Exercise 1: Big-O Proofs

For the following statements, either prove them or give a counterexample.

(a) $\max\left[|f(n)|, |g(n)|\right] \in \Theta(|f(n)| + |g(n)|)$.

(b) For eventually nonzero $f(n)$ and $g(n)$, $f(n) \in O(g(n))$ implies $g(n) \in O(f(n))$.

(c) For eventually nonzero $f(n)$ and $g(n)$, $f(n) \in O(g(n))$ if and only if $g(n) \in \Omega(f(n))$.

(d) $f(n) \in O(f(2n))$.

$f(n)$ being "eventually nonzero" here means that there is some n_0 so that for all $n > n_0$, $f(n) \neq 0$.

Exercise 2: Big-O Order

(a) Partition the following functions into Θ-equivalence classes, where $f \sim g$ means $f \in \Theta(g)$. List the classes in increasing order of asymptotic growth rate, denoted by $\prec$, where $C_i \prec C_j$ means $f(n) \in o(g(n))$ for all $f \in C_i$ and $g \in C_j$.

In all of these, log is the natural logarithm.

- $f(n) = n^2$
- $g(n) = n^2 + \log n$
- $h(n) = n^n$
- $i(n) = \log n$
- $j(n) = (\log n)^2$
- $k(n) = \log(n^2)$
- $l(n) = 2^{2^n}$
- $m(n) = 2^n$
- $n(n) = n!$
- $p(n) = 2^{\log n}$

You may need to use the following approximation of $n!$, called Stirling's approximation:

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + O\left(\frac{1}{n}\right)\right)$$

(b) In which of the cases in (a) does the base of the logarithm matter? I.e., in which case is a given function not in the Θ-equivalence class of the same function using a logarithm of a different base?

Exercise 3: Empirical Epsilon

(a) Write a function in R that computes, for a given positive floating point number x, the smallest number $u(x)$ such that $x + u(x) \neq x$ in machine arithmetic. Use a bisection method: start with candidates u_{lower}, for which you know that $x + u_{\text{lower}} = x$, and u_{upper}, for which you know that $x + u_{\text{upper}} \neq x$. Then, repeatedly check the midpoint of the interval $[u_{\text{lower}}, u_{\text{upper}}]$ on whether adding it to x yields x or not, and replace either the lower or upper bound by the midpoint. Make sure that your function works with very small numbers: The midpoint between u_{lower} and u_{upper} might not be representable as a value distinct from either u_{lower} or u_{upper}, your function should not get stuck in an infinite loop in this case.

(b) Use your function to calculate the value of $u(x)$ for a reasonably dense grid of values of x between 2^{-1024} and 2^{-1010} on a logarithmic scale. ("reasonably dense" here means you should not only consider integer powers of 2. Notice that these are all very small numbers.) Plot the result on a log-log plot. What do you observe?

Bonus: (think about this, but don't worry if you can't solve it) Why is $u(x)$ not monotonically non-decreasing? I.e., why are there apparently random fluctuations, with values of $x_1 < x_2$ for which $u(x_1) > u(x_2)$?