WEEK 5 : SOLVABLE GROUPS

Exercise 1: Let G be a group.

1. Show that for $g, x, y \in G$, $g[x, y]g^{-1}$ is a commutator.
2. Prove that $G^{(i)} \trianglelefteq G$ for $i \in \mathbb{N}$.
3. Show that if $H \leq G$, then $[H, H] \leq [G, G]$.
4. Let $f : G \to M$ be a surjective group homomorphism.
   Prove that $f([G, G]) = [M, M]$.
5. Let $f : G \to M$ be a surjective group homomorphism. Show that for all $k \geq 1$, $f(G^k) = M^{(k)}$. Use this to conclude that if the group G is solvable, then so is M.
6. If $N \trianglelefteq G$, prove that G is solvable if and only if both N and G/N are solvable.

Solution to Exercise 1 :

1. $g[x, y]g^{-1} = gxyx^{-1}y^{-1}g^{-1} = (gxg^{-1})(gyg^{-1})(gxg^{-1})^{-1}(gyg^{-1})^{-1} = [gxg^{-1}, gyg^{-1}]$.
2. Since $\{[x, y] \mid x, y \in H\} \subseteq \{[x, y] \mid x, y \in G\}$, $[H, H] = \langle\{[x, y] \mid x, y \in H\}\rangle \leq \langle\{[x, y] \mid x, y \in G\}\rangle = [G, G]$.
3. Proof by induction on i. Clearly true for $n = 0$. Assume $G^{(n)} \triangleleft G$. Consider $G^{(n+1)}$. Since, $G^{(n+1)} = [G^{(n)}, G^{(n)}]$, $G^{(n+1)} \leq G$. Now take any $g \in G$ and any $x \in G^{(n+1)}$. Then there exist $x_1, ..., x_k, y_1, ..., y_k \in G^{(n)}$ and $a_1, ..., a_k \in \mathbb{Z}$ such that $x = [x_1, y_1]^{a_1}...[x_k, y_k]^{a_k}$, and so $gxg^{-1} = g[x_1, y_1]^{a_1}...[x_k, y_k]^{a_k}g^{-1} = (g[x_1, y_1]g^{-1})^{a_1}...(g[x_k, y_k]g^{-1})^{a_k} = [gx_1g^{-1}, gy_1g^{-1}]^{a_1}...[gx_kg^{-1}, gy_kg^{-1}]^{a_k}$. Since $G^{(n)} \triangleleft G$, $gx_ig^{-1}, gy_ig^{-1} \in G^{(n)}$ for $1 \leq i \leq k$, and so $gxg^{-1} \in G^{(n+1)}$.
4. Observe that for $x, y \in G$, $f([x, y]) = f(xyx^{-1}y^{-1}) = f(x)f(y)f(x)^{-1}f(y)^{-1}) = [f(x), f(y)]$, and so $f([G, G]) \leq [M, M]$. Take any $m \in [M, M]$. Then there exist $a_1, ...a_k, b_1, ..., b_k \in M$ such that $m = [a_1, b_1]...[a_l, b_l]$. Since f is surjective, for $1 \leq i \leq l$, there exist $x_i, y_i \in G$ with $f(x_i) = a_i$ and $f(y_i) = b_i$. But then $[a_i, b_i] = f([x_i, y_i])$, and so $m = f([x_1, y_1]...[x_l, y_l])$. Thus $f([G, G]) = [M, M]$.
5. Proof is by induction on k. By the previous part, $f(G^{(1)}) = M^{(1)}$. Take any $i \geq 1$ and assume $f(G^{(i)}) = M^{(i)}$. Then the restriction $f : G^{(i)} \to M^{(i)}$ is a surjective homomorphism, and again applying part (4), we obtain that $f(G^{(i+1)}) = M^{(i+1)}$. Thus for all k, $f(G^k) = M^{(k)}$. If G is solvable, then there exists $n \in \mathbb{N}$ with $G^n = \{1_G\}$. It follows that for all k, $M^{(n)} = f(G^n) = \{1_H\}$, and so M is solvable.
6. Assume that G is solvable. We have seen in class that N is solvable. Now denote by $\pi : G \to G/N$ the natural homomorphism.

Let us first show that for each $m \geq 0$, $(G/N)^{(m)} = \pi(G^{(m)})$. Indeed, if $m = 0$ it is obviously true. Assume $(G/N)^{(m)} = \pi(G^{(m)})$. Then $(G/N)^{(m+1)} = [(G/n)^{(m)}, (G/N)^{(m)}] = [\pi(G^{(m)}), \pi(G^{(m)})] = \langle\{[x, y] \mid x, y \in \pi(G^{(m)}\}\rangle = \langle\{[\pi(a), \pi(b)] \mid a, b \in G^{(m)}\}\rangle = \langle\{\pi([a, b]) \mid a, b \in G^{(m)}\}\rangle = \pi([G^{(m)}, G^{(m)}]) = \pi(G^{(m+1)})$.
In particular, if n is a positive integer such that $G^{(n)} = \{1\}$, then $(G/N)^{(n)} = \{1_{G/N}\}$ and hence G/N is solvable by Proposition 1.5.3.

Conversely, assume that N and G/N are both solvable. Fix two integers n and m such that $N^{(m)} = \{1\}$ and $(G/N)^{(n)} = \{1_{G/N}\}$. Since $(G/N)^{(n)} = \pi(G^{(n)})$, we see that $G^{(n)} \subseteq N$. But then :

$$G^{(n+m)} = (G^{(n)})^{(m)} \subseteq N^{(m)} = \{1\}.$$

Hence, G is solvable.

Exercise 2: Let $n \in \mathbb{N}$ with $n \geq 5$.

1. Show that every 3-cycle is a commutator of two 3-cycles.
2. Show that $[A_n, A_n] = A_n$.
3. Prove that $[S_n, S_n] = A_n$.
4. Is A_n solvable ? Is S_n solvable ?

Solution to Exercise 2 :

1. Let $|\{i, j, k, l, m\}| = 5$. By direct calculations, $[(i, j, l), (i, k, m)] = (i, j, k)$.
2. By Q10 of Tutorial 2, A_n is generated by its 3-cycles. By Part (1), every 3-cycle in A_n is a commutator of two elements of A_n. Since $[A_n, A_n]$ is generated by commutators, we conclude $A_n = [A_n, A_n]$.
3. Recall that $|S_n : A_n| = 2$, so $A_n \trianglelefteq S_n$. Then $S_n/A_n \cong C_2$ is abelian, and so, by Lemma, $[S_n, S_n] \leq A_n$. But $A_n \leq S_n$, and so $A_n = [A_n, A_n] \leq [S_n, S_n] \leq A_n$.
4. A_n is not solvable, as the derived series of A_n does not terminate. Similarly, S_n is not solvable.

Exercise 3: Let $G = A_4$.

1. Determine the derived series of G. Decide whether the group is solvable.
2. Give a composition series of G. Explain your answer.
3. Is S_4 solvable ?

Solution to Exercise 3 :

1. Since G is non-abelian, $[G,G] \neq \{1\}$. By direct calculation, $V = \{1,(12)(34),(13)(24),(14)(23)\} \leq G$. By Tutorial 4, $V \cong C_2 \times C_2$. as $g^2 = 1$ for all $g \in V$.
   Let $\{i,j,k,l\} = \{1,2,3,4\}$. Observe that since $(ijk)(ij)(kl)(kji) \in V$, $V \trianglelefteq G$ (why?).
   Now $|G/V| = 3$, and so $G/V \cong C_3$ is abelian. Therefore $[G,G] \leq V$ (why?)
   Since $[(ijk),(j,k,l)] = (ij)(kl)$, every element of V is a commutator, and so $V \leq [G,G]$. Thus $V = G^{(1)}$. Since V is abelian, $[V,V] = G^{(2)} = 1$. In particular, G is solvable.
2. Consider a series $G > V > \{1,(12)(34)\} > \{1\}$. Then $V \trianglelefteq G$ (from part (1)), $N := \{1,(12)(34)\} \triangleleft V$ (as V is abelian). Moreover, $G/V \cong C_3$ is simple, $V/N \cong C_2$ is simple and $V/\{1\} \cong N \cong C_2$ is simple. This is indeed a composition series.
3. Since $A_4 \trianglelefteq S_4$ and $S_4/A_4 \cong C_2$ - simple, the series $S_4 \geq A_4 \geq V \geq N \geq \{1\}$ (with V and N as in parts (1) and (2)) is a composition series of S_4. Since every composition factor is cyclic of prime order, we obtain that S_4 is solvable by the result we proved in class.

Exercise 4: Let $G = D_8$.

1. Determine the derived series of G. Is G solvable?
2. Give a composition series of G. Explain your answer.

Do the same calculations for your favourite dihedral groups.

Solution to Exercise 4 : Recall that $G = \{1, r, r^2, r^3, s, rs, r^2s, r^3s\}$ in terms of the lecture notes and Tutorial 3.

1. From Tutorial 3, $\langle r^2\rangle \triangleleft G$, $|G/\langle r^2\rangle| = 4$ and every non-identity element in $G/\langle r^2\rangle$ has order 2. Hence, from Tutorial 1, we know that $G/\langle r^2\rangle$ is abelian. Hence, by Lemma 1.5.3, $[G,G] \leq \langle r^2\rangle$. But $|\langle r^2\rangle| = 2$, and so either $[G,G] = \{1\}$, or $[G,G] = \langle r^2\rangle$. Since G is non-abelian, $[G,G] = \langle r^2\rangle$. Now $G^{(1)} = [G,G] \cong C_2$ is abelian, and so $G^{(2)} = \{1\}$. Thus $G \geq \langle r^2\rangle = \{1\}$ is the derived series of G, so G is solvable.
2. For example, take $G \geq \langle r\rangle \geq \langle r^2\rangle \geq \{1\}$. Then $|G : \langle r\rangle| = 2$, and so $\langle r\rangle \triangleleft G$, and as $\langle r\rangle \cong C_4$ is abelian, $\langle r^2\rangle \triangleleft \langle r\rangle$, Clearly, $\{1\} \triangleleft \langle r^2\rangle$. Finally, $G/\langle r\rangle \cong C_2$ is simple, $\langle g\rangle/\langle r^2\rangle \cong C_2$ is simple and $\langle r^2\rangle/\{1\} \cong \langle r^2\rangle \cong C_2$ is simple. Hence, this series above is a composition series of G.

Exercise 5: Prove that if K and H are solvable groups, then $K \times H$ is solvable.

Solution to Exercise 5 : By Tutorial 4, $\{1\} \times H \triangleleft K \times H$, and as $\{1\} \times H \cong H$ (under the map $i : \{1\} \times H \to H$ with $i(1,h) = h$ for $h \in H$), $\{1\} \times H$ is solvable. Also, $(K \times H)(\{1\} \times H) \cong K$ is solvable. Hence, by Exercise 1, $K \times H$ is solvable.

Exercise 6: Show that the set B of upper-triangular 2×2 matrices over $\mathbb{R}$ with non-zero determinant, is a subgroup of $GL(2,\mathbb{R})$. Prove that this subgroup is solvable.

Solution to Exercise 6 : Clearly B is non-empty. Check by direct calculation that for $x, y \in B$, $xy^{-1} \in B$. Hence, B is a subgroup of $GL(2,\mathbb{R})$. Now by direct calculation show that $B^{(1)}$ is a set of upper triangular matrices with ones on the diagonal. From Tutorial 1, this group is abelian. Hence, $B^{(2)} = \{1\}$, and so G is solvable.

Exercise 7: $\star$ Let G be a solvable group and let N be a non-trivial normal subgroup of G. Proof that there exists a non-trivial normal subgroup K of G such that $K \leq N$ and K is abelian.

Solution to Exercise 7 : Consider the derived series $G > G^{(1)} > \ldots > G^{(n)} = \{1\}$ of G. Choose $k \in \{0,\ldots,n\}$ to be maximal among those i for which $N \cap G^{(i)} \neq \{1\}$. Then $K := N \cap G^{(k)}$ is a non-trivial normal subgroup of G (as the intersection of two normal subgroups). Consider a map $f : K \to G^{(k)}/G^{(k+1)}$ defined by $f(x) = xG^{(k+1)}$. Show that f is a group homomorphism. If $x \in \ker(f)$, then $x \in G^{(k+1)}$, and so $x = 1$ by the maximal choice of K. Thus f is injective, and so K is isomorphic to a subgroup of $G^{(k)}/G^{(k+1)}$, which is abelian.