Statistical Inference 2 Summer Term 2025

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Exercise Sheet 4

Exercise 4.1 - Bayesian Inference

An appliances company carries out a quality control check on the scales they produce. In particular, 27 scales are checked by using a standard weight of 100g. We assume the weights measured by the scales to be normally distributed with variance $0.1g^2$. However, it is not known whether the scales measure without a bias.

(a) We are interested in assessing the behavior of the mean weight μ. Specify the concept of a Bayesian credibility interval, elaborating on the underlying principles.

(b) Let us now assume that the posterior distribution $P(\mu|x_1, ..., x_{27})$ has the following form: $N(100.14g, 0.0049g^2)$. Specify the highest density 95% credibility interval (HDI) for μ. Do the scales conform to the standard of unbiasedness of the firm?

(c) Company management asks you to provide a point estimate of μ. Which are the possibilities? Which number do you give them?

(d) The prior distribution for μ is assumed to be $N(100g, 0.13g^2)$. Using results from the lecture, can you calculate the sample mean $\bar{x}$? If so, compute it, if not, explain why.

(e) Is the prior in subtask (d) an informative or uninformative prior, i.e. does it contain rather a lot or a little prior knowledge? Explain your thoughts.

Exercise 4.2 - Bayesian vs. Frequentist Inference

A quality control engineer tests $N = 500$ units of a newly developed safety-critical component. None of the tested units fail during the test period. That is, the number of observed failures is $x = 0$.

(a) You are interested in estimating the unknown failure probability θ of a single unit. First, consider a frequentist perspective:

1. Estimate θ using the maximum likelihood estimator (MLE).
2. Attempt to construct a 95% confidence interval for θ. Discuss the challenges that arise in this situation.

(b) Now consider a Bayesian approach. Assume a *Beta prior* for θ: $\theta \sim \text{Beta}(\alpha, \beta)$.

1. Choose a reasonable prior, justifying your choice (e.g., a non-informative prior or a weakly informative prior that reflects safety-critical caution).
2. Using results from the lecture, state the posterior distribution for θ, given the observed data.
3. Compute a 95% Bayesian credibility interval (Highest Density Interval if possible) for θ.

(c) Compare the frequentist and Bayesian intervals. Which one provides a more informative and realistic assessment of uncertainty in this context? Why?

Exercise 4.3 - Conjugacy vs. MCMC

One of the methods used by Astronomers to estimate the unknown mass μ of Stellar-mass black holes (1-100 solar masses) involves measuring X-rays from the accretion disc. However, actual measurements are costly and scientists thus have to work with as little data as possible. It is known that the measurements x_i are normally distributed with known measurement error (variance) $\sigma^2 = \frac{1}{5}$:

$$X_i \sim \mathcal{N}(\mu, \sigma^2) \quad (i = 1, \ldots, 5),$$

We want to estimate μ of one such black hole with the following observed data (in solar masses):

$$\boldsymbol{x} = (7.7,\ 8.3,\ 8.1,\ 8.1,\ 8.8).$$

We assume a prior distribution for μ with

$$\mu \sim \mathcal{N}(\mu_0, \tau_0^2),$$

with $\mu_0 = 8$ solar masses and $\tau_0^2 = 4$ (solar masses)^2.
Under the Normal-Normal conjugate model, the posterior for μ is

$$\mu \mid \boldsymbol{x} \sim \mathcal{N}\left(\mu_n,\ \tau_n^2\right),$$

with

$$\mu_n = \frac{\tau_0^{-2}\mu_0 + n\,\sigma^{-2}\,\bar{x}}{\tau_0^{-2} + n\,\sigma^{-2}}, \qquad \tau_n^2 = \frac{1}{\tau_0^{-2} + n\,\sigma^{-2}},$$

where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ is the sample mean and $n = 5$.

(a) Assume that point estimates $m_1, m_2, \ldots, m_{64}$ from 64 previously measured black holes are available. Explain how these could have been used to determine the prior hyperparameters μ_0 and τ_0^2. Does this approach contradict the Bayesian idea?

(b) How could empirical Bayes (methods) be used instead to specify the prior? Calculate.

(c) Compute the posterior mean μ_n and variance τ_n^2 given the observed data **and** interpret the result.

Suppose that we chose the normal likelihood only for convenience and actually, the measurements follow a Laplace distribution, i.e.

$$X_i \sim \text{Laplace}(\mu, b),$$

with known $b = \sqrt{\frac{1}{10}}$.

(d) Using this new likelihood, use MCMC to estimate the posterior of μ via Metropolis-Hastings. Implement the algorithm in R (choose reasonable settings or parameters, if necessary) and plot the posterior sample you obtain. How do we obtain a posterior point estimate for μ?