and B, denoted $A \cup B$, is the set

$$(A \cup B) = \{x : (x \in A) \vee (x \in B)\}$$

Notation. The **empty set**, denoted $\emptyset$, contains no members

Note.

$$\emptyset \subseteq A$$

Definition 0.1.5 (Power Set). Let A be a set; the **power set of** A, denoted $P(A)$, is the set of all subsets of A:

$$P(A) = \{S : S \subseteq A\}$$

Example.

$$\begin{aligned} A &= \{\smiley, \frownie, \bell\} \\ P(A) &= \{\emptyset, \{\smiley\}, \{\frownie\}, \{\bell\}, \{\smiley, \frownie\}, \{\smiley, \bell\}, \{\frownie, \bell\}, \{\smiley, \frownie, \bell\}\} \end{aligned}$$

Definition 0.1.6 (Cartesian Product). Let A, B be sets; the **Cartesian product** of A with B, denoted $A \times B$ is the set of all ordered pairs of items, the first taken from A and the second taken from B

$$A \times B = \{(x, y) : x \in A, y \in B\}$$

Example.

$$A = \{\text{☺}, \text{☹}, \text{🔔}\}, B = \{\text{⬡}, \text{𝅗𝅥}\}$$
$$A \times B = \{(\text{☺}, \text{⬡}), (\text{☺}, \text{𝅗𝅥}), (\text{☹}, \text{⬡}), (\text{☹}, \text{𝅗𝅥}), (\text{🔔}, \text{⬡}), (\text{🔔}, \text{𝅗𝅥})\}$$

Note. We denote an ordered pair with parenthesis, not curly braces

- (x, y) is not the same as x, y because order maters
- $x, y = y, x$ but $(x, y) \neq (y, x)$

0.2 Proofs

Lecture 02: Proofs

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0.2.1 Deductive Reasoning

Definition (Deductive Reasoning)**.** The process of making deductive arguments

Definition (Deductive Argument)**.** Process of making a logical inference

Definition (Inference)**.** Claim that a certain predicate, called the **conclusion**, follows from one or more predicates, called the **premises**.

Predicate B **follows from** A if it is impossible simultaneously for A to be True and B to be False

An inference is **valid** if the conclusion follows from the premises

A deductive argument is **sound** if the inference is valid and its premises are True

0.2.2 More Logic Notation and Terminology

Definition (Universal Quantification Symbol ($\forall$))**.** Denotes that a proposition is True for all members

Example.

$$\forall x \in \mathbb{N} : x^2 \geq x$$

Read: "**for all** (every, any, each) x in $\mathbb{N}$ the predicate $(x^2 \geq x)$ evaluates to True"

Definition (Existential Quantification Symbol($\exists$))**.** Denotes that a proposition is True for at least one member

Example.

$$\exists x \in \mathbb{N} : x^2 < x$$

Read: "**there exists** an x in $\mathbb{N}$ such that the predicate $x^2 < x$ evaluates to true"

Note. This is an example of a False proposition

0.2.3 Contraposition

Definition (Contraposition). Let p, q be predicates and consider the conditional statement

$$p \Rightarrow q$$

The **contrapositive** form of the statement is

$$\neg q \Rightarrow \neg p$$

A conditional statement and its contrapositive form are **equivalent**

$$(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$$

When judging truthfulness of a statement, it sometimes helps to consider its contrapositive

Also existing, but not as common are:

- The **inverse**: $\neg p \Rightarrow \neg q$
- The **converse**: $q \Rightarrow p$
- The **complement**: $\neg(p \Rightarrow q)$

0.2.4 Proofs

Definition (Mathematical Proof). A deductive argument about something related to math. Uses spken/written language, or even sketches/diagrams.

Usually rigorous (spells out assumptions and deductive steps as is convenient) but informal (some natural language with occasionally ambiguous symbols/rules) deductive reasoning

Proposition 0.2.1. Let $n, m \in \mathbb{N}$ and suppose n, m are even; then $(n + m)$ is even

Proof.

- Since n is even $\exists k \in \mathbb{N}$ such that $n = 2k$
- Since m is even $\exists q \in \mathbb{N}$ such that $n = 2q$
- Then $(n + m) = (2k + 2q) = 2(k + q)$
- Therefore, $(n + m)$ is even

Note. We used a method called **direct proof**

$\blacksquare$

0.2.5 Mathematical Induction

Definition (Mathematical Induction). If asked to prove that a certain proposition, $P(n)$ is true for any $n \in \mathbb{N}$, we will accept as proof the following inference, if sound:

$$\begin{matrix} P(1) = \mathrm{T} \\ \left(P(k) \overset{2}{=} \mathrm{T}\right) \Rightarrow (P(k+1) = \mathrm{T}) \end{matrix} \overset{3}{\Rightarrow} P(n) = \mathrm{T}, \forall n \in \mathbb{N} \tag{1}$$

Where (1) is called the **base case**; (2) is called the **induction hypothesis**; (3) is the **induction step**