Exercice 1

$R_1(x,y) = x^2 + y^2 + 1$

$\frac{\delta f}{\delta x} = 2x$, $\frac{\delta f}{\delta y} = 2y$

$$H_{f(x,y)} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$f(u) = \frac{1}{2}uAu + b + u + c$$

$$f(x,y) = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + 1$$

$h(x,y) = x^2 - 2xy + 5y^2 + y$

$$H = \begin{pmatrix} 2 & -2 \\ -2 & 10 \end{pmatrix}$$

$\frac{\delta h}{\delta x} = 2x - 2y$, $\frac{\delta h}{\delta y} = 10y + 1 - 2x$, $\frac{\delta^2 h}{\delta x \delta y} = -2, \frac{\delta^2 h}{\delta y \delta x} = -2$, $\frac{\delta^2 h}{\delta^2 x} = 2, \frac{\delta^2 h}{\delta^2 y} = 10$

$$f(x,y) = \frac{3}{2}x^2 + y^2 - x + 2xy - 2y + \frac{5}{2}$$

$\frac{\delta f}{\delta x} = 3x - 1 + 2y, \frac{\delta f}{\delta y} = 2y + 2x - 2, \frac{\delta^2 f}{\delta x^2} = 3, \frac{\delta^2 f}{\delta y^2} = 2, \frac{\delta^2 f}{\delta xy} = 2, \frac{\delta^2 f}{\delta yx} = 2$

$$\begin{aligned}
&f(x,y) = \frac{1}{2}(x,y)\begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + (-1,-2)\begin{pmatrix} x \\ y \end{pmatrix} + \frac{5}{2} \\
&f(x,y) = \frac{1}{2}(x,y)\begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + (-1,-2)\begin{pmatrix} x \\ y \end{pmatrix} + \frac{5}{2} \\
&\Delta = 25 - 8 = 17 > 0
\end{aligned}$$

$$\begin{aligned}
x_1 &= \tfrac{-b-\sqrt{\Delta}}{2a} & x_2 &= \tfrac{-b+\sqrt{\Delta}}{2a} \\
&= \tfrac{5-\sqrt{17}}{2} & &= \tfrac{5+\sqrt{17}}{2}
\end{aligned}$$

$$\nabla f = 0 \implies \begin{cases} 3x - 1 + 2y = 0 \\ 2y + 2x - 2 = 0 \end{cases} \qquad \begin{aligned} y - 4 &= 0 \\ 2y &= 4 \\ y &= 2 \end{aligned}$$

$$x+1=0$$

$$x=0$$

$$f(-1,2) = \frac{3}{2} + \cancel{4 - 4} - 4 + \frac{5}{2} = 4 - 4 = 0$$

Erreur sur le calcul de fin, se référer à l'exercice 6

Exercice 2

$g(x,y) = \frac{e^{x+y}}{\sqrt{x+y}}$
$D_g = \{(x,y) \in \mathbb{R}^2 | x+y > 0\}$

$u = x+y$, $u > 0$, $g(u) = \frac{e^u}{\sqrt{u}}$,

$$\begin{aligned}
g'(u) = & \qquad \frac{e^u - \frac{1}{2\sqrt{u}}e^u}{u} = e^u u^{\frac{1}{2}} - \frac{1}{2}e^u u^{-\frac{3}{2}} \\
g''(u) = & \qquad e^u \left(u^{-\frac{1}{2}} - \frac{u^{-2/3}}{2} - \frac{1}{2}u^{-3/2} + \frac{3}{4}u^{-5/2} \right) \\
g''(u) = & \qquad e^u \left(u^{-1/2} - u^{-3/2} + \frac{3}{4}u^{-5/2} \right) \\
g''(u) = & \qquad e^u u^{-5/2} \left(u^2 - u + \frac{3}{4} \right)
\end{aligned}$$

$\Delta = 1^2 - 4 \times \frac{3}{4} = -2 < 0$, donc pas de racine réelle donc le polynôme est toujours positif donc g est convexe.

$$\begin{aligned}
\frac{\delta g}{\delta x} &= \frac{e^{x+y} \times \sqrt{x+y} - \frac{1}{2\sqrt{x+y}}e^{x+y}}{x+y} = \frac{e^{x+y}\left(\sqrt{x+y} - \frac{1}{2\sqrt{x+y}}\right)}{x+y} \\
\frac{\delta g}{\delta y} &= \frac{e^{x+y}\left(\sqrt{x+y} - \frac{1}{2\sqrt{x+y}}\right)}{x+y} \\
\frac{\delta g}{\delta x} &= \frac{\delta g}{\delta y}
\end{aligned}$$

$$\begin{aligned}
Z &= f(a,b) + \frac{\delta f}{\delta x}(a,b)(x-a) + \frac{\delta f}{\delta y}(a,b)(y-b) \\
Z &= e^1 + \frac{1}{2}e^1(x-1) + \frac{1}{2}ye^1
\end{aligned}$$

Exercice 3

$f(x, y) = xe^y + ye^x$
$\begin{cases} xy = 1 \\ xe^{1/x} + e^x = 0 \end{cases}$

$\dfrac{\delta f}{\delta x} = e^y + ye^x$
$\dfrac{\delta f}{\delta y} = xe^y + e^x$

$$\nabla f = \begin{pmatrix} e^y + ye^x \\ xe^y + e^x \end{pmatrix}$$

$$\nabla f = 0 \Leftrightarrow \begin{cases} e^y + ye^x = 0 \\ xe^y + e^x = 0 \end{cases} \Leftrightarrow \begin{cases} e^y = -ye^x \\ xye^x + e^x = 0 \end{cases} \Leftrightarrow e^x(1 - xy) = 0$$
$$\Leftrightarrow xy = 1 \Leftrightarrow y = \frac{1}{x} \Leftrightarrow xe^{1/x} + e^x = 0$$

$\phi(x) = xe^{1/x} + e^x$
$\phi'(x) = \dfrac{e_1 1}{x} - \dfrac{e^{1/x}}{x} + e^x$
$= e^{1/x} + \left(1 - \dfrac{1}{x}\right) + e^x$

$\phi(-1) = -e^{-1} + e^{-1} = 0$
on peut déduire que la seule racine est -1 sur $x \in]-\infty; 0[$
$1 - \dfrac{1}{x} > 0$ car $\dfrac{1 > 1}{x}$

$\dfrac{\delta^2 f}{\delta^2 x} = ye^x = -e^{-1}$
$\dfrac{\delta^2 f}{\delta^2 y} = xe^y = -e^{-1}$
$\dfrac{\delta^2 f}{\delta y \delta x} = e^y + e^x = 2e^{-1}$
$\dfrac{\delta f}{\delta x \delta y} = e^y + e^x$

$$H = \begin{pmatrix} e^{-1} & 2e^{-1} \\ 2e^{-1} & -e^{-1} \end{pmatrix}$$

$\det H = e^{-2} - 4e^{-2} = -3e^{-2}$

Exercice 4

$$f(x) = \frac{1}{1+e^x}$$
$$Df = \mathbb{R}$$
$$f'(x) = \frac{-e^x}{(1+e^x)^2}$$

$$f''(x) = \frac{-e^x(1+e^x)^2 + e^x(2e^{2x} + 2e^x)}{(1+e^x)^4}$$
$$f''(x) = \frac{e^x(1+e^x)^{\cancel{2}} + \cancel{1+e^x}\, 2e^{2x}}{(1+e^x)^{\cancel{4}3}}$$
$$f''(x) = \frac{-e^x(1+e^x) + 2e^{2x}}{(1+e^x)^3} = \frac{e^x(e^x-1)}{(1+e^x)^3}$$

$$\boxed{f''(x) = \frac{e^x(e^x-1)}{(1+e^x)^3} = \frac{-e^x + e^{2x}}{(1+e^x)^3}}$$

$$f''(x) > 0 \implies e^x - 1 > 0$$

$$f''(x) < 0 \implies e^x - 1 < 0$$
$$\implies e^x < 1$$

$f(x)$ convexe sur $]0; +\infty[$
$f(x)$ concave sur $]-\infty, 0[$

$$f\left(\sum \lambda; x;\right) \leq \sum \lambda; f(x;)$$

Soit $n \geq 2, \quad f(x) \mapsto \mathbb{R}$
Une fonction convexe alors $\forall(\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}_+^n$
De plus, $\sum_{i=1}^{n} \lambda_i = 1$ et $\forall(x_1, \dots, x_n) \in I^n$

$$f\left(\sum_{i=1}^{n} \lambda_1 x_1\right) \leq \sum_{i=1}^{n} \lambda_{i_1} f(xi)$$

Non fini

Exercice 6

$$f : \mathbb{R}^2 \to \mathbb{R}$$
$$f(x,y) = \frac{3}{2}x^2 + y^2 - x + 2xy - 2y + \frac{5}{2}$$
$$H = \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}$$
$$\frac{1}{2}(x,y)\begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - (-1,2)\begin{pmatrix} x \\ y \end{pmatrix} + \frac{5}{2}$$

$$\det\begin{pmatrix} 3-\lambda & 2 \\ 2 & 2-\lambda \end{pmatrix} = (3-\lambda)(2-\lambda) - 4 = \lambda^2 - 5\lambda 2$$
$$\delta = 17$$
$$x_1 = \frac{5+\sqrt{17}}{2}$$
$$x_2 = \frac{5-\sqrt{17}}{2}$$

$$\nabla f = 0 \Leftrightarrow \begin{pmatrix} 3x+2y-1 \\ 2y+2x-2 \end{pmatrix} = 0 \Leftrightarrow \begin{cases} 3x+2y-1=0 \\ 2y+2x-2=0 \end{cases}$$
$$\Leftrightarrow \begin{cases} 3x-2x+2-1=0 \\ y=-x+1 \end{cases} \Leftrightarrow \begin{cases} x=-1 \\ y=0 \end{cases} \quad (-1;2) \text{ Unique}$$

$$X^{(K+1)} = X^{(K)} + \alpha W^{(K)}$$
$$W^{(K)} = -\nabla f(X^{(K)})$$
$$X^0 = (0,0)$$

$$\begin{cases} 3x+2y-1 \\ -2x+2y-2 \end{cases}$$

$$(0,0) + \begin{pmatrix} 12 \\ 1 \end{pmatrix}$$

$$X^{(1)} = X^{(0)} + \alpha W^{(0)}$$
$$W^{(0)} = -\nabla f(X^{(0)})$$
$$= -\nabla f(0,0)$$
$$= -1\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$
$$X^{(1)} = \left(\frac{1}{2}, 1\right)$$
$$X^{(1)} = (0,0) + \frac{1}{2}\begin{pmatrix} \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ 1 \end{pmatrix}$$
$$f(X^{(1)}) = ?$$

$$\text{GPO}, \forall \alpha > 0, f(X^{(k)} + \alpha W^{(K)}) \leq f(X^K + \alpha^* W^{(K)})$$
$$f(x) = \langle \nabla f(U^{(0)} + \alpha U^{(0)}) | W^{(0)} \rangle$$
$$= \langle \nabla f(\alpha, 2\alpha) | (1,2) \rangle$$

$f(\alpha, 2\alpha)$ doit être minimisée
$= f_K(\alpha) = f(X^{(k)} + \alpha W^{(k)})$

Minimiser $f(\alpha, 2\alpha) = \frac{3}{2}\alpha^2 + 4\alpha^2 - \alpha + 4\alpha^4 - 4\alpha + \frac{5}{2}$
$= \frac{19}{2}\alpha^2 - 5\alpha + \frac{5}{2}$

$\Delta = 25 - 4\left(\frac{19}{2}\right)\left(\frac{5}{2}\right) = 25 - 4\left(\frac{95}{4}\right) = 25 - 95 = -70 < 0$
$\frac{5}{2\left(\frac{19}{2}\right)} = \frac{5}{19}$

$f(\alpha, 2\alpha) = 19\alpha - 5$
$19\alpha - 5 = 0$
$19\alpha = 5$
$\alpha = \frac{5}{19}$

$\alpha^* = \frac{5}{19}$
$X^{(1)} = X^{(0)} + \frac{5}{19W^{(0)}}$
$f(X^{(1)}) = f\left(\frac{5}{19}, \frac{10}{19}\right)$