Contents

Foreword

The course FMA_3F003_EP is a continuation of the Discrete Mathematics and of the second year Algebra courses. There the basic algebraic structures (groups, rings, fields) were introduced. In this course we will go deeper into the understanding of those structures and their relationships, and we will see some geometric and arithmetical situations in which they arise.

All pictures of mathematicians in these course notes are taken from Wikipedia.

Chapter 1

Introduction to Group Theory

1.1 Definitions and Examples

Definition 1.1.1

A **group** is a set G endowed with a binary operation:

$$G \times G \to G$$
$$(g_1, g_2) \mapsto g_1 g_2$$

called the **group law** of G and satisfying the following conditions:

(G1) *Associativity:*

$$\forall (g_1, g_2, g_3) \in G^3, \quad (g_1 g_2) g_3 = g_1 (g_2 g_3). \tag{1.1}$$

(G2) *Identity:* there is an element $e \in G$ such that:

$$\forall g \in G, \quad g = eg = ge. \tag{1.2}$$

The element e is called the **identity element** of G and is often denoted by 1_G or simply by 1.

(G3) *Inverse:* for each $g \in G$, there is an element $h \in G$ such that:

$$e = gh = hg. \tag{1.3}$$

The element h is called an **inverse** of g.

A group G is said to be **abelian** (or **commutative**) if $gh = hg$ for all $g, h \in G$. In that case, the operation on G is generally denoted additively and the identity element by 0_G.

Remark 1.1.2. Let G be a group. As we saw in the previous courses (and/or in Tutorial 1), G has a unique identity, and every $g \in G$ has a unique inverse. We will denote the inverse of $g \in G$ by g^{-1}. If G is an abelian group, the inverse of $g \in G$ is often denoted by $-g$.

Example 1.1.3.

(i) The pair $(\mathbb{Z}, +)$ is an abelian group.

(ii) For each $n > 0$, the pair $(\mathbb{Z}/n\mathbb{Z}, +)$ is an abelian group.

(iii) For each field K (e.g. $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$), the pairs $(K, +)$ and $(K^\times, \times)$ are abelian groups.

(iv) Let K be a field and n a positive integer.

1. The set of invertible $n \times n$ matrices with coefficients in K endowed with matrix multiplication is a group, called the **general linear group** and denoted $GL_n(K)$ or $GL(n,K)$. This group is not abelian for $n \geq 2$. More generally, if K is a field and V is any K-vector space, the set of K-linear automorphisms $GL(V)$ of V endowed with composition is a group.
2. The $\{A \in GL_n(K) \mid \det(A) = 1\}$ endowed with matrix multiplication is a group called the **special linear group** and denoted either by $SL_n(K)$ or $SL(n,K)$.

(v) Given a set X, the set $\mathrm{Sym}(X) := \{f : X \to X \mid f \text{ is a bilection}\}$ of bijections from X to X endowed with composition is a group. It is called the **symmetric group on** X.

(v') When $X = X_n := \{1,...,n\}$, the group $\mathrm{Sym}(X_n)$ is denoted by $\mathscr{S}_n$ and is called the **symmetric group** of degree n or the symmetric group on n letters. Let us recall few facts about $\mathscr{S}_n$ that we learned in the previous courses.

An element $\sigma \in \mathscr{S}_n$ is called a **permutation** and is often denoted by:

$$\begin{pmatrix} 1 & 2 & ... & n \\ \sigma(1) & \sigma(2) & ... & \sigma(n) \end{pmatrix}.$$

Given r pairwise distinct integers $a_1,...,a_r$ in $\{1,2,...,n\}$, the permutation that sends a_i to a_{i+1} for $1 \leq i \leq r-1$, that sends a_r to a_1, and that fixes all other elements of $\{1,2,...,n\}$ is also denoted by:

$$(a_1\ a_2\ ...\ a_r)$$

and is called an r**-cycle** or a cycle of length r. Its inverse if again an r-cycle, $(a_r\ ...\ a_2\ a_1)$.

• Any permutation in S_n can be written as a product of disjoint cycles (i.e., cycles with disjoint supports) in a unique way up to the order.

• For $n \geq 3$, the symmetric group $\mathscr{S}_n$ is not abelian. For instance, we have:

$$(1\ 2)(2\ 3) = (1\ 2\ 3) \neq (1\ 3\ 2) = (2\ 3)(1\ 2).$$

Notation 1.1.4. Let G be a group and $g \in G$. We define $g^0 = 1_G$. For $n \in \mathbb{N}$, $n \geq 1$, we define inductively $g^1 = g$ and $g^{n+1} = g^n g$. Also, we define g^{-n} to be the inverse of g^n.

Definition 1.1.5

Let G be a group. If G is finite, $|G|$ is the number of elements of G and is called the **order** of G.
Let $g \in G$. The **order** of g denoted by $|g|$ or $o(g)$ is the least positive integer n such that $g^n = 1_G$, if such n exists. If there is no such n, then g has an infinite order.

Exercise 1.1.6. If $g \in G$, prove that $o(g) = 1$ if and only if $g = 1_G$.

Example 1.1.7. (1) The group $(\mathbb{Z}, +)$ is infinite and if $g \in \mathbb{Z} \setminus \{0\}$, then g has infinite order.

(2) For each $n > 0$, the group $(\mathbb{Z}/n\mathbb{Z}, +)$ is finite of order n.

(3) The group $\mathscr{S}_n$ is finite of order $n!$.

(4) Let $n \geq 3$ be an integer. In the plane $\mathbb{R}^2$, consider a regular n-gon P_n with center $(0,0)$. The set D_{2n} of isometries of P_n (under the operation of compoisition) forms a group called the **dihedral group** of order $2n$. Let us find its elements.

Observe first that the (counterclockwise) rotation r through the angle $\frac{2\pi}{n}$ about $(0,0)$ is in D_{2n}. Hence we get n rotations in D_{2n}:

$$1, r, r^2, ..., r^{n-1}$$

Let A be a fixed vertex of P, the reflection s with respect to the line passing through A and the center of P is in D_{2n}. Then D_{2n} also contains the following n symmetries:

$$s, rs, r^2 s, ..., r^{n-1} s.$$

Let's check that $D_{2n} = \{1, r, r^2, ..., r^{n-1}, s, rs, r^2 s, ..., r^{n-1} s\}$.

To do so, denote by $A_0 = A, A_1 ..., A_{n-1}$ the vertices of P in clockwise order and fix an element $\sigma \in D_{2n}$. Let $i \in \{0, ..., n-1\}$ be such that $\sigma(A_0) = A_i$ and set $\tau := r^i \sigma$. We have $\tau(A_0) = A_0$. Since τ preserves distances,

$$\tau(A_1) \in \{A_1, A_{n-1}\}.$$

If $\tau(A_1) = A_1$, then τ fixes both A_0 and A_1: hence $\tau = 1$ and $\sigma = r^{-i}$.

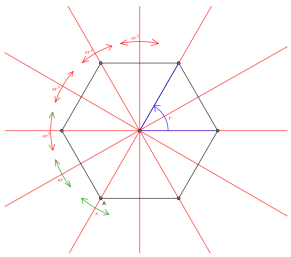
If $\tau(A_1) = A_{n-1}$, then $s\tau$ fixes both A_0 and A_1: hence $s\tau = 1$ and $\sigma = r^{-i} s$.

Since $o(r) = n$, in all cases, $\sigma \in \{1, r, r^2, ..., r^{n-1}, s, rs, r^2s, ..., r^{n-1}s\}$, and hence:

$$D_{2n} = \{1, r, r^2, ..., r^{n-1}, s, rs, r^2s, ..., r^{n-1}s\}.$$

Observe that $r^i r^{n-i} = 1$, $r^i s = sr^{n-i}$ and $(r^i s)^2 = r^i sr^i s = r^i ssr^{n-i} = 1$. One can conclude that D_{2n} is indeed a group of order $2n$. Also remark that $o(r^i s) = 2$ and $r^i s$ is a reflection for all $0 \leq i < n$.

(4a) If $n = 6$, P_n is a regular hexagon.



In this case the group D_{12} contains six rotations and six reflections.

Definition 1.1.8

A group G is called **cyclic** if there exists $g \in G$ such that for every $h \in G$, $h = g^k$ for some $k \in \mathbb{Z}$. In this case g is called a generator of G.

Example 1.1.9.

1. The group $(\mathbb{Z}, +)$ is an infinite cyclic group, $g_1 = 1$ is a generator of $\mathbb{Z}$, $g_2 = -1$ is a generator of $\mathbb{Z}$.
2. For $n \geq 1$, $(\mathbb{Z}/n\mathbb{Z}, +)$ is a cyclic group of order n.

Definition 1.1.10

Let G and H be two groups.
A **group homomorphism** (or a **group morphism**) from G to H is a map $f : G \to H$ such that:

$$\forall (g_1, g_2) \in G^2, f(g_1 g_2) = f(g_1) f(g_2).$$

If f is bijective, then its inverse is also a homomorphism and we say that f is an **isomorphism**. Two groups G and H are called **isomorphic** if there exists an isomorphism between them. In this case we write $G \cong H$.

Finally, if $G = H$ and $f : G \to G$ is an isomorphism, we say that f is an **automorphism** of G.

Exercise 1.1.11. If $f : G \to H$ is a group homomorphism, show that $f(1_G) = 1_H$ and for all $g \in G$, $f(g^{-1}) = f(g)^{-1}$.

The following proposition gives a useful criterion to check whether a homomorphism is injective:

Proposition 1.1.12

Let G and H be two groups, and let $f : G \to H$ be a homomorphism. Define the **kernel** of f by

$$\ker(f) := f^{-1}(\{1_H\}) = \{g \in G \mid f(g) = 1_H\}.$$

Then f is injective if, and only if, $\ker(f) = \{1\}$.

Proof. Assume that $\ker(f) = \{1\}$ and let x and y be elements of G such that $f(x) = f(y)$. Then $f(xy^{-1}) = f(x)f(y^{-1}) = f(x)f(y)^{-1} = 1$ and so $xy^{-1} \in \ker(f)$. Then $xy^{-1} = 1$ and $x = y$. This settles the injectivity of f. The converse follows immediately from the definition of injectivity and the fact that $f(1) = 1$. □

Exercise 1.1.13. Prove that any two infinite cyclic groups are isomorphic. Moreover, prove that if G and H are two cyclic groups with $|G| = |H| = n$, $n \geq 1$, then $G \cong H$.

Example 1.1.14.

1. The exponential map:

$$\begin{aligned} \exp : \mathbb{C} &\to \mathbb{C}^\times \\ z &\mapsto e^z \end{aligned}$$

   is a surjective group homomorphism. Its kernel is $2\pi i\mathbb{Z}$.

2. For $n \geq 2$, the signature map $\operatorname{sign} : \mathscr{S}_n \to \{-1, 1\}$ is a surjective group homomorphism. Its kernel is the **alternating group** $\mathscr{A}_n$.

3. Let K be a field and let $n \geq 1$ be an integer. The determinant map

$$\det : GL_n(K) \to K^\times$$

   is a surjective group homomorphism. Its kernel is $SL_n(K)$.

4. The group homomorphisms from $\mathbb{Z}$ to $\mathbb{Z}$ are the maps $n \mapsto an$ for $a \in \mathbb{Z}$.

5. Let G be a group and let $n \in \mathbb{Z}$ be an integer. The map:

$$\begin{aligned} G &\to G \\ g &\mapsto g^n \end{aligned}$$

is a group homomorphism if G is abelian, but not in general.

6. Let G be a group. Given $g \in G$, the map $\phi_g : G \to G$ such that

$$\phi_g(h) = ghg^{-1}$$

for $h \in G$, is a group automorphism. Such an automorphism is called an **inner** automorphism of G (Tutorial Sheet 1).

1.2 Subgroups

Definition 1.2.1

Let G be a group. A **subgroup** of G is a subset H of G such that the group law $G \times G \to G$ restricts to a group law $H \times H \to H$.
If H is a subgroup of G, we will write $H \leq G$.

Remark 1.2.2. Equivalently, a subset H of G is a subgroup of G if

$$\begin{gathered} 1_G \in H, \\ \forall (h_1, h_2) \in H^2, \quad h_1 h_2 \in H, \\ \forall h \in H, \quad h^{-1} \in H. \end{gathered}$$

Example 1.2.3.

1. The subgroups of $\mathbb{Z}$ are the subsets of the form $n\mathbb{Z}$ for some $n \in \mathbb{Z}$.

2. The subgroups of $\mathbb{Z}/n\mathbb{Z}$ are the subsets of form $\{[0], [d], [2d], ..., [n-d]\}$ with d a divisor of n.

3. If K is a field and $n \in \mathbb{N}^*$, $SL_n(\mathbb{F}) \leq GL_n(\mathbb{F})$.

4. Let G and M be two groups and $f : G \to M$ a homomorphism.
Then $\ker(f) \leq G$ and $\mathrm{im}(f) \leq M$.

   For instance, the special linear group $SL_n(K)$ is a subgroup of $GL_n(K)$ since it is the kernel of the determinant map, and the alternating group $\mathscr{A}_n$ is a subgroup of $\mathscr{S}_n$ since it is the kernel of the signature map.

5. Let $f : G \to M$ be a group homomorphism. If $H \leq G$, then $f(H) \leq M$. Also, if $K \leq M$, then $f^{-1}(K) = \{g \in G \mid f(g) \in K\}$ is a subgroup of G.

6. Let $(G_i)_{i \in I}$ be a family of subgroups of a group G. Then the intersection of all the G_i is a subgroup of G. Note however that the union need not be a subgroup of G.

The following lemma gives us a quicker way to check whether a non-empty subset of a group is a subgroup.

Lemma 1.2.4

Let G be a group. If H is a non-empty subset of G, then H is a subgroup of G iff for all $a, b \in H$, $ab \in H$ and $a^{-1} \in H$.

Proposition 1.2.5

Let G be a group and let A be a subset of G. There exists a smallest subgroup H of G containing A. It is called the **subgroup generated by** A and it is usually denoted by $\langle A \rangle$.

Proof. Let H be the intersection of all subgroups of G containing A. It is a subgroup of G containing A and every other subgroup of G containing A contains H. Hence it is the smallest subgroup of G containing A. □

Remark 1.2.6. More explicitly, if $A \neq \emptyset$, the subgroup $\langle A \rangle$ can be described as the set S of elements of G that can be written as $g_1^{\epsilon_1} ... g_r^{\epsilon_r}$ for some $r \in \mathbb{N}$, some $\epsilon_1, ..., \epsilon_r \in \{-1, 1\}$ and some $g_1, ..., g_r \in A$. If $A = \emptyset$, $\langle A \rangle = \{1_G\}$.

Example 1.2.7.

1. If G is a cyclic group and g is a generator of G, then $G = \langle g \rangle$.

2. Let $a_1, ..., a_r$ be r integers. The subgroup of $\mathbb{Z}$ generated by $a_1, ..., a_r$ is:

$$a_1\mathbb{Z} + ... + a_r\mathbb{Z} = \gcd(a_1, ..., a_r)\mathbb{Z}.$$

3. The group $\mathcal{S}_3$ is generated by the elements:

$$\sigma = (1\,2\,3), \quad \tau = (1\,2).$$

Indeed, one can easily check that $\mathcal{S}_3 = \{\mathrm{Id}, \sigma, \sigma^2, \tau, \sigma\tau, \sigma^2\tau\}$. More generally, the following sets generate the symmetric group $\mathcal{S}_n$:

- the set of all transpositions;
- the set $\{(1\,2), (2\,3), ..., ((n-1)\ n)\}$.
- the transposition $(1\,2)$ and the cycle $(1\,2\,...\,n)$.

The alternating group $\mathscr{A}_n$ is generated by the set of 3-cycles. See Tutorial Sheet 2 for more details.

4. The dihedral group D_{2n} is generated by the rotation r and a reflection s. These two generators satisfy the relations $r^n = s^2 = 1$ and $srs^{-1} = r^{-1}$.

5. Consider the following complex matrices:

$$A = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Observe that:

$$A^4 = I_2, \quad A^2 = B^2, \quad BA = A^3B.$$

The subgroup of $GL_2(\mathbb{C})$ generated by A and B is:

$$\{I_2, A, A^2, A^3, B, AB, A^2B, A^3B\}.$$

It is called the **quaternion group** and will be denoted by Q_8. It can also be described as the 8-element group $\{\pm 1, \pm i, \pm j, \pm k\}$ subject to the following relations:

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad ki = -ik = j, \quad jk = -kj = i.$$

1.2.1 Cosets

Let G be a group and H a subgroup of G. Define a relation $\mathscr{R}$ on G by:

$$x \mathscr{R} y \Leftrightarrow x^{-1}y \in H.$$

We have checked in class that this is an equivalence relation and that given $x \in G$, the equivalence class of x is the set

$$xH = \{xh | h \in H\},$$

which is called a **left coset** of H in G. The set of all left cosets of H in G is usually denoted by G/H. The number of distinct left cosets is denoted by $|G : H|$ and is called the **index** of H in G.

Remark 1.2.8. One can also define an equivalence relation:

$$x\mathscr{R}'y \Leftrightarrow xy^{-1} \in H.$$

In this case, the equivalence class of an element $x \in G$ for $\mathscr{R}'$ is

$$Hx = \{hx | h \in H\}$$

called a **right coset** of H in G. The set of all right cosets of H in G is usually denoted by $H\backslash G$.

One can define a bijection between G/H and $H\backslash G$ by:

$$\begin{aligned} G/H &\to H\backslash G \\ xH &\mapsto Hx^{-1}. \end{aligned}$$

Exercise 1.2.9. Check that this is indeed a bijection.

Theorem 1.2.10 – Lagrange's theorem

Let G be a finite group and let H be any subgroup of G. Then:

$$|G| = |H| \cdot |G : H|.$$

Proof. The left cosets form a partition of G, so that:

$$|G| = \sum_{Y \in G/H} |Y|.$$

But, for each $x \in G$, there is a bijection:

$$\begin{aligned} H &\to xH \\ h &\mapsto xh, \end{aligned}$$

and hence each left coset has $|H|$ elements. We deduce that:

$$|G| = \sum_{Y \in G/H} |H| = |G : H| \cdot |H|.$$

□

Joseph-Louis Lagrange (Turin 1736 - Paris 1813)

Corollary 1.2.11

Let G be a finite group and let $x \in G$. The order of x divides $|G|$.

Proof. Apply Lagrange's theorem to $H = \langle x \rangle$. □

1.3 Normal Subgroups and Quotient Groups

Definition 1.3.1

Let G be a group and let N be a subgroup of G. Then N is called a **normal** subgroup of G if for all $g \in G$,

$$gN = Ng$$

If N is a normal subgroup of G, we will write either $N \triangleleft G$ or $N \trianglelefteq G$.

Example 1.3.2. 1. If G is a group, the subgroups $\{1_G\}$ and G are both normal subgroups of G.

2. If G is an abelian group and N is a subgroup of G, then $N \trianglelefteq G$.

3. Let $G = S_3$. Then $N = \{1, (1,2,3), (3,2,1)\} \trianglelefteq G$, while subgroup $H = \{1, (1,2)\}$ is not normal in G.

In fact, there is a different way to decide whether a given subgroup is normal or not.

Proposition 1.3.3

Let G be a group and N a subgroup of G. Then N is a normal subgroup of G if and only if for all $g \in G$ and $n \in N$, $gng^{-1} \in N$.

Proof. Let N be a normal subgroup of G. Take any $g \in G$ and $n \in N$. Then $gN = Ng$, and so $gn \in gN = Ng$. Thus there exists $n' \in N$ such that $gn = n'g$, and so $gng^{-1} = n' \in N$.

Assume now that $gng^{-1} \in N$ for all $g \in G$ and $n \in N$. Then for any $gn \in gN$, as $gng^{-1} \in N$, there exists $n_1 \in N$ with $gng^{-1} = n_1$. Hence, $gn = n_1 g \in Ng$, and so $gN \subseteq Ng$. Now, $g^{-1}ng \in N$ and so there exists $n_2 \in N$ such that $g^{-1}ng = n_2$. Hence, $ng = gn_2 \in gN$, and so $Ng \subseteq Ng$. Thus $gN \subseteq Ng$ and $Ng \subseteq gN$, and so $gN = Ng$. □

Example 1.3.4. 1. For $n \in \mathbb{N}$, $n \geq 1$, $A_n \trianglelefteq S_n$.

2. Let $n \in \mathbb{N}$, $n \geq 1$. Then $SL(n,\mathbb{R}) \trianglelefteq GL(n,\mathbb{R})$.

These examples can be generalised in the following way.

Lemma 1.3.5

Let $f : G \to M$ be a group homomorphism. Then $\ker(f) \trianglelefteq G$.

Proof. Let $f : G \to M$ be a group homomorphism. We have seen last time that a kernel of a homomorphsim is a subgroup of G. Now take any $g \in G$ and any $k \in \ker(f)$ and consider gkg^{-1}. Since $f(k) = 1$, we have

$$f(gkg^{-1}) = f(g)f(k)f(g^{-1}) = f(g)f(k)f(g)^{-1} = f(g)1_H f(g)^{-1} = 1_H,$$

and so $gkg^{-1} \in \ker(f)$. Using Proposition 1.3.3, we obtain that $\ker(f) \trianglelefteq G$. □

Notation 1.3.6. Let G be a group and $A \subseteq G$ and $B \subseteq G$. Then

$$AB := \{ab \mid a \in A, b \in B\}$$

Lemma 1.3.7

Let G be a group and N a normal subgroup of G. Then for $x, y \in G$,

$$(Nx)(Ny) = N(xy)$$

Proof. Take any $x, y \in G$ and $n, n_1, n_2 \in N$. Then $(n_1x)(n_2y) = n_1(xn_2)y$. Now $xn_2 \in xN = Nx$ (as $N \lhd G$), and so there exists $n_2' \in N$ such that $xn_2 = n_2'x$. Therefore $n_1(xn_2)y = n_1(n_2'x)y = (n_1n_2')(xy)$. We conclude that $(Nx)(Ny) \subseteq N(xy)$.

Now $n(xy) = (nx)(1_Gy) \in (Nx)(Ny)$, and so $N(xy) \subseteq (Nx)(Ny)$.

We may now conclude that $(Nx)(Ny) = N(xy)$. □

Remark 1.3.8. Observe that in the proof Lemma 1.3.7, we used the fact that N was a normal subgroup of G. Can you think of an example of a group and a subgroup for which Lemma 1.3.7 does not hold?

Theorem 1.3.9

Let G be a group and let N be a normal subgroup of G. Then the set of cosets of N in G forms a group under the operation of multiplication of cosets. We denote this group by G/N and call it the **quotient group** of G by N.

Proof. As we in Lemma 1.3.7, the set G/N of cosets of N in G is indeed closed under the operation of multiplication of cosets. Now it is easy to check that this operation is associative (check!). Also, as $N = N1_G$, for all $x \in G$, we have

$$(N1_G)(Nx) = N(1_Gx) = Nx = N(x1_G) = (Nx)(N1_G)$$

and

$$(Nx)(Nx^{-1}) = N(xx^{-1}) = N = N(x^{-1}x) = (Nx^{-1})(Nx).$$

Therefore, G/N is a group under multiplication of cosets with

$$1_{G/N} = N \text{ and } (Nx)^{-1} = Nx^{-1}.$$

□

Remark 1.3.10. If G is a finite group and $N \trianglelefteq G$, Lagrange's Theorem implies that that $|G/N| = |G : N| = \frac{|G|}{|N|}$.

Theorem 1.3.11 – First Isomorphism Theorem

Let $f : G \to M$ be a group homomorphism with kernel $\ker(f) := K$. Then $K \trianglelefteq G$ and

$$G/K \cong Im(f)$$

More precisely, there is an isomorphism $\overline{f} : G/K \to Im(f)$ defined by $\overline{f}(Kg) = f(g)$ for all $g \in G$.

Proof. Let us first show that $\overline{f}$ is well-defined. Take any $x, y \in G$ with $Kx = Ky$. Then $\overline{f}(Kx) = f(x)$ and $\overline{f}(Ky) = f(y)$. Since $Kx = Ky$, $x = ky$ for some $k \in K$. Hence, $f(x) = f(ky) = f(k)f(y) = 1_M f(y) = f(y)$ and so, indeed, $\overline{f}(Kx) = \overline{f}(Ky)$. Now

$$\overline{f}((Kx)(Ky)) = \overline{f}(K(xy)) = f(xy) = f(x)f(y) = \overline{f}(Kx)\overline{f}(Ky)$$

for all $Kx, Ky \in G/K$, and so $\overline{f}$ is a homomorphism. Since $Im(\overline{f}) = Im(f)$, $\overline{f}$ is surjective, and as $\ker(\overline{f}) = \{Kx \mid f(x) = 1_M\} = \{Kx \mid x \in K\} = \{1_{G/K}\}$, f is injective. Thus $\overline{f}$ is an isomorphism. □

Lemma 1.3.12

Let G be a group and N a normal subgroup of G. Then the map

$$\pi : G \to G/N$$

defined by $\pi(g) = Ng$ is a surjective homomorphism with kernel N.
This map is called the **natural homomorphism** from G to G/N.

Proof. Take any $x, y \in G$. Then $\pi(xy) = N(xy) = (Nx)(Ny) = \pi(x)\pi(y)$ and so π is a homomorphism. Finally,

$$\ker(\pi) = \{g \in G \mid \pi(g) = 1_{G/N}\} = \{g \in G \mid Ng = N\} = N.$$

□

Theorem 1.3.13 – Second Isomorphism Theorem

Let G be a group, $N \trianglelefteq G$ and $H \leq G$. Then the following conditions hold:

1. $NH \leq G$,
2. $N \cap H \trianglelefteq H$, and

3. $NH/N \cong H/(N \cap H)$.

Proof. As $N \subseteq NH$, $NH \neq \emptyset$. Take any $n_1h_1, n_2h_2 \in NH$. Then

$$(n_1h_h)(n_2h_2) \in Nh_1Nh_2 = N(h_1h_2) \subseteq NH$$

and

$$(n_1h_1)^{-1} = h_1^{-1}n_1^{-1} = (h_1^{-1}n_1^{-1}h_1)h_1^{-1} \in Nh_1^{-1} \subseteq NH$$

proving that $NH \leq G$.

Take the natural map $\pi : G \to G/N$, and consider its restriction to H:

$$\pi_H : H \to G/N.$$

Then π_H is a homomorphism, $Im(\pi_H) = \{Nh \mid h \in H\} = NH/N$. Finally,

$$\ker(\pi_H) = \ker(\pi) \cap H = N \cap H.$$

Therefore, by the First Isomorphism Theorem, $N \cap H \trianglelefteq H$ and $H/(N \cap H) \cong NH/H$. □

1.4 Direct products and finite abelian groups

1.4.1 Direct products

Let G_1 and G_2 be two groups. Consider the set

$$G_1 \times G_2 = \{(g_1, g_2) \mid g_i \in G_i\}$$

and let us define an operation on this set by

$$(a_1, b_1)(a_2, b_2) = (a_1a_2, b_2b_2) \text{ for all } a_1, b_1 \in G_1 \text{ and } a_2, b_2 \in G_2.$$

It is easy to see that this operation is associative (check!). Moreover, for all $(g_1, g_2) \in G_1 \times G_2$, $(1_{G_1}, 1_{G_2})(g_1, g_2) = (g_1, g_2)$ and $(g_1^{-1}, g_2^{-1})(g_1, g_2) = (1_{G_1}, 1_{G_2})$. Hence, by Question 1 of Tutorial 1, $G_1 \times G_2$ together with the operation defined above is a group. This group is called the *direct product* of G_1 and G_2.

Remark 1.4.1. Let G_1 and G_2 be two groups. It is easy to check that the following hold:

1. The group $G_1 \times G_2$ is abelian if and only if G_1 is abelian and G_2 is abelian.

2. The group $G_1 \times G_2$ is finite if and only if G_1 is finite and G_2 is finite.

3. If $G_1 \times G_2$ is finite, then $|G_1 \times G_2| = |G_1| \cdot |G_2|$.

4. Finally, if $n \in \mathbb{N}$ with $n \geq 2$, and $G_1, ..., G_n$ are n groups, we can define a direct product of G_1, G_2, ... G_n.

   If $G_1 = G_2 = ... = G_n = G$, we will denote the direct product of G_1, ... , G_n by G^n.

Lemma 1.4.2

Let G be a group, $N \trianglelefteq G$ and $K \trianglelefteq G$. Suppose further that $NK = G$ and $N \cap K = \{1\}$. Then $G \cong N \times K$.

Proof. Consider a map $\phi : N \times K \to G$ defined by $\phi((n,k)) = nk$ for $(n,k) \in N \times K$.
Take any two elements $(n_1, k_1), (n_2, k_2) \in N \times K$. Then

$$\phi((n_1,k_1)(n_2,k_2)) = \phi((n_1n_2, k_1k_2)) = n_1n_2k_1k_2 =$$

$$= n_1n_2k_1(n_2^{-1}k_1^{-1}k_1n_2)k_2 = n_1(n_2k_1n_2^{-1}k_1^{-1})k_1n_2k_2$$

Now $n_2k_1n_2^{-1}k_1^{-1} = (n_2k_1n_2^{-1})k_1^{-1} \in K$ as $k_1 \in K$ and $K \trianglelefteq G$.
Similarly, $n_2k_1n_2^{-1}k_1^{-1} = n_2(k_1n_2^{-1}k_1^{-1}) \in N$ as $n_2 \in N$ and $N \trianglelefteq G$, and so $n_2k_1n_2^{-1}k_1^{-1} \in K \cap N = \{1_G\}$. Therefore

$$\phi((n_1,k_1)(n_2,k_2)) = n_1(n_2k_1n_2^{-1}k_1^{-1})k_1n_2k_2 = n_1k_1n_2k_2 = \phi((n_1,k_1))\phi((n_2,k_2)),$$

and so ϕ is a homomorphism.

Clearly, ϕ is surjective as $G = NK$. Finally, if $\phi((n_1,k_1)) = \phi((n_2,k_2))$ for some $(n_1,k_1),(n_2,k_2) \in N \times K$, then $n_1k_1 = n_2k_2$, and so $n_2^{-1}n_1 = k_2k_1^{-1} \in N \cap K = \{1_G\}$. Thus $n_1 = n_2$, $k_1 = k_2$, and ϕ is injective. The result now follows. □

1.4.2 Cyclic groups

Recall that we proved (see Tutorial 2) that any two infinite cyclic groups are isomorphic. In particular, every infinite cyclic group os isomorphic to $(\mathbb{Z}, +)$. We also proved that any two finite cyclic groups of the same order are isomorphic. In particular, every cyclic group of order n is isomorphic to $(\mathbb{Z}/n\mathbb{Z}, +)$. Let us now look at some properties of cyclic groups.

Notation 1.4.3. We will denote a cyclic group of order n by C_n.

Lemma 1.4.4

Every subgroup of a cyclic group is cyclic.

Proof. Let G be a cyclic group with generator g. Let $H \leq G$. If $H = \{1\}$, H is obviously cyclic. Let H be a non-trivial subgroup of G and Let m be the smallest positive integer such that $g^m \in H$. Take any $1 \neq h \in H$. Since G is generated by g, there exists $a \in \mathbb{Z}$ such that $h = g^a$. But then there exist $q, r \in \mathbb{Z}$ such that $a = mq + r$ and $0 \leq r \leq m-1$. Hence,

$$h = g^a = g^{mq+r} = (g^m)^q g^r$$

As $h \in H$ and $g^m \in H$, $(g^m)^q \in H$ and so $(g^{mq})^{-1}h = g^r \in H$. But $0 \leq r \leq m-1$, and so $r = 0$ because of our choice of m. Therefore m divides a, and so $h = g^{mq}$ and $H = \langle g^m \rangle$. □

Lemma 1.4.5

Let $n \in \mathbb{N}$, $n \geq 1$, and let G be a cyclic group of order n. Then there exists a unique subgroup of G of order d for every divisor d of n.

Proof. Let g be a generator of G. Then $H_d := \langle g^{\frac{n}{d}} \rangle = \{(g^{\frac{n}{d}})^i \mid 1 \leq i \leq d\}$ is a subgroup of G of order d.

Let $X \leq G$ be a subgroup of order d. By the previous lemma, X is cyclic. Hence, $X = \langle x \rangle$ where $o(x) = d$. Since $x \in G$, $x = g^a$ for some $a \in \mathbb{Z}$. Then $1 = x^d = g^{ad}$. In fact, $n \mid (ad)$, for if not, $ad = nq + r$ for some $0 < r \leq n-1$, and so $g^{ad} = (g^n)^q g^r = g^r \neq 1$, a contradiction. Hence, there exists $k \in \mathbb{Z}$ such that $nk = ad$. But now $x = g^a = g^{\frac{nk}{d}} = (g^{\frac{n}{d}})^k \in H_d$. Since $o(x) = d$ and $|H_d| = d$, $H_d = \langle x \rangle = X$. □

1.4.3 Finite Abelian groups

Lemma 1.4.6

Let $m, n \in \mathbb{N}^*$. Then m and n are co-prime if and only if $C_{nm} \cong C_n \times C_m$.

Proof. Suppose $gcd(m,n) = 1$. Let G be a cyclic group of order mn and let g be a generator of G. Consider subgroups $N := \langle g^n \rangle$ and $M := \langle g^m \rangle$ of G. By Lemma 1.4.4, $N \cong C_n$ and $M \cong C_m$. Since G is abelian, $N \trianglelefteq G$ and $M \trianglelefteq G$. By the Second Isomorphism Theorem, $NM \leq G$. Hence, $N \leq NM$ and $M \leq NM$, and

so, by Lagrange's Theorem, $n \mid |NM|$ and $m \mid |NM|$. Since $gcd(m,n) = 1$, $nm \leq |NM|$, and so $nm \leq |NM| \leq |G| = nm$. Thus $G = NM$. Again using Lagrange's Theorem, we obtain that $N \cap M = \{1_G\}$ (for $N \cap M \leq N$ and $N \cap M \leq M$). By Lemma 1.4.2, we conclude that $G \cong C_n \times C_m$.

For the rest of the proof, see Tutorial 3. If $(m,n) \neq 1$, there exists a prime p dividing both n and m. Let $H = C_n$ and $K = C_m$. By Lemma 1.4.4, there exist $H_p \leq H$ with $H_p \cong C_p$ and $K_p \leq K$ with $K_p \cong C_p$. Then the group $H \times K$ contains more than one subgroup of order p (subgroups $H_p \times \{1_K\}$ and $\{1_H\} \times K_p$), and so $H \times K$ is not cyclic. In particular, $C_n \times C_m \ncong C_{nm}$. □

Proposition 1.4.7. *If p is a prime and G is a group whose order is a power of p, then $G \cong C_{p^{a_1}} \times ... \times C_{p^{a_k}}$ for some $k \geq 1$ and $a_1, ..., a_k \in \mathbb{N}^*$.*

Proof. Let us prove the result by induction on n where $|G| = p^n$. If $n = 1$, $G \cong C_p$ (see Tutorial 2). Assume the result is true if $1 \leq k < n$. Suppose now that G is a group of order p^n.

Choose an element $g \in G$ so that $p^a := o(g) \geq o(x)$ for all $x \in G$. Choose $K \leq G$ maximal subject to $\langle g \rangle \cap K = \{1\}$. If $G = \langle g \rangle K$, then using Lemma 1.4.2, we obtain that $G \cong \langle g \rangle \times K$. Since $|K| < |G|$, by induction, $K \cong C_{p^{a_2}} \times ... \times C_{p^{a_k}}$ for some $k \geq 1$ and $a_2, ..., a_k \in \mathbb{N}^*$, and so $G \cong C_{p^a} \times C_{p^{a_2}} \times ... \times C_{p^{a_k}}$ which proves the result.

Thus it remains to show that $G = \langle g \rangle K$. Assume not. Choose $x \in G \setminus \langle g \rangle K$ so that $o(x) \leq o(y)$ for all $y \in G \setminus \langle g \rangle K$. Remark that $o(x) \neq 1$, and by the corollary to the Lagrange's theorem, $o(x)$ is a power of p. Then $o(x^p) < o(x)$, and so $x^p \in \langle g \rangle K$. Hence, there exist $m \in \mathbb{Z}$ and $k \in K$ such that $x^p = g^m k$. Raising both sides to the power p^{a-1}, we obtain that

$$x^{p^a} = g^{mp^{a-1}} k^{p^{a-1}},$$

and as $o(x) \leq o(g) = p^a$, we obtain that $g^{mp^{a-1}} k^{p^{a-1}} = 1$. Hence, $g^{mp^{a-1}} \in K \cap \langle g \rangle$, and so $m = pm_0$ with $m_0 \in \mathbb{Z}$. Then $x^p = g^{pm_0} k$. Denote by $c = x^{-1} g^{m_0}$. Then $c \notin K$ (as $x \notin \langle g \rangle K$), but $c^p = x^{-p} g^m = k^{-1} \in K$.

Consider $K_0 := \langle c, K \rangle$. Clearly, $K \leq K_0$, but $K \neq K_0$. From the maximal choice of K, we have that $\langle g \rangle \cap K_0 \neq \{1\}$. Hence, there exist $t, l \in \mathbb{Z}$ and $k_0 \in K$ such that $1 \neq g^t = c^l k_0 = x^{-l} g^{m_0 l} k_0$. If $p \mid l$, then $l = pl_0$ with $l_0 \in \mathbb{Z}$, and so

$$g^t = x^{-pl_0} g^{m_0 pl_0} k_0 = (x^{-p} g^m)^{l_0} k_0 = (c^p)^{l_0} k_0 \in K,$$

and so $1 \neq g^t \in K \cap \langle g \rangle$, a contradiction.

Thus $(p,l) = 1$, and so, as $g^t = x^{-l} g^{m_0 l} k_0$, $x^l = g^{m_0 l - t} k_0 \in \langle g \rangle K$, we have $x \in \langle g \rangle K$, a contradiction. This completes the proof. □

In fact, more general result holds.

Theorem 1.4.8 – Classification of finite abelian groups

Any finite abelian group G can be written as:

$$G \cong \mathbb{Z}/n_1\mathbb{Z} \times ... \times \mathbb{Z}/n_k\mathbb{Z}$$

for some $n_1, ..., n_k > 1$. Moreover:

(i) the integers $n_1, ..., n_k$ can be chosen to be powers of prime numbers, and then they are uniquely determined by G up to permutation.

Alternatively:

(ii) the integers $n_1, ..., n_k$ can be chosen so that $n_k | n_{k-1} | ... | n_1$, and then they are uniquely determined by G.

Leopold Kronecker (Liegnitz 1823 - Berlin 1891)

We proceed in four steps:

Step 1: *Let G be an abelian group and let $g_1, ..., g_k$ be generators of G. Let $n_1, ..., n_k$ be non-negative integers such that* $\gcd(n_1, ..., n_k) = 1$. *Then one can*

find generators $h_1, ..., h_k$ of G with $h_1 = n_1 g_1 + ... + n_k g_k$.

Proof. We proceed by induction on $s := n_1 + ... + n_k$. If $s = 1$, there is nothing to prove. Assume now that $s > 1$. Without loss of generality, we can assume that $n_1 \geq n_2 \geq ... \geq n_k$, so that $n_2 > 0$. Observe that:

$$\begin{cases} G = \langle g_1, g_1 + g_2, g_3, ..., g_k \rangle, \\ \gcd(n_1 - n_2, n_2, ..., n_k) = 1, \\ (n_1 - n_2) + n_2 + ... + n_k = s - n_2 < s. \end{cases}$$

Hence, by the inductive assumption, we can find generators $h_1, ..., h_k$ of G with:

$$h_1 = (n_1 - n_2) g_1 + n_2 (g_1 + g_2) + n_3 g_3 + ... + n_k g_k = n_1 g_1 + ... + n_k g_k.$$

□

Step 2: *Any finite abelian group is a direct product of cyclic groups.*

Proof. Let G be an abelian group and let k be the minimal number of generators of G. We proceed by induction on k. For $k = 1$, the group G is cyclic and therefore there is nothing to prove. Now assume that $k \geq 2$ and consider the set $\mathscr{E}$ of k-tuples $(g_1, ..., g_k) \in G$ such that:

$$\begin{cases} G = \langle g_1, ..., g_k \rangle, \\ \mathrm{ord}(g_1) \leq ... \leq \mathrm{ord}(g_k) \end{cases}.$$

Fix an element $(g_1, ..., g_k)$ of $\mathscr{E}$ for which $\mathrm{ord}(g_1)$ is minimal. By the inductive assumption, we know that $\langle g_2, ..., g_k \rangle$ is a product of cyclic groups. Hence it is enough to prove that:

$$G = \langle g_1 \rangle \times \langle g_2, ..., g_k \rangle.$$

To do so, we proceed by contradiction and assume that the previous equality is not true. We can therefore find integers $m_1, ..., m_k$ such that:

$$m_1 g_1 = m_2 g_2 + ... + m_k g_k \neq 0.$$

We may and do assume that $0 < m_1 < \mathrm{ord}(g_1)$ and that $m_i < 0$ for $i \geq 2$. Set $d := \gcd(m_1, ..., m_k)$. By step 1, we can find $h_1, ..., h_k \in G$ such that $G = \langle h_1, ..., h_k \rangle$ and

$$h_1 = \frac{m_1}{d} g_1 - \frac{m_2}{d} g_2 - .... - \frac{m_k}{d} g_k.$$

But then $d h_1 = 0$ and hence $\mathrm{ord}(h_1) < \mathrm{ord}(g_1)$. This contradicts the minimality of $\mathrm{ord}(g_1)$. □

Step 3: *Proof of (i).*

Proof. By the second step, G can be written as:

$$G \cong \mathbb{Z}/n_1\mathbb{Z} \times ... \times \mathbb{Z}/n_k\mathbb{Z}$$

for some $n_1, ..., n_k > 1$. By Lemma 1.4.3, for any $n > 1$, we have:

$$\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/p_1^{\alpha_1}\mathbb{Z} \times ... \times \mathbb{Z}/p_r^{\alpha_r}\mathbb{Z},$$

where $n = p_1^{\alpha_1}...p_r^{\alpha_r}$ is the prime decomposition of n. This proves that $n_1, ..., n_k$ can be chosen to be powers of prime numbers. In other words, G can be written as:

$$G \cong \prod_p \prod_{i=1}^{\infty} (\mathbb{Z}/p^i\mathbb{Z})^{\beta_{p,i}},$$

where p runs through the set of prime numbers and each $\beta_{p,i}$ is a non-negative integer. We then have, for each p and each i:

$$\beta_{p,i} = \dim_{\mathbb{F}_p}(p^iG/p^{i+1}G) - \dim_{\mathbb{F}_p}(p^{i-1}G/p^iG).$$

Hence $n_1, ..., n_k$ are uniquely determined by G up to permutation. □

Step 4: *Proof of (ii).*

Proof. By step 3, we can write:

$$G \cong \prod_{p \in \mathscr{P}} \prod_{j=1}^{+\infty} \mathbb{Z}/p^{\alpha_j^{(p)}}\mathbb{Z},$$

where $\mathscr{P}$ is a finite set of prime numbers and for each $p \in \mathscr{P}$, the sequence $(\alpha_j^{(p)})_{j \in \mathbb{N}}$ is a non-increasing sequence of non-negative integers that stabilizes to 0. Let k be the largest integer such that not all the $\alpha_k^{(p)}$ are 0. By the Chinese remainder theorem, we then have:

$$G \cong \prod_{i=1}^{k} \mathbb{Z}/n_i\mathbb{Z}$$

with $n_i = \prod_{p \in \mathscr{P}} p^{\alpha_i^{(p)}}$ for each i. By construction, $n_k | n_{k-1} | ... | n_1$. For the uniqueness, let $\mathbb{P}$ be the set of all prime numbers and observe that:

$$k = \max\{\dim_{\mathbb{F}_p} G/pG | p \in \mathbb{P}\}.$$

The integer k is therefore uniquely determined by G. We denote it by $k(G)$. Now, for each positive integer n, we have:

$$nG \cong \prod_{i=1}^{k} \mathbb{Z}/(n_i \gcd(n_i, n)^{-1})\mathbb{Z}.$$

Hence, for each $i \in \{1, ..., k\}$, we have $n_i = \min\{n | k(nG) = i - 1\}$ and n_i is completely determined by G. □

1.4.4 The Jordan-Hölder decomposition

Definition 1.4.9

We say that a group G is **simple** if it is not trivial and its only normal subgroups are $\{1\}$ and G.

Example 1.4.10. Let p be a prime. A cyclic group of order p is simple (proved).

Lemma 1.4.11

An abelian group is simple if, and only if, it is a cyclic group of prime order.

Proof. If G is a cyclic group of prime order, it is abelian and simple (see example above).

Assume that G is abelian and simple. If G is non-cyclic, for any $g \in G \setminus \{1\}$, $G \neq \langle g \rangle$, and, as G is abelian, $\langle g \rangle \lhd G$, contradicting the simplicity of G. Thus G is cyclic.

If G is infinite, $G \cong \mathbb{Z}$. Since $2\mathbb{Z}$ is a proper subgroup of G and G is abelian, G has a proper non-trivial normal subgroup, contradicting the simplicity of G.

Hence, G is a finite cyclic group. Let $n = |G|$. If n is not prime, take a non-trivial proper divisor d of G. By Lemmas 1.4.5, G has a non-trivial subgroup of order d. Since G is abelian, this subgroup is normal, a contradiction. The result now follows. □

Definition 1.4.12

(i) A **composition series** of a group G is a (finite) series of subgroups

$$G = G_0 \supset G_1 \supset ... \supset G_r = \{1_G\}$$

such that for each $i \in \{1, ..., r\}$, $G_i \trianglelefteq G_{i-1}$ and the quotient G_{i-1}/G_i

is simple.

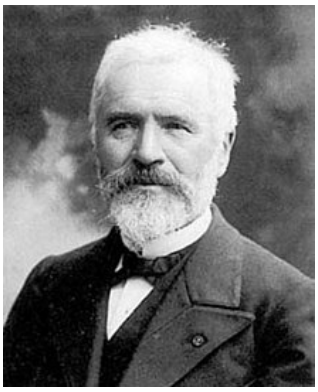
(ii) Two composition series $G = G_0 \supset G_1 \supset \ldots \supset G_r = \{1_G\}$ and $G = \widetilde{G}_0 \supset \widetilde{G}_1 \supset \ldots \supset \widetilde{G}_s = \{1_G\}$ are **equivalent** if $r = s$ and there exists a permutation $\sigma \in \mathscr{S}_r$ such that $G_{i-1}/G_i \cong \widetilde{G}_{\sigma(i)-1}/\widetilde{G}_{\sigma(i)}$.

Example 1.4.13. 1. If $G = S_3$, take $G_1 = A_3$. Then $G > G_1\{1_G\}$ is a composition series of G.

2. If $G = \langle g \rangle \cong C_6$, then $G_1 := \langle g^2 \rangle \cong C_3$ and $\widetilde{G}_1 := \langle g^3 \rangle \cong C_2$. Then $G > G_1 > \{1_G\}$ is a composition series of G and $G > \widetilde{G}_1 > \{1_G\}$ is a composition series of G.

Theorem 1.4.14 – Jordan-Hölder decomposition

Every finite group has a composition series, which is unique up to equivalence.



Camille Jordan
(Lyon 1838 - Paris 1922)

Otto Hölder
(Stuttgart 1859 - Leipzig 1937)

Proof of existence in theorem 1.4.14. Let G be a finite group. Let us prove that G has a composition series by induction on the order of G. If G is simple, then $G \supset \{1_G\}$ is a composition series of G. Otherwise, we can find a maximal normal proper subgroup G_1 of G. By the inductive assumption, the group G_1 has a composition series. Hence it suffices to check that the quotient G/G_1 is simple. To do so, denote by $\pi : G \to G/G_1$ the natural projection and let H be a normal subgroup of G/G_1. The pre-image $\pi^{-1}(H)$ is then a normal subgroup of G that contains G_1 (see Tutorial 3). By maximality of G_1, we deduce that $\pi^{-1}(H) = G$ or $\pi^{-1}(H) = G_1$. Hence either $H = G/G_1$ or H is trivial, as wished. □

Lemma 1.4.15

Let G be a group and let H and K be distinct normal subgroups of G such that G/H and G/K are simple. Then $H \cap K$ is normal in H and there is an isomorphism:

$$H/(H \cap K) \cong G/K.$$

Proof. The normality of $H \cap K$ in H is straightforward. By composing the inclusion $H \subseteq G$ with the projection $G \to G/K$, we get a morphism:

$$H \to G/K.$$

whose kernel is $H \cap K$. We can therefore see $H/(H \cap K)$ as a subgroup of G/K. It is moreover a normal subgroup since H is normal in G. Hence, by simplicity of G/K, we get that $H/(H \cap K) = G/K$ or $H/(H \cap K) = \{1\}$.

Assume that $H/(H \cap K) = \{1\}$. Then $H \subseteq K$ and K/H is a normal subgroup of the simple group G/H. This implies that $K = H$ or $K = G$: contradiction! Hence $H/(H \cap K) \neq \{1\}$, and the only remaining possibility is that $H/(H \cap K) = G/K$. □

Proof of uniqueness in theorem 1.4.14. Let G be a finite group. Let's prove the uniqueness of its composition series. To do so, let $r(G)$ be the minimal length of a composition series of G, and let's proceed by induction on $r(G)$. If $r(G) = 1$, then G is simple and hence there is nothing to prove.

Now assume that $r(G) \geq 2$ and consider two composition series:

$$G = H_0 \rhd H_1 \rhd \ldots \rhd H_{r(G)} = 1,$$
$$G = K_0 \rhd K_1 \rhd \ldots \rhd K_s = 1,$$

with $s \geq r(G)$. We want to check that these two composition series are equivalent.

First case. Assume that $H_1 = K_1$. Then the sequences $(H_i)_{i \geq 1}$ and $(K_i)_{i \geq 1}$ are both composition series of H_1 and hence, by the inductive assumption, they are equivalent. We deduce that the composition series $(H_i)_{i \geq 0}$ and $(K_i)_{i \geq 0}$ of G are also equivalent.

Second case. Assume now that $H_1 \neq K_1$ and consider a composition series of $H_1 \cap K_1$:

$$H_1 \cap K_1 = L_2 \rhd L_3 \rhd \ldots \rhd L_t = 1.$$

By lemma 1.4.15, we have two isomorphisms:

$$H_1/(H_1 \cap K_1) \cong G/K_1 \quad \text{and} \quad K_1/(H_1 \cap K_1) \cong G/H_1. \tag{1.4}$$

In particular, the quotients $H_1/(H_1 \cap K_1)$ and $K_1/(H_1 \cap K_1)$ are simple, so that the four sequences:

$$G \rhd H_1 \rhd H_2 \rhd \ldots \rhd H_{r(G)} = 1, \tag{1.5}$$
$$G \rhd H_1 \rhd H_1 \cap K_1 = L_2 \rhd L_3 \rhd \ldots \rhd L_t, \tag{1.6}$$
$$G \rhd K_1 \rhd H_1 \cap K_1 = L_2 \rhd L_3 \rhd \ldots \rhd L_t, \tag{1.7}$$
$$G \rhd K_1 \rhd K_2 \rhd \ldots \rhd K_s = 1, \tag{1.8}$$

are all composition series of G. By the first case, the composition series (1.5) and (1.6) are equivalent, as well as (1.8) and (1.7). Moreover, the isomorphisms (1.4) show that (1.6) and (1.7) are also equivalent. Hence (1.5) and (1.8) are equivalent. □

Definition 1.4.16

Let G be a finite group with a composition series:

$$G = G_0 \trianglerighteq G_1 \trianglerighteq \ldots \trianglerighteq G_r = \{1\}$$

Then r is called the *length* of the composition series, and the quotients G_0/G_1, G_1/G_2, ..., G_{r-1}/G_r are called the **composition factors** of G.

1.4.5 Commutators and commutator subgroups

Definition 1.4.17

Let G be a group. For $x \in G$ and $y \in G$, the **commutator** of x and y is the element $[x,y] = xyx^{-1}y^{-1}$.
The **derived subgroup** of G (or the **commutator subgroup** of G) is the subgroup $[G,G]$ generated by the set of all commutators of G, i.e.,

$$[G,G] = \langle \{[x,y] \mid x,y \in G\} \rangle$$

Example 1.4.18.

1. If G is a group and $x, y \in G$, then $[x,y] = 1$ iff $xy = yx$.
2. If G is a group and $x, y \in G$, then $[x,y]^{-1} = [y,x]$.

3. A group G is abelian iff $[G,G] = \{1\}$.

4. If $G = A_n$, $n \geq 5$, $[(1,2,4),(1,3,5)] = (1,2,3)$.

Lemma 1.4.19

The commutator subgroup of a group is normal.

In fact, the following result holds.

Lemma 1.4.20

Let G be a group and N a normal subgroup of G. Then G/N is abelian if and only if $[G,G] \leq N$.

Proof. Take any $x, y \in G$. Then $[Nx, Ny] = NxNyNx^{-1}Ny^{-1} = N[x,y]$.

Then G/N is abelian iff $[Nx, Ny] = 1_{G/N}$ for all $x, y \in G$ iff $[x,y] \in N$ for all $x, y \in G$ iff $[G,G] \leq N$. □

1.4.6 Solvable groups

Definition 1.4.21

Let G be a group. Define the following series of subgroups:

$$G^{(0)} = G,$$
$$G^{(i)} = [G^{(i-1)}, G^{(i-1)}], \quad i \in \mathbb{N}^*.$$

This series is called the **derived series** of G.

Exercise 1.4.22. For each $i \in \mathbb{N}$, $G^{(i)} \trianglelefteq G$.

Definition 1.4.23

A group G is solvable if there exists a positive integer n with $G^{(n)} = \{1\}$.

Example 1.4.24.

1. Every abelian group is solvable.

2. The group $G = S_3$ is solvable.

3. For $n \geq 5$, the group S_n is not solvable and the group A_n is not solvable.

Lemma 1.4.25

If a group G is both solvable and simple, then G is cyclic of prime order.

Proof. Since G is solvable, $G^{(1)}$ is a proper normal subgroup of G. But G is simple, and so $G^{(1)} = \{1\}$. Now $G/G^{(1)}$ is abelian, hence, by Lemma 1.4.11, $G \cong C_p$ for some prime p. □

Proposition 1.4.26

Let G be a solvable group.

(i) If H is a subgroup of G, then H is also solvable.

(ii) If $f : G \to M$ is a surjective group homomorphism, then M is solvable.

(iii) Every homomorphic image of G is solvable.

(iv) If $N \trianglelefteq G$, then G/N is solvable.

(v) Assume that H is normal in G. The group G is solvable if, and only if, H and G/H are solvable.

Proof. (i) By definition, there exists a positive integer n such that $G^{(n)} = \{1\}$. Since $H^{(n)} \subseteq G^{(n)}$, we deduce that $H^{(n)} = \{1\}$ and so the group H is solvable.

(ii) Observe that for $x, y \in G$, $f([x,y]) = [f(x), f(y)]$, and so $f([G,G]) \leq [M,M]$. Take any $m \in [M,M]$. Then there exist $a_1, ...a_k, b_1, ..., b_k \in M$ such that $m = [a_1, b_1]...[a_k, b_k]$. Since f is surjective, for $1 \leq i \leq k$, there exist $x_i, y_i \in G$ with $f(x_i) = a_i$ and $f(y_i) = b_i$. But then $[a_i, b_i] = f([x_i, y_i])$, and so $m = f([x_1, y_1]...[x_k, y_k])$. Thus $f([G,G]) = [M,M]$. Similarly, one can show that $f(G^{(i)}) = M^{(i)}$ for any $i \in \mathbb{N}$. Since G is solvable, $G^{(n)} = \{1_G\}$ for some $n \in \mathbb{N}$, and so $M^{(n)} = \{1_H\}$ proving that M is solvable.

(iii) Let $\phi : G \to L$ be a group homomorphism. Apply part (ii) to $\phi_0 : G \to Im(\phi)$ where $\phi_0(g) = \phi(g)$ for every $g \in G$.

(iv) Consider the natural map $\pi : G \to G/N$. Since π is a group homomorphism, we may apply part (ii) to obtain the desired conclusion.

(v) Assume that G is solvable. We have just seen that H is solvable. Now denote by $\pi : G \to G/H$ the natural homomorphism, and observe (see Tutorial 5) that for each $m \geq 1$,

$$(G/H)^{(m)} = \pi(G^{(m)})$$

In particular, if n is a positive integer such that $G^{(n)} = \{1\}$, then $(G/H)^{(n)} = \{1_{G/H}\}$ and hence G/H is solvable by Proposition **??**.

Conversely, assume that H and G/H are both solvable. Fix two integers n and m such that $H^{(m)} = \{1\}$ and $(G/H)^{(n)} = \{1_{G/H}\}$. Since $(G/H)^{(n)} = \pi(G^{(n)})$, we see that $G^{(n)} \subseteq H$. But then:

$$G^{(n+m)} = (G^{(n)})^{(m)} \subseteq H^{(m)} = \{1\}.$$

Hence, G is solvable.

□

Corollary 1.4.27

Let G be a finite group. Then G is solvable if, and only if, its composition factors are cyclic groups of prime order.

Proof. Assume that G is solvable. Let

$$G \geq G_1 \geq \ldots \geq G_r = \{1\}$$

be a composition series of G. Hence, for $1 \leq i \leq r$, $G_i \trianglelefteq G_{i-1}$ and G_{i-1}/G_i is simple. Since G is solvable, by Proposition 1.4.26, G_{i-1}/G_i is solvable. Hence, by Lemma 1.4.25, each G_{i-1}/G_i is cyclic of prime order.

Assume now each composition factor of G is cyclic of prime order. Let

$$G \geq G_1 \geq \ldots \geq G_r = \{1\}$$

be a composition series of G. Let us show that $G^{(k)} \leq G_k$ for all k. Since G/G_1 is cyclic, it is abelian, and so, by Lemma 1.4.20, $G^{(1)} \leq G_1$. Take any $i \geq 1$ and assume that $G^{(i)} \leq G_i$. Then $G^{(i+1)} = [G^{(i)}, G^{(i)}] \leq [G_i, G_i]$, and as G_i/G_{i+1} is abelian, by Lemma 1.4.20, $[G_i, G_i] \leq G_{i+1}$. Thus $G^{(i+1)} \leq G_{i+1}$. Hence, indeed, $G^{(k)} \leq G_k$. In particular, $G^{(r)} = \{1\}$, and so G is solvable. □

1.5 Group actions

Definition 1.5.1

Let G be a group and let X be a set. A (left) **action** of G on X is a map:

$$\begin{aligned} \cdot : G \times X &\to X \\ (g, x) &\mapsto g \cdot x \end{aligned}$$

satisfying the following conditions:

$$\forall x \in X,\ 1 \cdot x = x, \tag{1.9}$$

$$\forall (g_1, g_2) \in G^2, \forall x \in X,\ g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x. \tag{1.10}$$

The **kernel** of the action is defined as

$$K := \{g \in G \mid g \cdot x = x\ \forall x \in X\},$$

and the action is called **faithful** if its kernel is trivial, i.e., if $K = \{1\}$.

Remark 1.5.2. Analogously one can define a **right** action of G on X as a map $X \times G \to X$. In this course we will mostly look at the left actions.

Example 1.5.3.

1. The group $\mathscr{S}_n$ acts on the set $\{1, ..., n\}$.
2. More generally, if X is a set, the group $Sym(X)$ acts on X.
3. Let K be a field. The group $GL_n(K)$ acts on K^n.
4. The group $SL_2(\mathbb{R})$ acts on the Poincaré plane $\mathscr{H} := \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}$ by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}.$$

5. The group $G = D_{2n}$ acts on a set of vertices of the regular n-gon.
6. The group $G = D_8$ acts on a set of the diagonals of a square.

Proposition 1.5.4. *Let $\cdot$ be an action of a group G on a set X. For $g \in G$, define the map $\varphi(g) : X \to X$ by*

$$\varphi(g)(x) = g \cdot x$$

Then $\varphi(g) \in Sym(X)$, and $\varphi : G \to Sym(X)$ is a homomorphism.

Proof. Assume that we are given a group G together with an action on a set X. For each $g \in G$, we can consider the map:

$$\begin{aligned} \varphi(g) : X &\to X \\ x &\mapsto g \cdot x. \end{aligned}$$

Then, for every $x \in X$ and every $g, h \in G$, we have:

$$\begin{gathered} (\varphi(g^{-1}) \circ \varphi(g))(x) = \varphi(g^{-1})(g \cdot x) = g^{-1} \cdot (g \cdot x) = (g^{-1}g) \cdot x = 1 \cdot x = x, \\ \text{similarly } (\varphi(g) \circ \varphi(g^{-1}))(x) = x, \text{ and} \\ \varphi(g) \circ \varphi(h)(x) = g \cdot (h \cdot x) = (gh) \cdot x = \varphi(gh)(x). \end{gathered}$$

Hence we can define a group homomorphism from G to $Sym(X)$ by:

$$\begin{aligned} \varphi : G &\to Sym(X) \\ g &\mapsto \varphi(g). \end{aligned}$$

□

Remark 1.5.5. Observe that conversely, for any group homomorphism

$$\varphi : G \to \mathrm{Sym}(X)$$

we can define a group action of G on X by:

$$g \cdot x = \varphi(g)(x) \quad \text{for } x \in X \text{ and } g \in G.$$

Thus to have an action of a group G on a set X is the same as to have a group homomorphism from G to $\mathrm{Sym}(X)$.

Lemma 1.5.6

Let $\cdot$ be an action of a group G on a set X and let $\varphi : G \to \mathrm{Sym}(X)$ the corresponding group homomorphism (as defined in Proposition 1.5.4). Then the kernel K of the action equals $\ker(\varphi)$, the kernel of the homomorphism φ.

Proof. Recall that $K = \{g \in G \mid g \cdot x = x \ \forall x \in X\} = \{g \in G \mid \varphi(g)(x) = x \ \forall x \in X\} = \{g \in G \mid \varphi(g) = id_X\} = \ker(\varphi)$. □

Remark 1.5.7. If K is the kernel of the action $\cdot$ of G on X and if $\varphi : G \to \mathrm{Sym}(X)$ is the corresponding group homomorphism, then the image $\varphi(G)$ of G in $\mathrm{Sym}(X)$ is isomorphic to G/K. In particular, if the action of G on X is faithful, then $\varphi(G) \cong G/\{1\} \cong G$.

Example 1.5.8.

1. The group $\mathscr{S}_n$ acts on the set $\{1,...,n\}$. The action is faithful.
2. More generally, if X is a set, the group $Sym(X)$ acts on X. The action is then faithful.
3. Let K be a field. The group $GL_n(K)$ acts on K^n. The action is faithful
4. The group $SL_2(\mathbb{R})$ acts on the Poincaré plane $\mathscr{H} := \{z \in \mathbb{C} | \text{Im}(z) > 0\}$ by:
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$
5. The group $G = D_{2n}$ acts on a set of vertices of the regular n-gon. The action is faithfull.
6. The group $G = D_8$ acts on a set of the diagonals of a square. The action is not faithful.

Definition 1.5.9

Let $\cdot$ be an action of a group G on a set X. The relation:
$$x \mathscr{R} y \Leftrightarrow \exists g \in G, x = g \cdot y$$
is an equivalence relation on X. The equivalence class of an element $x \in X$, denoted by $G \cdot x$ or by $\text{Orb}_\text{G}(\text{x})$ is the set
$$\text{Orb}_\text{G}(\text{x}) = \{g \cdot x \mid g \in G\}$$
called the **orbit** of x.
Let $x \in X$. The set $\text{Stab}_G(x) = \{g \in G \mid g \cdot x = x\}$ (also denoted by G_x) is called the **stabilizer of** x in G.

Lemma 1.5.10

Let G be a group acting on a set X and $x \in X$. Then $\text{Stab}_G(x)$ is a subgroup of G.

Proof. Since for all $x \in X$, $1_G \cdot x = x$ by Definition 1.5.1, $1_G \in \text{Stab}_G(x)$. Moreover, again using Definition 1.5.1, we obtain that for $x \in X$ and $g, h \in \text{Stab}_G(x)$, $(gh) \cdot x = g \cdot (h \cdot x) = g \cdot x = x$, i.e, $gh \in \text{Stab}_G(x)$, and $g^{-1} \cdot x = g^{-1} \cdot (g \cdot x) = (g^{-1}g) \cdot x = 1_G \cdot x = x$, i.e., $g^{-1} \in \text{Stab}_G(x)$. Thus $\text{Stab}_G(x) \leq G$. □

If $x \in X$, its orbit and stabilizer are connected by the following result.

Theorem 1.5.11 – Orbit-Stabilizer Theorem

Let G be a group acting on a set X and $x \in X$. Then there exists a bijection between the set of left cosets of $\mathrm{Stab}_G(x)$ in G and the set $\mathrm{Orb}_G(x)$.
In particular, if G is a finite group,

$$|\mathrm{Orb}_G(x)| = |G : \mathrm{Stab}_G(x)|$$

Proof. Tutorial 6.

□

Definition 1.5.12

We say that the action of G on X is **transitive** (or that G acts **transitively** on X) if G has a unique orbit on X.

Remark 1.5.13. If G acts transitively on X, then $X = \mathrm{Orb}_G(x)$ for some (and also for every) $x \in X$. Moreover, in this case, for each $x \in X$, the map f_x induces a bijection between the sets $G/\mathrm{Stab}_G(x)$ and X.

Example 1.5.14.

1. The action of $\mathcal{S}_n$ on $\{1, ..., n\}$ is transitive. The stabilizer of any element in $\{1, ..., n\}$ is isomorphic to $\mathcal{S}_{n-1}$.
2. Let K be a field. The action of $GL_n(K)$ on K^n has two orbits: $\{0\}$ and $K^n \setminus \{0\}$. The element 0 is fixed by the whole group $GL_n(K)$.
3. The action of $SL_2(\mathbb{R})$ on $\mathcal{H}$ is transitive.

1.5.1 Group Actions, Examples: Left Regular Action

Let G be a group. Take $X = G$ and define the map

$$\cdot : G \times G \to G$$

so that $g \cdot x = gx$ for every $g \in G$ and $x \in X = G$. Then for all $x, g, h \in G$,

$$1_G \cdot x = 1_G x = x \text{ and } (gh) \cdot x = (gh)x = g(hx) = g \cdot (h \cdot x)$$

Hence, G acts on itself . This action is called the **left regular** action.

Observe that for every $x \in G$,

$$\mathrm{Stab}_G(x) = \{g \in G \mid g \cdot x = x\} = \{g \in G \mid gx = x\} = \{1_G\}$$

and so the action is faithful, as

$$K = \{g \in G \mid g \cdot x = x \ \forall x \in X\} = \bigcap_{x \in X} \mathrm{Stab}_G(x) = \{1_G\}.$$

Remark that this action is transitive:if $x, y \in G$, then there exists $g = yx^{-1} \in G$ such that

$$g \cdot x = gx = (yx^{-1})x = y.$$

Observe that Proposition 1.5.4 implies the following result:

Theorem 1.5.15– Cayley Theorem

Every group is isomorphic to a subgroup of $\mathrm{Sym}(X)$ for some set X.
In particular, if $|G| = n$, then G is isomorphic to a subgroup of S_n.

1.5.2 Group Actions, Examples: Conjugation Action

Let G be a group and $X = G$. This time define the map $\cdot : G \times X \to X$ by

$$g \cdot x = gxg^{-1}$$

for $g \in G$ and $x \in X = G$. Then for $g, h, x \in G$,

$$1_G \cdot x = 1_G x 1_G^{-1} = x$$

and

$$(gh) \cdot x = (gh)x(gh)^{-1} = g(hxh^{-1})g^{-1} = g \cdot (h \cdot x)$$

Thus we defined a group action. In this case we say that G acts on itself by **conjugation**.

Let $x \in G$. The orbit of x, $\mathrm{Orb}_G(x) = \{gxg^{-1} \mid g \in G\}$ is called the **conjugacy class** of x in G and is often denoted by $Cl_G(x)$ or by x^G. Observe that if G is non-trivial, the action is not transitive (e.g., $Cl_G(1_G) = \{1_G\}$).

The stabilizer of x,

$$\mathrm{Stab}_G(x) = \{g \in G \mid gxg^{-1} = x\} = \{g \in G \mid gx = xg\}$$

is called the **centralizer** of x in G and is often denoted by $C_G(x)$.

The kernel of this action,

$$K = \{g \in G \mid gxg^{-1} = x \text{ for all } x \in G\} = \{g \in G \mid gx = xg \text{ for all } x \in G\} = Z(G)$$

is called the **center** of G.

1.5.3 Application to p-groups

Given a prime number p, a **p-group** is a finite group whose order is a power of p.

Proposition 1.5.16

Let p be a prime number and let G be a p-group.

(i) Let X be a finite set endowed with a G-action. Denote by

$$\mathrm{Fix}_X(G) = \{x \in X \mid g \cdot x = x \text{ for all } g \in G\}$$

Then:

$$|\mathrm{Fix}_X(G)| \equiv |X| \mod p.$$

(ii) If G is non-trivial, then the center $Z(G)$ of G is non-trivial.

Proof. (i) By the class formula, if R is a subset of X containing exactly one element of each orbit, we have:

$$|X| = |\mathrm{Fix}_X(G)| + \sum_{x \in R \setminus \mathrm{Fix}_X(G)} [G : \mathrm{Stab}(x)].$$

But for any $x \in R \setminus \mathrm{Fix}_X(G)$, the number $[G : \mathrm{Stab}(x)]$ is a divisor of $|G|$ different from 1 and hence is divisible by p. The desired congruence follows:

$$|X| \equiv |\mathrm{Fix}_X(G)| \mod p.$$

(ii) We consider the action of G on itself by conjugation:

$$\begin{aligned} G \times G &\to G \\ (g, h) &\mapsto ghg^{-1}. \end{aligned}$$

We then have $Z(G) = \mathrm{Fix}_G(G)$, so that, by (i):

$$|Z(G)| = |\mathrm{Fix}_G(G)| \equiv 0 \mod p.$$

Hence $Z(G)$ is not trivial.

□