Statistical Inference 2 Summer Term 2025

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Exercise Sheet 8

Exercise 8.1 - ARMA processes

(a) Let $\{X_t\}_{t=-\infty}^{\infty}$ be a white noise process, meaning that $X_t \sim \mathcal{N}(0, \sigma^2)$ for each t. Let $Y_t = X_t^2$ and $Z_t = |X_t|$. Are $\{Y_t\}_{t=-\infty}^{\infty}$ and $\{Z_t\}_{t=-\infty}^{\infty}$ also white noise processes? Why, or why not?

(b) Consider the MA(1) process:

$$Y_t = m + \epsilon_t + \theta\epsilon_{t-1}; \qquad m \in \mathbb{R},\ -1 < \theta < 1,\ \epsilon_t \sim \mathcal{N}(0, \sigma^2),\ \sigma > 0.$$

Evaluate $\mathbb{E}[Y_t], Var(Y_t)$ and $\rho_1 = Cor(Y_t, Y_{t-1})$.

(c) Consider the AR(1) process:

$$Y_t - m = \phi(Y_{t-1} - m) + \epsilon_t; \qquad m \in,\ -1 < \phi < 1,\ \epsilon_t \sim \mathcal{N}(0, \sigma^2),\ \sigma > 0.$$

Evaluate $\mathbb{E}[Y_t], Var(Y_t)$ and $\rho_j = Cor(Y_t, Y_{t-j})$ for $j = 1, \ldots, 5$.

(d) Use the `R` function `arima.sim` to simulate and plot 250 occurrences of the processes in (c) and (d), for different values of parameters θ, ϕ. Interpret the effect of the parameters on the behaviour of the time series.

Exercise 8.2 - Stationary Time Series

(a) Suppose X_t and Y_t are two uncorrelated weakly stationary time series, meaning that $Cov(X_t, Y_s) = 0$ for each t, s. Show that $(X + Y)$ is also weakly stationary.

(b) Suppose ϵ_t to be a white noise with variance 1, for $t = 1, 2, \ldots$. Which of the following time series are stationary? Explain your reasoning in detail.

1. $X_t = \epsilon_t - \epsilon_{t+2}$;
2. $X_t = \epsilon_t + t\epsilon_{t+2}$;
3. A time series X_t such that $X_t = X_{t-1} + \epsilon_t$;
4. A time series X_t such that $X_t = \phi X_{t-1} + 1 + \epsilon_t$, with $|\phi| < 1$.

(c) Suppose X_t is a stationary time series with autocorrelation function $\rho_k = Cor(X_t, X_{t-k})$. Which of the following values are possible for the autocorrelation function?

$$(1)\ \rho_k = \begin{cases} 1 & \text{if } k = 0 \\ 0.8 & \text{if } k = 1 \\ 0.2 & \text{if } k = 2 \\ 0 & \text{if } k \geq 3 \end{cases} \qquad (2)\ \rho_k = \begin{cases} 1 & \text{if } k = 0 \\ 0.8 & \text{if } k = 1 \\ 0.5 & \text{if } k = 2 \\ 0 & \text{if } k \geq 3 \end{cases}$$

Hint: The autocorrelation matrix is positive definite.

Exercise 8.3 - Fitting Models to Time Series Data

Simulated time series data can be found in the file `Ex_8_3.txt` on Moodle. Load the data into `R` to solve the following exercises:

(a) Plot the ACF and PACF of the time series. Which process would you choose to model the data and why? Specify the process formally.

(b) Fit a linear model with two lags (i.e. with covariates X_{t-1} and X_{t-2}) to determine the coefficients for these lags. State the point estimates and compute 95% confidence intervals for them.

(c) Quickly explain what an ARMA process is. Fit an ARMA(2, 1) process to the data and state the point estimates and 95% confidence intervals.

(d) Compare the point estimates and confidence intervals from (b) and (c). Identify similarities and differences and explain their causes. Recall the underlying models' assumptions and think about what might be violated.

Hint: What if the true data-generating process is an ARMA(2, 1) process where θ_1 is relatively small?