MAA304: Asymptotic Statistics

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Homework 1

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Exercise 1. Suppose that R and U are independent continuous random variables where U has a Uniform distribution on $[0, 1]$ and R has the density function

$$f_R(x) = x \exp\left(-x^2/2\right) \qquad \text{for } x \geq 0.$$

1. Show that R^2 has an Exponential distribution.
2. Define $X = R\cos(2\pi U)$ and $Y = R\sin(2\pi U)$. Show that X and Y are independent standard Normal random variables.
3. Suggest a method for generating Normal random variables based on the results in parts (a) and (b).

Solution. 1. Let $Z = R^2$. Then $R = \sqrt{Z}$, and $\frac{dR}{dZ} = \frac{1}{2\sqrt{Z}}$. The density of R is

$$f_R(x) = xe^{-x^2/2}, \quad x \geq 0.$$

By the transformation formula:

$$f_Z(z) = f_R(\sqrt{z}) \cdot \left|\frac{dR}{dz}\right| = (\sqrt{z}e^{-z/2}) \cdot \frac{1}{2\sqrt{z}} = \frac{1}{2}e^{-z/2}, \quad z \geq 0.$$

This is the density of an Exponential distribution with rate parameter $\lambda = \frac{1}{2}$ (or mean 2). Thus,

$$\boxed{R^2 \sim \mathrm{Exp}(1/2)}$$

2. Let $\Theta = 2\pi U$, so $\Theta \sim \mathrm{Uniform}(0, 2\pi)$. The joint density of (R, Θ) is

$$f_{R,\Theta}(r, \theta) = f_R(r) \cdot \frac{1}{2\pi} = \frac{1}{2\pi} \cdot re^{-r^2/2}, \quad r \geq 0, \theta \in [0, 2\pi).$$

Transform $(R, \Theta) \to (X, Y)$ by $X = R\cos\Theta$, $Y = R\sin\Theta$.

The Jacobian of the transformation $(r, \theta) \to (x, y)$ is:

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

Thus, the Jacobian determinant is $|J| = r$, and since $dx\,dy = r\,dr\,d\theta$:

$$f_{X,Y}(x, y) = \frac{f_{R,\Theta}(r, \theta)}{r} = \frac{1}{2\pi}e^{-r^2/2}, \quad \text{where } r^2 = x^2 + y^2.$$

So,

$$f_{X,Y}(x, y) = \frac{1}{2\pi}e^{-(x^2+y^2)/2} = \left(\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\right)\left(\frac{1}{\sqrt{2\pi}}e^{-y^2/2}\right).$$

Therefore,

$$\boxed{X \text{ and } Y \text{ are independent } N(0, 1)}$$

3. Generate U_1, U_2 independent Uniform$(0, 1)$. Let $R^2 = -2\ln U_1$, so $R = \sqrt{-2\ln U_1}$. Let $\Theta = 2\pi U_2$. Then set

$$X = R\cos\Theta, \quad Y = R\sin\Theta.$$

Then X and Y are i.i.d. $N(0, 1)$. This is the **Box–Muller method**:

$$\boxed{X = \sqrt{-2\ln U_1}\cos(2\pi U_2), \quad Y = \sqrt{-2\ln U_1}\sin(2\pi U_2)}$$

Exercise 2. Suppose that $X_1, \cdots, X_n$ are i.i.d. continuous random variables with distribution function $F(x)$ and density function $f(x)$; let $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ be the order statistics.

1. Show that the distribution function of $X_{(k)}$ is

$$G_k(x) = \sum_{j=k}^{n} \binom{n}{j} F(x)^j (1 - F(x))^{n-j}.$$

2. Show that the density function of $X_{(k)}$ is

$$g_k(x) = \frac{n!}{(n-k)!(k-1)!} F(x)^{k-1} (1 - F(x))^{n-k} f(x).$$

Solution.

1. Let $Y_i = \mathbb{1}(X_i \leq x)$, and $S = \sum_{i=1}^n Y_i$. Then Y_i are i.i.d. Bernoulli$(F(x))$, so $S \sim$ Binomial$(n, F(x))$. The event $\{X_{(k)} \leq x\}$ means at least k of the X_i are $\leq x$, i.e., $S \geq k$. Thus,

$$\mathbb{P}(X_{(k)} \leq x) = \mathbb{P}(S \geq k) = \sum_{j=k}^{n} \binom{n}{j} F(x)^j (1 - F(x))^{n-j}.$$

$$\boxed{G_k(x) = \sum_{j=k}^{n} \binom{n}{j} F(x)^j (1 - F(x))^{n-j}}$$

2. Differentiate $G_k(x)$ term by term:

$$G_k(x) = \sum_{j=k}^{n} \binom{n}{j} F(x)^j (1 - F(x))^{n-j}.$$

For a fixed j, the derivative of $F^j(1-F)^{n-j}$ is

$$jF^{j-1}f\cdot(1-F)^{n-j} + F^j\cdot(n-j)(1-F)^{n-j-1}(-f) = f\left[jF^{j-1}(1-F)^{n-j} - (n-j)F^j(1-F)^{n-j-1}\right].$$

So,

$$g_k(x) = f(x) \sum_{j=k}^{n} \binom{n}{j} \left[jF^{j-1}(1-F)^{n-j} - (n-j)F^j(1-F)^{n-j-1}\right].$$

Note: $\binom{n}{j}j = n\binom{n-1}{j-1}$, and $\binom{n}{j}(n-j) = n\binom{n-1}{j}$. Thus, the first sum becomes

$$\sum_{j=k}^{n} n\binom{n-1}{j-1} F^{j-1}(1-F)^{n-j}.$$

Let $m = j-1$, then m runs from $k-1$ to $n-1$:

$$n \sum_{m=k-1}^{n-1} \binom{n-1}{m} F^m (1-F)^{n-1-m}.$$

The second sum becomes

$$\sum_{j=k}^{n} n\binom{n-1}{j} F^j(1-F)^{n-j-1} = n\sum_{m=k}^{n} \binom{n-1}{m} F^m (1-F)^{n-1-m}.$$

Subtracting the second from the first gives a telescoping sum, leaving only the $m = k-1$ term:

$$n\binom{n-1}{k-1} F^{k-1}(1-F)^{n-k}.$$

Therefore,

$$g_k(x) = f(x) \cdot n\binom{n-1}{k-1} F(x)^{k-1}(1-F(x))^{n-k} = \frac{n!}{(k-1)!(n-k)!} F(x)^{k-1}(1-F(x))^{n-k} f(x).$$

$$\boxed{g_k(x) = \frac{n!}{(n-k)!(k-1)!} F(x)^{k-1}(1-F(x))^{n-k} f(x)}$$

Exercise 3. Suppose that $X_1, \cdots, X_n$ are i.i.d. Exponential random variables with parameter λ. Let $X_{(1)} < \cdots < X_{(n)}$ be the order statistics and define

$$\begin{aligned} Y_1 &= nX_{(1)} \\ Y_2 &= (n-1)(X_{(2)} - X_{(1)}) \\ Y_3 &= (n-2)(X_{(3)} - X_{(2)}) \\ &\vdots \\ Y_n &= X_{(n)} - X_{(n-1)}. \end{aligned}$$

Show that $Y_1, \cdots, Y_n$ are i.i.d. Exponential random variables with parameter λ.

Solution. The joint density of the order statistics $X_{(1)}, \dots, X_{(n)}$ for i.i.d. Exponential(λ) is

$$f_{X_{(1)},\dots,X_{(n)}}(x_1, \dots, x_n) = n!\lambda^n e^{-\lambda(x_1+\cdots+x_n)}, \quad 0 \le x_1 \le \cdots \le x_n.$$

Define the transformation:

$$\begin{aligned} y_1 &= nx_1 \\ y_2 &= (n-1)(x_2 - x_1) \\ y_3 &= (n-2)(x_3 - x_2) \\ &\vdots \\ y_n &= x_n - x_{n-1}. \end{aligned}$$

Inverting:

$$\begin{aligned} x_1 &= \frac{y_1}{n} \\ x_2 &= x_1 + \frac{y_2}{n-1} = \frac{y_1}{n} + \frac{y_2}{n-1} \\ x_3 &= x_2 + \frac{y_3}{n-2} = \frac{y_1}{n} + \frac{y_2}{n-1} + \frac{y_3}{n-2} \\ &\vdots \\ x_k &= \sum_{i=1}^{k} \frac{y_i}{n-i+1}. \end{aligned}$$

The Jacobian matrix $\frac{\partial x}{\partial y}$ is lower triangular:

$$\frac{\partial x}{\partial y} = \begin{pmatrix} \frac{1}{n} & 0 & 0 & \cdots & 0 \\ \frac{1}{n} & \frac{1}{n-1} & 0 & \cdots & 0 \\ \frac{1}{n} & \frac{1}{n-1} & \frac{1}{n-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n-1} & \frac{1}{n-2} & \cdots & 1 \end{pmatrix}$$

The determinant is the product of the diagonal entries:

$$\left|\frac{\partial x}{\partial y}\right| = \frac{1}{n} \cdot \frac{1}{n-1} \cdots \frac{1}{1} = \frac{1}{n!}.$$

Now we compute $\sum_{i=1}^n x_i$:

$$\sum_{i=1}^n x_i = \sum_{i=1}^n \sum_{j=1}^i \frac{y_j}{n-j+1}.$$

Changing the order of summation (for each j, sum over all $i \geq j$):

$$= \sum_{j=1}^n \sum_{i=j}^n \frac{y_j}{n-j+1} = \sum_{j=1}^n \frac{y_j}{n-j+1} \sum_{i=j}^n 1 = \sum_{j=1}^n \frac{y_j}{n-j+1} \cdot (n-j+1) = \sum_{j=1}^n y_j.$$

Thus, the joint density of $Y_1, \ldots, Y_n$ is

$$f_Y(y_1, \ldots, y_n) = n!\lambda^n e^{-\lambda \sum y_i} \cdot \frac{1}{n!} = \lambda^n e^{-\lambda \sum_{i=1}^n y_i}, \quad y_i \geq 0.$$

This factors as $\prod_{i=1}^n \left[\lambda e^{-\lambda y_i}\right]$.

Therefore,

$$\boxed{Y_1, \ldots, Y_n \text{ are i.i.d. Exponential}(\lambda)}$$

Exercise 4. Let (X, Y) be a 2-dimensional Gaussian vector with probability density

$$f(x,y) = \frac{1}{2\pi\left(1-\rho^2\right)^{1/2}} \exp\left\{-\frac{1}{2\left(1-\rho^2\right)}\left(x^2 - 2\rho xy + y^2\right)\right\}$$

where $|\rho| < 1$.

1. Show that X and $(Y - \rho X) / \left(1-\rho^2\right)^{1/2}$ are independent Gaussian random variables with mean 0 and variance 1.

2. Show that
$$\begin{aligned}\mathbb{P}(X > 0, Y > 0) &= \mathbb{P}\left(X > 0, \rho X + Z\left(1-\rho^2\right)^{1/2} > 0\right) \\ &= \mathbb{P}\left(X > 0, Z > -\frac{\rho}{\left(1-\rho^2\right)^{1/2}} X\right).\end{aligned}$$

3. Compute
$$\mathbb{P}(X > 0, Y > 0) = \frac{1}{4} + \frac{1}{2\pi}\sin^{-1}(\rho).$$

Solution. 1. Complete the square in the exponent:

$$x^2 - 2\rho xy + y^2 = (y - \rho x)^2 + x^2(1-\rho^2).$$

So the exponent becomes

$$-\frac{(y-\rho x)^2}{2(1-\rho^2)} - \frac{x^2}{2}.$$

Thus,

$$f(x,y) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \cdot \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left[-\frac{(y-\rho x)^2}{2(1-\rho^2)}\right].$$

Let $Z = \frac{Y - \rho X}{\sqrt{1-\rho^2}}$. Given $X = x$, $Y \sim N(\rho x, 1-\rho^2)$, so $Z \sim N(0,1)$ and is independent of X because the conditional density of Z given X is standard normal.

Therefore,

$$\boxed{X \text{ and } Z = \frac{Y - \rho X}{\sqrt{1-\rho^2}} \text{ are independent } N(0,1)}$$

2. Since $Y = \rho X + \sqrt{1-\rho^2} Z$, the condition $Y > 0$ is equivalent to

$$Z > -\frac{\rho X}{\sqrt{1-\rho^2}}.$$

Thus,

$$\mathbb{P}(X > 0, Y > 0) = \mathbb{P}\left(X > 0, Z > -\frac{\rho X}{\sqrt{1-\rho^2}}\right).$$

3. Let (X, Z) be i.i.d. $N(0,1)$. Transform to polar coordinates: $X = R\cos\Phi$, $Z = R\sin\Phi$, with $R > 0$, $\Phi \in [0, 2\pi)$. The condition $X > 0$ means $\cos\Phi > 0$, i.e., $\Phi \in (-\pi/2, \pi/2)$. The condition $Z > -\frac{\rho}{\sqrt{1-\rho^2}} X$ becomes

$$R\sin\Phi > -R\cos\Phi \cdot \frac{\rho}{\sqrt{1-\rho^2}} \implies \tan\Phi > -\frac{\rho}{\sqrt{1-\rho^2}}.$$

Let $\theta = \arcsin(\rho)$, so $\sin\theta = \rho$, $\cos\theta = \sqrt{1-\rho^2}$, and $\tan\theta = \frac{\rho}{\sqrt{1-\rho^2}}$. Then the inequality becomes $\Phi > -\theta$. Within $(-\pi/2, \pi/2)$, this means $\Phi \in (-\theta, \pi/2)$. The length of this interval is $\pi/2 + \theta$. By circular symmetry of the bivariate standard normal, the probability is

$$\frac{\pi/2 + \theta}{2\pi} = \frac{1}{4} + \frac{\theta}{2\pi}.$$

Since $\theta = \arcsin(\rho)$, we have:

$$\boxed{\mathbb{P}(X > 0, Y > 0) = \frac{1}{4} + \frac{1}{2\pi}\arcsin(\rho)}$$