WEEK 3 : NORMAL SUBGROUPS AND QUOTIENTS

Exercise 1: Let G be a group. Decide whether the following statements are true. Justify all your answers.

1. If N is a normal subgroup of G and H is a subgroup of G with $N \subseteq H$, then $N \trianglelefteq H$.
2. If H is a normal subgroup of G and K is a normal subgroup of H, then K is a normal subgroup of G.
3. If H and K are subgroups of G, then $\langle H \cup K \rangle = HK$.

Solution to Exercise 1 :

1. True. As $N \subseteq H$ and $N \leq G$,$N \leq H$. Take any $h \in H$ and any $n \in N$. Since $H \subseteq G$, $hnh^{-1} \in N$ by Proposition 1.3.4, and so applying the same proposition, we conclude that $N \trianglelefteq H$.
2. False. To see this, just take $G = D_8$, $H = \left\langle r^2, s \right\rangle$ and $K = \langle s \rangle$. Observe that $H = \{1, r^2, s, r^2 s\}$. Since $|H| = 4$ and $|G| = 8$, $|G : H| = 2$, and so $H \triangleleft G$ by Lemma 1.3.3. Notice that H is abelian as $r^2 s = sr^2$, and so $K \trianglelefteq H$. But $K \ntrianglelefteq G$ as $rsr^{-1} = rrs = r^2 s \notin K$.
3. False. To see this, take $G = \mathscr{S}_3$, $H = \langle (1\,2) \rangle$ and $K = \langle (2\,3) \rangle$. Then $HK = \{1, (1\,2), (2\,3), (1\,2\,3)\}$ and HK is not a subgroup of G.

Exercise 2: Make the list of all the subgroups of D_8.Decide which subgroups are normal. Can you calculate the corresponding quotient groups?

Solution to Exercise 2 : Let $G = D_8$ and let r and s be as introduced in Section 1 of the notes. Clearly, $G \leq G$ and $\{1\} \leq G$.

Let H be a proper non-trivial subgroup of D_8. The intersection $H \cap \langle r \rangle$ is a subgroup of $\langle r \rangle$. By Lagrange's Theorem, $n := |H \cap \langle r \rangle| \in \{1, 2, 4\}$. Also, by Lagrange's Theorem, $|H| \in \{2, 4\}$.

$n = 4$: Then $\langle r \rangle \subseteq H$. Since $|\langle r \rangle| = 4$ while $|H| \in \{2, 4\}$, we conclude that $|H| = 4$ and $H = \langle r \rangle$. Observe that as $|H| = 4$, $|G : H| = 2$, and so $H \triangleleft G$ by Lemma 1.3.3. Moreover, $|G/H| = 2$, and so by HW2, G/H is a cyclic group of order 2.

$n = 1$: Then $H \subseteq \{1, s, rs, r^2 s, r^3 s\}$. If $r^i s \in H$ and $r^j s \in H$ with $0 \leq i \neq j \leq 3$, then $r^{i+j} = r^i s (r^j s)^{-1} \in H$ and $r^i s r^j s = r^i r^{4-j} = r^{4+i-j} \in H$. Since $H \cap \langle r \rangle = \{1\}$, this is not possible. Thus $H = \langle r^i s \rangle$ with $i \in \{0, 1, 2, 3\}$. As $rr^i sr^{-1} = r^{i+1} rs = r^{i+2} s \notin H$, H is not a normal subgroup of G.

$n = 2$ In this case $r^2 \in H$. If $|H| = 2$, then $H = \langle r^2 \rangle = \{1, r^2\}$. Moreover, as $r^i r^2 r^{n-i} = r^2$ and $(r^i s) r^2 s r^{n-i} = r^i s s r^{n-i} r^2 = r^2$ for $0 \leq i \leq 3$, $H \trianglelefteq G$. Consider G/H. Then $|G/H| = 4$. Now $HrHr = Hr^2 = H$ and $Hr^i sHr^i s = Hr^i sr^i s = Hr^i r^{n-i} ss = H$

for $0 \le i \le 3$. Hence, G/H is a group of order 4 such that every non-identity element has order 2. We will discuss this group next time.

Suppose finally that $|H| = 4$. Then there exists $i \in \{0,1,2,3\}$ such that $r^i s \in H$. Hence, $r^2 r^i s = r^{i+2} s \in H$.

If $i = 0$ or 2, $H = \{1, r^2, s, r^2 s\}$. If $i = 1$ or 3, $H = \{1, r^2, rs, r^3 s\}$. In all the cases H is an abelian group and every non-identity element has order 2.

Exercise 3: ⋆

Let G be a group and let H be a normal subgroup of G. Let $\pi : G \to G/H$ be the natural homomorphism.

1. Let G' be a subgroup of G containing H. Prove that $\pi(G') \cong G'/H$.
2. Prove that the map φ that sends G' to $\pi(G')$ is a bijection between the subgroups of G containing H and the subgroups of G/H.
3. Prove that a subgroup G' of G containing H is normal if, and only if, $\varphi(G')$ is normal in G/H. Check that, in that case, $G/G' \cong (G/H)/\varphi(G')$.

Solution to Exercise 3 :

1. We have :

$$p(G') = \mathrm{im}(p|_{G'}) \cong G'/\ker(p|_{G'}) = G'/(G' \cap \ker(p)) = G'/H.$$

2. Consider the map ψ that sends a subgroup K of G/H to the subgroup $p^{-1}(K)$ in G. Note that $p^{-1}(K)$ automatically contains H.

   Let's prove that ψ and φ are mutual inverses. Since p is surjective, we have $\varphi(\psi(K)) = p(p^{-1}(K)) = K$ for any subgroup K of G/H. Now take a subgroup G' of G containing H. We have $\psi(\varphi(G')) = p^{-1}(p(G')) \supseteq G'$. Conversely, let $x \in p^{-1}(p(G'))$. Then we can find $y \in G'$ such that $p(x) = p(y)$. We deduce that $xy^{-1} \in \ker(p) = H$, and hence $x \in yH \subseteq yG' = G'$. This settles the equality $\psi(\varphi(G')) \supseteq G'$.

3. Assume that G' is a normal subgroup of G, and take $g \in G$ and $g' \in G'$. Then, in the group G/H, we have the following equalities :

$$(gH)^{-1}(g'H)(gH) = (g^{-1}g'g)H = p(g^{-1}g'g) \in p(G') = \varphi(G').$$

   Hence $\varphi(G')$ is normal in G/H.

   Conversely, assume that $\varphi(G')$ is normal in G/H, and take $g \in G$ and $g' \in G'$. Then :

$$p(g^{-1}g'g) = p(g)^{-1}p(g')p(g) \in \varphi(G') = p(G').$$

Hence $g^{-1}g'g \in p^{-1}(p(G')) = G'$ by question **2.**. We deduce that G' is normal in G.

Now consider the composite $p' \circ p$ of the projection $p : G \to G/H$ and the projection $p' : G/H \to (G/H)/\varphi(G')$. It is a surjective group homomorphism, and its kernel is :

$$p^{-1}(\varphi(G')) = p^{-1}(p(G')) = G'.$$

It therefore induces an isomorphism $G/G' \cong (G/H)/\varphi(G')$.

Exercise 4: $\star$ Let G be a finite group and let S be a non-empty subset of G. We set :

$$N_G(S) := \{g \in G | gSg^{-1} = S\},$$
$$C_G(S) := \{g \in G | \forall s \in S, gs = sg\}.$$

We say that $N_G(S)$ is the *normalizer* of S and that $C_G(S)$ is the *centralizer* of S.

1. Check that $N_G(S)$ is a subgroup of G and that $C_G(S)$ is a normal subgroup of $N_G(S)$.
2. Prove that $N_G(S) = G$ if, and only if, $S = \bigcup_{g \in G} gSg^{-1}$.
3. Let H be a normal subgroup of G. Prove that $C_G(H)$ is a normal subgroup of G.
4. Let H be a subgroup of G. Prove that $N_G(H)$ is the largest subgroup of G that has H as a normal subgroup.

Solution to Exercise 4 :

1. Of course, $1 \in N(S)$. Moreover, for any $g, h \in N(S)$, we have :

$$(gh)S(gh)^{-1} = g(hSh^{-1})g^{-1} = gSg^{-1} = S,$$
$$g^{-1}Sg = S.$$

Hence $N(S)$ is a subgroup of G.

Of course, $1 \in C(S)$. Moreover, for any $g, h \in C(S)$ and any $s \in S$, we have :

$$1s = s1,$$
$$(gh)s = g(hs) = g(sh) = (gs)h = (sg)h = s(gh),$$
$$g^{-1}s = sg^{-1}.$$

Since $C(S) \subseteq N(S)$, we deduce that $C(S)$ is a subgroup of $N(S)$. Moreover, for $g \in N(S)$, $h \in C(S)$ and $s \in S$, we have

$$(ghg^{-1})s = g\underbrace{h}_{\in Z(S)}\underbrace{g^{-1}sg}_{\in S}g^{-1} = g(g^{-1}sg)hg^{-1} = s(ghg^{-1}).$$

Hence $ghg^{-1} \in C(S)$ and $C(S)$ is normal in $N(S)$.

2. Assume that $S = \bigcup_{g \in G} gSg^{-1}$. Then for any $g \in G$, we have $gSg^{-1} \subseteq S$, and by cardinality, we deduce that $gSg^{-1} = S$. This being true for every $g \in G$, we get $N(S) = G$. The converse is immediate.
3. Since H is a normal subgroup of G, we have $H = \bigcup_{g \in G} gHg^{-1}$. By question **2.**, we deduce that $N(H) = G$. Question **1.** then implies that $C(H)$ is normal in $N(H) = G$.
4. By definition of $N(H)$ the group H is a normal subgroup of $N(H)$. Conversely, let K be a subgroup of G that has H as a normal subgroup. Then, for any $g \in K$, we have $gHg^{-1} \subseteq H$, and hence, by cardinality, $gHg^{-1} = H$. Consequently, K is contained in $N(H)$.

Exercise 5: Let G and H be two finite groups with co-prime orders. Find all group homomorphisms from G to H.

Solution to Exercise 5 : Let f be such a homomorphism. Let $x \in G$ and let n be the order of x. Observe that, by Lagrange's theorem n divides $|G|$. Moreover $f(x)^n = f(x^n) = f(1) = 1$, and hence the order m of $f(x)$ divides n. We deduce that m divides $|G|$. But by Lagrange's theorem, it also divides $|H|$. Since $|G|$ and $|H|$ are coprime, we conclude that $m = 1$. This means that $f(x) = 1$, and hence f is the trivial morphism.

Exercise 6: $\star$ Let $(G_i)_{i \in I}$ be a family of groups. We consider a normal subgroup H_i of G_i for each $i \in I$. Prove that $\prod_{i \in I} H_i$ is a normal subgroup of $\prod_{i \in I} G_i$ and that $(\prod_{i \in I} G_i)/(\prod_{i \in I} H_i) \cong \prod_{i \in I} G_i/H_i$.

Solution to Exercise 6 : For each family $(h_i)_{i \in I} \in \prod_{i \in I} H_i$ and each family $(g_i)_{i \in I} \in \prod_{i \in I} G_i$, we have $(g_i^{-1} h_i g_i)_{i \in I} \in \prod_{i \in I} H_i$. Hence $\prod_{i \in I} H_i$ is a normal subgroup of $\prod_{i \in I} G_i$. Now consider the natural map $p : \prod_{i \in I} G_i \to \prod_{i \in I} G_i/H_i$ induced by the projections $p_i : G_i \to G_i/H_i$. It is a surjective group homomorphism with kernel $\prod_{i \in I} H_i$. It therefore induces an isomorphism $(\prod_{i \in I} G_i)/(\prod_{i \in I} H_i) \cong \prod_{i \in I} G_i/H_i$.

Exercise 7: $\star$ Let G be group. We say that two subgroups H and H' of G are *commensurable* if $H \cap H'$ has finite index in both H and H'. Prove that commensurability is an equivalence relation.

Solution to Exercise 7 : Reflexivity and symmetry are immediate. Before starting the proof of transitivity, observe that, if H and G' are subgroups of G and H has finite index in G, then $H \cap G'$ has finite index in G' because $G'/(G' \cap H)$ embeds into G/H. Now consider three subgroup H, H' and H'' of G such that H and H' are commensurable and H' and H'' are commensurable. Then $H \cap H' \cap H''$ has finite index in both $H \cap H'$ and in $H' \cap H''$, and hence in H and in H''. Since $H \cap H''$ contains $H \cap H' \cap H''$, we deduce that $H \cap H''$ has finite index in H and in H'', and hence H and H'' are commensurable.

Exercise 8: ⋆ Let G be a group with order mn and let H be a normal subgroup of G of order n.

1. Prove that $x^m \in H$ for all $x \in G$.
2. Assume that m and n are coprime. Prove that H is the only subgroup of G that has order n.

Solution to Exercise 8 :

1. Let $x \in G$ and let $p : G \to G/H$ be the natural projection. By Lagrange's theorem :
$$|G/H| = |G| \cdot |H|^{-1} = m.$$
Hence again by Lagrange's theorem, we deduce that $p(x^m) = p(x)^m = 1$. Hence $x^m \in \ker(p) = H$.
2. Let H' be a subgroup of G with order n other than H. We can then find $x \in H' \setminus H$. By the first question, we know that $x^m \in H$, and by Lagrange's theorem, $x^n = 1$. But n and m are coprime, so by Bezout's theorem, we can find two integers u and v such that $mu + nv = 1$. Hence :
$$x = x^{mu+nv} = (x^m)^u \in H.$$
Contradiction!

Exercise 9: ⋆ Let G be a finite group and let H and K be two subgroups of G such that $[G : H]$ and $[G : K]$ are coprime. Prove that H and K generate G.

Solution to Exercise 9 : Let G' be the subgroup of G generated by H and K. By Lagrange's theorem, $|G'|$ is divisible by :

$$\operatorname{lcm}(|H|, |K|) = \operatorname{lcm}\left(\frac{|G|}{[G : H]}, \frac{|G|}{[G : K]}\right) = \frac{|G|}{\gcd([G : H], [G : K])} = |G|.$$

Hence $G' = G$.

Exercise 10: ⋆⋆ Let G be the subgroup of $GL_2(\mathbb{R})$ generated by the matrices :

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let H be the subgroup of G generated by AB.

1. Compute the order of H.

2. Prove that H is normal in G.

3. Compute the quotient G/H.

4. Deduce the order of G. Do you recognize the group G?

Solution to Exercise 10 :

1. One computes :

$$AB = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad (AB)^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (AB)^4 = -I_2, \quad (AB)^8 = I_2.$$

Hence H has order 8.

2. Since $A = A^{-1}$ and $B = B^{-1}$, one computes :

$$A(AB)A^{-1} = A^{-1}(AB)A = BA = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = (AB)^{-1} \in H,$$
$$B^{-1}(AB)B = B(AB)B^{-1} = BA = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = (AB)^{-1} \in H.$$

But G is generated by A and B and H by AB. Hence H is a normal subgroup of G.

3. Let $\pi : G \to G/H$ be the natural projection. Since G is generated by A and B, the quotient G/H is generated by $\pi(A)$ and $\pi(B)$. Moreover, $p(B) = p(A^{-1}AB) = p(A)^{-1}p(AB) = p(A)^{-1}$. Hence G/H isgenerated by $p(A)$. One then easily checks that $A \notin H$ and that :

$$p(A)^2 = p(A^2) = p(1) = 1.$$

Hence G/H is cyclic of order 2.

4. By Lagrange's theorem :

$$|G| = |H| \cdot |G/H| = 16.$$

In fact, in polar coordinates, A is the symmetry with respect to the line $\theta = \frac{3\pi}{8}$ and B is the symmetry with respect to the line $\theta = \frac{\pi}{2}$. Hence G is the dihedral group D_{16}. The element AB is none other than the rotation of angle $-\frac{\pi}{4}$.

Exercise 11: Let G be a finitely generated group and let H be a finite index subgroup of G. Prove that H is finitely generated. *Remark : the result does not remain true if one removes the assumption that H has finite index in G.*

Solution to Exercise 11 : Let $g_1, \dots, g_r \in G$ be such that $g_1 = 1$ and $G/H = \{g_1 H, \dots, g_r H\}$. Also let $g'_1, \dots, g'_s \in G$ such that any element of G can be written as a product of elements in $\{g'_1, \dots, g'_s\}$. For each i and j, we can find an element $k \in \{1, \dots, r\}$ and $h_{i,j} \in H$

such that $g'_i g_j = g_k h_{i,j}$.

Let's prove that the $h_{i,j}$'s span H. Take an element $h \in H$ and write $h = g'_{i_1} ... g'_{i_t}$ for some $i_1, ..., i_t \in \{1, ..., s\}$. We have $g'_{i_t} = g'_{i_t} g_1 = g_{k_t} h_{i_t,1}$ for some $k_t \in \{1, ..., r\}$, hence :

$$h = g'_{i_1} ... g'_{i_{t-1}} g_{k_t} h_{i_t,1}.$$

Now we can find $k_{t-1} \in \{1, ..., r\}$ such that $g'_{i_{t-1}} g_{k_t} = g_{k_{t-1}} h_{i_{t-1},k_t}$. Hence :

$$h = g'_{i_1} ... g'_{i_{t-2}} g_{k_{t-1}} h_{i_{t-1},k_t} h_{i_t,1}.$$

By repeating this procedure, we get that :

$$h = g_{k_1} h_{i_1,k_2} h_{i_2,k_3} ... h_{i_{t-1},k_t} h_{i_t,1}.$$

Since h and $h_{i_1,k_2} h_{i_2,k_3} ... h_{i_{t-1},k_t} h_{i_t,1}$ are both in H, we also have $g_{k_1} \in H$. Hence $k_1 = 1$, and :

$$h = h_{i_1,k_2} h_{i_2,k_3} ... h_{i_{t-1},k_t} h_{i_t,1}.$$