1 Exercice 1

$R_1(x,y) = x^2 + y^2 + 1$

$\frac{\partial f}{\partial x} = 2x,\ \frac{\partial f}{\partial y} = 2y$

$$H_{f(x,y)} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$f(u) = \frac{1}{2}u^T A u + b^T u + c$$

$$f(x,y) = \frac{1}{2}\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + 1$$

$h(x,y) = x^2 - 2xy + 5y^2 + y$

$$H = \begin{pmatrix} 2 & -2 \\ -2 & 10 \end{pmatrix}$$

$\frac{\partial h}{\partial x} = 2x - 2y,\ \frac{\partial h}{\partial y} = 10y + 1 - 2x,\ \frac{\partial^2 h}{\partial x \partial y} = -2,\ \frac{\partial^2 h}{\partial y \partial x} = -2,\ \frac{\partial^2 h}{\partial x^2} = 2,\ \frac{\partial^2 h}{\partial y^2} = 10$

$$f(x,y) = \frac{3}{2}x^2 + y^2 - x + 2xy - 2y + \frac{5}{2}$$

$\frac{\partial f}{\partial x} = 3x - 1 + 2y,\ \frac{\partial f}{\partial y} = 2y + 2x - 2,\ \frac{\partial^2 f}{\partial x^2} = 3,\ \frac{\partial^2 f}{\partial y^2} = 2,\ \frac{\partial^2 f}{\partial x \partial y} = 2$

$$f(x,y) = \frac{1}{2}\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -1 \\ -2 \end{pmatrix}^T \begin{pmatrix} x \\ y \end{pmatrix} + \frac{5}{2}$$

$$\Delta = 25 - 8 = 17 > 0$$

$$x_1 = \frac{-b - \sqrt{\Delta}}{2a}, \quad x_2 = \frac{-b + \sqrt{\Delta}}{2a}$$

$$\nabla f = 0 \implies \begin{cases} 3x - 1 + 2y = 0 \\ 2y + 2x - 2 = 0 \end{cases} \implies x = -1, y = 2$$

$$f(-1, 2) = 0 \quad \text{(erreur sur le calcul de fin, se référer à l'exercice 6)}$$

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2 Exercice 2

$g(x,y) = \frac{e^{x+y}}{\sqrt{x+y}}$, $D_g = \{(x,y) \in \mathbb{R}^2 \mid x + y > 0\}$

$u = x + y$, $u > 0$, $g(u) = \frac{e^u}{\sqrt{u}}$

$$g'(u) = e^u u^{-1/2} - \frac{1}{2} e^u u^{-3/2} = e^u \left(u^{-1/2} - \frac{1}{2} u^{-3/2} \right)$$

$$g''(u) = e^u \left(u^{-1/2} - u^{-3/2} + \frac{3}{4} u^{-5/2} \right) = e^u u^{-5/2} \left(u^2 - u + \frac{3}{4} \right)$$

$\Delta = 1^2 - 4 \times \frac{3}{4} = -2 < 0$, donc le polynôme est toujours positif et g est convexe.

$$\frac{\partial g}{\partial x} = \frac{\partial g}{\partial y} = \frac{e^{x+y} \left(\sqrt{x+y} - \frac{1}{2\sqrt{x+y}} \right)}{x+y}$$

$$Z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

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3 Exercice 3

$$f(x,y) = xe^y + ye^x$$

$$\begin{cases} xy = 1 \\ xe^{1/x} + e^x = 0 \end{cases}$$

$$\frac{\partial f}{\partial x} = e^y + ye^x, \quad \frac{\partial f}{\partial y} = xe^y + e^x$$

$$\nabla f = \begin{pmatrix} e^y + ye^x \\ xe^y + e^x \end{pmatrix}$$

$$\nabla f = 0 \iff \begin{cases} e^y + ye^x = 0 \\ xe^y + e^x = 0 \end{cases} \iff y = \frac{1}{x}, \quad xe^{1/x} + e^x = 0$$

$$\phi(x) = xe^{1/x} + e^x, \quad \phi'(x) = -\frac{1}{x^2}e^{1/x} + e^{1/x} + e^x$$

$$\frac{\partial^2 f}{\partial x^2} = ye^x = -e^{-1}, \quad \frac{\partial^2 f}{\partial y^2} = xe^y = -e^{-1}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = e^y + e^x = 2e^{-1}$$

$$H = \begin{pmatrix} -e^{-1} & 2e^{-1} \\ 2e^{-1} & -e^{-1} \end{pmatrix}, \quad \det H = -3e^{-2}$$

4 Exercice 4

$$f(x) = \frac{1}{1+e^x}, \quad Df = \mathbb{R}$$

$$f'(x) = \frac{-e^x}{(1+e^x)^2}, \quad f''(x) = \frac{e^x(e^x-1)}{(1+e^x)^3}$$

$$f''(x) > 0 \implies x > 0, \quad f''(x) < 0 \implies x < 0$$

$$f(x) \text{ convexe sur } (0,+\infty), \quad f(x) \text{ concave sur } (-\infty, 0)$$

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5 Exercice 6

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = \frac{3}{2}x^2 + y^2 - x + 2xy - 2y + \frac{5}{2}$$

$$H = \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}, \quad \det(H - \lambda I) = (3-\lambda)(2-\lambda) - 4 = \lambda^2 - 5\lambda + 2$$

$$\Delta = 17, \quad x_1 = \frac{5+\sqrt{17}}{2}, \quad x_2 = \frac{5-\sqrt{17}}{2}$$

$$\nabla f = 0 \implies \begin{cases} 3x + 2y - 1 = 0 \\ 2x + 2y - 2 = 0 \end{cases} \implies x = -1, y = 0$$

$$X^{(K+1)} = X^{(K)} + \alpha W^{(K)}, \quad W^{(K)} = -\nabla f(X^{(K)}), \quad X^0 = (0,0)$$

$$X^{(1)} = X^{(0)} + \frac{5}{19}W^{(0)}, \quad f(X^{(1)}) = f\left(\frac{5}{19}, \frac{10}{19}\right)$$