WEEK 4 : DIRECT PRODUCTS AND FINITE ABELIAN GROUPS

Exercise 1: Let $n \in \mathbb{N}$, $n \geq 1$, and let G be a cyclic group of order n. Then there exists a unique subgroup of G of order d for every divisor d of n.

Solution to Exercise 1 : Let g be a generator of G. Then $H_d := \langle g^{\frac{n}{d}} \rangle = \{(g^{\frac{n}{d}})^i \mid 1 \leq i \leq d\}$ is a subgroup of G of order d.

Let $X \leq G$ be a subgroup of order d. By the previous lemma, X is cyclic. Hence, $X = \langle x \rangle$ where $o(x) = d$. Since $x \in G$, $x = g^a$ for some $a \in \mathbb{Z}$. Then $1 = x^d = g^{ad}$. In fact, $n \mid (ad)$, for if not, $ad = nq + r$ for some $0 < r \leq n - 1$, and so $g^{ad} = (g^n)^q g^r = g^r \neq 1$, a contradiction. Hence, there exists $k \in \mathbb{Z}$ such that $nk = ad$. But now $x = g^a = g^{\frac{nk}{d}} = (g^{\frac{n}{d}})^k \in H_d$. Since $o(x) = d$ and $|H_d| = d$, $H_d = \langle x \rangle = X$.

Exercise 2: Let H and K be groups and $G = H \times K$. Decide whether the following statements are true or false. Explain your answers.

1. $H \leq G$ and $K \leq G$.

2. $H \times \{1_K\}$ is a normal subgroup of G.

3. $\{1_H\} \times K$ is a normal subgroup of G.

4. $G/(H \times \{1_K\}) \cong K$ and $G/(\{1_H\} \times K) \cong H$.

5. If G is finite and $(h, k) \in G$, then the order of (h, k) is the $lcm(o(h), o(k))$.

Solution to Exercise 2 :

1. False.

2. True. Use the definition.

3. True. Use the definition.

4. True. Use the Second Isomorphism Theorem.

5. True. Let $n := o(h, k)$. If $l := lcm(o(h), o(k))$, then $(a, b)^l = (a^l, b^l) = (1_H, 1_K)$. Hence, $n \mid l$. But $1_G = (h, k)^n = (h^n, k^n) = (1_H, 1_K)$. Hence, $o(h) \mid n$ and $o(k) \mid n$, and so $lcm(o(h), o(k)) \mid l$. Thus $n = l$.

Exercise 3: Let $m, n \in \mathbb{N}^*$. Prove that $(m, n) = 1$ if and only if $C_{nm} \cong C_n \times C_m$.

Solution to Exercise 3 : Recall that we proved in class that if $(m, n) = 1$, $C_{nm} \cong C_n \times C_m$.

If $(m, n) \neq 1$, there exists a prime p dividing both n and m. Let $H = C_n$ and $K = C_m$. By Lemma **??**, there exist $H_p \leq H$ with $H_p \cong C_p$ and $K_p \leq K$ with $K_p \cong C_p$. Then the group $H \times K$ contains more than one subgroup of order p (subgroups $H_p \times \{1_K\}$ and $\{1_H\} \times K_p$) which contradicts Exercise 1, and so $H \times K$ is not cyclic. In particular, $C_n \times C_m \not\cong C_{nm}$.

Exercise 4: Prove that :

$$\mathbb{Z}/12\mathbb{Z} \times \mathbb{Z}/90\mathbb{Z} \times \mathbb{Z}/25\mathbb{Z} \cong \mathbb{Z}/100\mathbb{Z} \times \mathbb{Z}/30\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}.$$

Solution to Exercise 4 : We use Lemma 1.4.3 :

$$\mathbb{Z}/12\mathbb{Z} \times \mathbb{Z}/90\mathbb{Z} \times \mathbb{Z}/25\mathbb{Z} \cong (\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}) \times \mathbb{Z}/25\mathbb{Z}$$
$$\cong (\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/25\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}) \times \mathbb{Z}/9\mathbb{Z}$$
$$\cong \mathbb{Z}/100\mathbb{Z} \times \mathbb{Z}/30\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}.$$

Exercise 5:

1. Find all abelian groups of order 72 (up to isomorphism).
2. $\star\star$ Let n be a positive integer. How many abelian groups of order n are there ?

Solution to Exercise 5 :

1. Any abelian group of order 72 can be written as :

$$G \cong \mathbb{Z}/n_1\mathbb{Z} \times ... \times \mathbb{Z}/n_r\mathbb{Z}$$

for some integers $n_1, ..., n_r \geq 2$ such that

$$\begin{cases} n_1 | n_2 | ... | n_r \\ n_1 ... n_r = 72. \end{cases} \tag{1}$$

Moreover, r and the n_i's are uniquely determined up to permutation. The possible solutions of (1) are :

$$\begin{cases} r = 1 \\ n_1 = 72 \end{cases} , \quad \begin{cases} r = 2 \\ (n_1, n_2) = (2, 36) \end{cases} , \quad \begin{cases} r = 2 \\ (n_1, n_2) = (3, 24) \end{cases} ,$$
$$\begin{cases} r = 2 \\ (n_1, n_2) = (6, 12) \end{cases} , \quad \begin{cases} r = 3 \\ (n_1, n_2, n_3) = (2, 2, 18) \end{cases} , \quad \begin{cases} r = 3 \\ (n_1, n_2, n_3) = (2, 6, 6) \end{cases} .$$

Hence the abelian groups of order 72 are :

$$\mathbb{Z}/72\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/36\mathbb{Z}, \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/24\mathbb{Z},$$
$$\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z}, (\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/18\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/6\mathbb{Z})^2.$$

2. In this question, for each positive integer m, we denote by $p(m)$ the number of partitions of m. Let $n = p_1^{\alpha_1}...p_r^{\alpha_r}$ be the prime factorization of n. By the

structure theorem of finite abelian groups, any abelian group G can be uniquely written as $G_1 \times ... \times G_r$ where each G_i is an abelian group of order $p_i^{\alpha_i}$. Moreover, the group G_1 can be written as :

$$\mathbb{Z}/p_1^{a_1}\mathbb{Z} \times \mathbb{Z}/p_1^{a_2}\mathbb{Z} \times ... \times \mathbb{Z}/p_1^{a_s}\mathbb{Z}$$

for some $s \geq 1$ and some integers $1 \leq a_1 \leq ... \leq a_s$ such that $a_1 + ... + a_s = \alpha_1$. Since s and $a_1,...,a_s$ are uniquely determined by G_1, we deduce that there are $p(\alpha_1)$ possibilities for the group G_1. Similarly, there are $p(\alpha_2)$ possibilities for G_2, $p(\alpha_3)$ possibilities for G_3, ..., and $p(\alpha_s)$ possibilities for G_s. Hence there are $p(\alpha_1)...p(\alpha_s)$ abelian groups of order n.

Exercise 6: Find all abelian groups of order 144.
Solution to Exercise 6 : As in the previous exercise, one checks that the finite abelian groups of order 144 are :

$$(\mathbb{Z}/2\mathbb{Z})^4 \times (\mathbb{Z}/3\mathbb{Z})^2, \quad (\mathbb{Z}/2\mathbb{Z})^4 \times \mathbb{Z}/9\mathbb{Z},$$
$$(\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^2, \quad (\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z},$$
$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^2, \quad \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z},$$
$$(\mathbb{Z}/4\mathbb{Z})^2 \times (\mathbb{Z}/3\mathbb{Z})^2, \quad (\mathbb{Z}/4\mathbb{Z})^2 \times \mathbb{Z}/9\mathbb{Z},$$
$$\mathbb{Z}/16\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^2, \quad \mathbb{Z}/16\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}.$$

Exercise 7: Let G be a group of order $n \geq 2$. Suppose that for every $g \in G$, $g^2 = 1$. Show that $n = 2^a$ for some $a \in \mathbb{N}$ and $G \cong \mathbb{Z}/2\mathbb{Z} \times ... \times \mathbb{Z}/2\mathbb{Z}$ (a times).
Solution to Exercise 7 : From Q3 Tutorial 1, G is an abelian group. Moreover, by definition of the order of an element, for every $g \in G \setminus \{1_G\}$, $o(g) = 2$.

Now by Theorem 1.4.4(a), $G \cong \mathbb{Z}/n_1\mathbb{Z} \times ... \times \mathbb{Z}/n_k\mathbb{Z}$ where for $1 \leq i \leq k$, $n_i = p_i^{a_i}$ with p_i a prime, $a_i \in \mathbb{N}^*$, and $n = n_1...n_k$. It follows that there exists $1_G \neq g_i \in (\{1\} \times ... \times \{1\} \times \mathbb{Z}/n_i\mathbb{Z} \times \{1\}... \times \{1\})$ such that $o(g_i) = n_i$. As $o(g_i) = 2$, we obtain that $n_i = 2$ for all $1 \leq i \leq k$ and the result follows.

Exercise 8: Let G be a group. Assume that $G/Z(G)$ is cyclic. Prove that G is abelian.
Solution to Exercise 8 : Recall that $Z(G) = \{z \in G \mid zg = gz \text{ for all } g \in G\}$ and that $Z(G) \leq G$. Then for $g \in G$ and $z \in Z(G)$, $gzg^{-1} = gg^{-1}z = z \in Z(G)$, and so $Z(G) \trianglelefteq G$. Let $\pi : G \to G/Z(G)$ be the natural homomorphism. Let $\overline{a}$ be a generator of $\overline{G} := G/Z(G)$ and consider $a \in G$ with $\pi(a) = \overline{a}$. Any element of G can then be written as $a^n x$ for some $n \in \mathbb{Z}$ and $x \in Z(G)$. But for any $n, m \in \mathbb{Z}$ and any $x, y \in Z(G)$:

$$(a^n x)(a^m y) = a^{m+n} xy = (a^m y)(a^n x).$$

Hence G is abelian.

Exercise 9: Prove that a non-cyclic finite abelian group has a subgroup isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ for some prime number p.

Solution to Exercise 9 : Let G be a non-cyclic finite abelian group. By the structure theorem of finite abelian groups, we can write :

$$G \cong \mathbb{Z}/n_1\mathbb{Z} \times ... \times \mathbb{Z}/n_r\mathbb{Z}$$

for some integers $n_1, ..., n_r \geq 2$ such that $n_1 | n_2 | ... | n_r$. Since G is not cyclic, $r \geq 2$. Fix a prime factor p of n_1. The prime p divides both n_1 and n_2, and hence $\mathbb{Z}/p\mathbb{Z}$ is a subgroup of both $\mathbb{Z}/n_1\mathbb{Z}$ and $\mathbb{Z}/n_2\mathbb{Z}$. We deduce that $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ is a subgroup of G.

Exercise 10: $\star$ Let G be an abelian group. Let e be the lcm of the orders of the elements of G. Prove that G contains an element of order e.

Solution to Exercise 10 : We can write :

$$G \cong \mathbb{Z}/n_1\mathbb{Z} \times ... \times \mathbb{Z}/n_r\mathbb{Z}$$

for some integers $n_1, ..., n_r \geq 2$ such that $n_1 | n_2 | ... | n_r$. For each element $x \in G$, we have $n_r x = 0$ and hence the order of x divides n_r. Since the element $(0, ..., 0, 1)$ has order n_r, we deduce that $e = n_r$ and G has an element of order e.

Exercise 11: $\star$ Classify up to isomorphism all groups of order 8.

Solution to Exercise 11 : Suppose first that G is abelian. Then G is isomorphic to one of the following groups : C_2^3, $c_2 \times C_4$ or C_8. Assume now that G is not abelian. By the Corollary to the Lagrange's Theorem, if $1_G \neq g \in G$, then $o(g) \in \{2, 4, 8\}$. If there exists $x \in G$ with $o(x) = 8$, then $G = \langle x \rangle$ is a cyclic group, and thus is abelian, a contradiction. If for every $1 \neq g \in G$, $o(g) = 2$, then by Q3 from Tutorial 1, G is abelian, again a contradiction. Therefore there exists $a \in G$ with $o(a) = 4$ and if $1 \neq g \in G$, $o(g) = 2$ or 4. Let $N = \langle a \rangle$. Then $|G : N| = 2$, and so $N \trianglelefteq G$. Now take an element $b \in G \setminus N$. Since $\langle a, b \rangle \leq G$ and $N \subset \langle a, b \rangle$, $|N| < |\langle a, b \rangle| \leq G$. Since $|N| = 4$, Lagrange's Theorem implies that $|\langle a, b \rangle| = |G|$, and so $G = \langle a, b \rangle$. In fact, $G = \{1, a, a^2, a^3, b, ab, a^2b, a^3b\}$.

As N is a normal subgroup of G, $bab^{-1} \in N$. Clearly, $bab^{-1} \neq 1$ (for otherwise, $a = 1$). If $ba = ab$, $G = \langle a, b \rangle$ is abelian, a contradiction, and so $bab^{-1} \neq a$. If $bab^{-1} = a^2$, $ba^2b^{-1} = bab^{-1}bab^{-1} = a^4 = 1$, and so $a^2 = 1$, a contradiction. Thus $bab^{-1} = a^{-1} = a^3$ (equivalently, $ba = a^{-1}b$).

Finally, let us determine b^2. Observe that $b^2 \in N$, for otherwise $b^2 \in \{b, ab, a^2b, a^3b\}$, and so $b \in \{1, a, a^2, a^3\} = N$, a contradiction. If $b^2 = a$ or $b^2 = a^3$, $o(b) = 8$, a contradiction. Thus $b^2 = 1$ or $b^2 = a^2$. In the former case, we observe that $G \cong D_8$. In the latter case one need to write few more elements to notice that $G \cong Q_8$.

Exercise 12: Write down a composition series for the dihedral group D_8. Is it unique? What are the corresponding simple factors?

Solution to Exercise 12 : We use the notation of Section 1. Then $D_8 = \{1, r, r^2, r^3, s, rs, r^2 s, r^3 s\}$. Take $G_1 = \langle r \rangle \cong C_4$. Then $|G : G_1| = 2$, and so $G_1 \lhd G$ and $|G/G_1| = 2$. Hence, $G/G_1 \cong C_2$ is simple. Take $G_2 = \langle r^2 \rangle \cong C_2$. As G_1 is abelian, $G_2 \lhd G_1$ and as $|G_1/G_2| = 2$, $G_1 \cong G_2 \cong C_2$ is simple. Finally, $G_2 \cong C_2$ is simple, and so $G > G_1 > G_2 > \{1_G\}$ is a composition series of G. The corresponding simple factors are $G/G_1 \cong C_2$, $G_1/G_2 \cong C_2$ and $G_2/\{1\} \cong C_2$.

This is not a unique composition series. For example, we could have taken $\widetilde{G}_1 = \langle r^2, s \rangle \cong C_2 \times C_2$ and $\widetilde{G}_2 = \langle s \rangle$ (check!).

Exercise 13: Write down a composition series of D_{12}.
Solution to Exercise 13 : This time take $G_1 = \langle r \rangle \cong C_6$, $G_2 = \langle r^2 \rangle \cong C_3$ and show that $G = G_0 > G_1 > G_2 > \{1_G\}$ is a composition series of G.

Exercise 14: Let G and H be finite groups such that $|G| = |H|$. Suppose that G has a composition series $G \geq G_1 \geq ... \geq G_r = \{1_G\}$ and H has a composition series $H \geq H_1 \geq ... \geq H_r\{1_H\}$ of the same length and such that $G_{i-1}/G_i \cong H_{i-1}/H_i$ for $1 \leq i \leq r$. Are G and H isomorphic?
Solution to Exercise 14 : No. For example, take $G \cong C_4$ and $H \cong C_2$, check that the conditions of the exercise are satisfied.