# WEEK 6 : GROUP ACTIONS

**Exercise 1:** Let $G$ be a group, let $X$ be a set and $x \in X$. Assume that the map $\cdot : G \times X \to X$ is a group action.

1. Show that the relation $\mathscr{R}$ on $X$ defined by $x\mathscr{R}y \iff \exists g \in G : y = g\cdot x$, is an equivalence relation.
2. Prove the Orbit-Stabilizer Theorem, that is, prove that there exists a bijection between the set of left cosets of $\mathrm{Stab}_G(x)$ in $G$ and the set $\mathrm{Orb}_G(x)$. In particular, if $G$ is a finite group, conclude that $|\mathrm{Orb}_G(x)| = |G : \mathrm{Stab}_G(x)|$.
3. Show that the kernel of the action $K = \bigcap_{x\in X} \mathrm{Stab_G(x)}$.
4. Show that if the action is transitive, $X = \mathrm{Orb_G(x)}$ for every $x \in X$.
5. Prove the following statement called the *Class Formula* : Let $R \subseteq X$ be a subset of $X$ containing exactly one element of each orbit. Then

$$|X| = \sum_{x\in R} |G : \mathrm{Stab}_G(x)|.$$

**Solution to Exercise 1** :

1. Take any $x \in X$. Since $1_g \cdot x = x$ (as $\cdot$ is a group action), $x\mathscr{R}x$, and so $\mathscr{R}$ is reflexive. Assume that $x\mathscr{R}y$. Then thereexists $g \in G$ such that $y = g \cdot x$. Using the definition of a group action we have

$$g^{-1} \cdot y = g^{-1} \cdot (g \cdot x) = (gg^{-1}) \cdot x = 1_G \cdot x = x$$

and so $y\mathscr{R}x$. Thus $\mathscr{R}$ is reflexive. Finally, if $x, y, z \in X$ are such that $x\mathscr{R}y$ and $y\mathscr{R}z$, there exist $g, h \in G$ with $y = g \cdot x$ and $z = h \cdot y$. Then

$$z = h \cdot y = h \cdot (g \cdot x) = (hg) \cdot x$$

and so $\mathscr{R}$ is transitive.

2. Take $x \in X$ and define a map $f_x : \mathrm{Orb}_G(x) \to G/\mathrm{Stab}_G(x)$ by $f_x(g \cdot x) = g\mathrm{Stab}_G(x)$. If $g \cdot x = h \cdot x$ for some $g, h \in G$, then $g^{-1}h \in \mathrm{Stab}_G(x)$, and so $g\mathrm{Stab}_G(x) = h\mathrm{Stab}_G(x)$. Hence, the map is well-defined. Clearly, the map is surjective. Now, if $f_x(a\cdot x) = f_x(b\cdot x)$, then $a\mathrm{Stab}_G(x) = b\mathrm{Stab}_G(x)$, and so there exists $h \in \mathrm{Stab}_G(x)$ such that $a = bh$. Hence,

$$a \cdot x = (bh) \cdot x = b \cdot (h \cdot x) = b \cdot x$$

and so $f_x$ is injective. Thus $f_x$ is a bijection. Finally if $G$ is finite, $|\mathrm{Orb}_G(x)| = |G : \mathrm{Stab}_G(x)|$.

3. By definition, $K = \{g \in G \mid g \cdot x = x \ \forall x \in X\} = \bigcap x \in X\{g \in G \mid g \cdot x = x\} = \bigcap_{x\in X} \mathrm{Stab_G(x)}$.
4. By definition, $G$ acts transitively on $X$ means that $G$ has a unique orbit on $X$. Take any element $x \in X$. Then $\mathrm{Orb_G(x)} \subseteq X$. If there exists $y \in X \setminus \mathrm{Orb_G(x)}$, $\mathrm{Orb_G(y)} \cap \mathrm{Orb_G(x)} = \emptyset$, and so $G$ has more than one orbit on $X$, a contradiction.

**5.** The orbits of the action form a partition of $X$, so

$$|X| = \sum_{x \in R} |\mathrm{Orb}_G(x)|.$$

But for each $x \in R$, $|G : \mathrm{Stab}_G(x)| = |\mathrm{Orb}_G(x)|$. Hence $|X| = \sum_{x \in R} |G : \mathrm{Stab}_G(x)|$.

**Exercise 2:** Let $n \in \mathbb{N}$, $n \geq 2$, $G = GL(n,\mathbb{R})$ and $X = \mathbb{R}^n$. Consider a map $\cdot : G \times X \to X$ given by $(A, v) \to Av$. Show that this map is a group action. Determine its kernel. Is the action transitive?

**Solution to Exercise 2** : Recall that$1_G = I_n$. Then $I_n v = v$ for all $v \in X$, and for $(AB)v = A(Bv)$ for all $A, B \in G$ and $v \in X$. Hence, this map defines a group action.

Let $A \in K$, the kernel of the action. Then $Av = v$ for all $v \in X$. Take a basis $e_1, \ldots, e_n$ of $X$. Then $A$ fixes the basis, and so $A$ is the identity. Hence, the action is faithful.

Notice that $A\overline{0} = \overline{0}$, and so $\mathrm{Orb_G}(\overline{0}) = \{\overline{0}\}$, so the action is not transitive.

**Exercise 3:** Let $G$ be a group and $H$ a subgroup of $G$. Recall that $G/H$ denotes the set of left cosets of $H$ in $G$. Show that the map $\cdot : G \times G/H \to G/H$ given by $(g, xH) \mapsto gxH$ defines a group action. Determine the kernel of this action. Is the action transitive?

**Solution to Exercise 3** : Clearly, $(1x)H = xH$ and $(gh)xH = g(hx)H$ for all $g, h, x \in G$. Hence, $G$ acts on $G/H$. Take any $xH, yH \in G/H$. Then $(yx^{-1})xH = yH$, and so the action is transitive. Finally, $K = \bigcap_{x \in G}\{g \in G \mid gxH = xH\} = \bigcap_{x \in G}\{g \in G \mid x^{-1}gx \in H\} = \bigcap_{x \in G}\{g \in G \mid g \in xHx^{-1}\} = \bigcap_{x \in G} xHx^{-1}$.

**Exercise 4:** Show that if $G$ is an infinite simple group, then $G$ has no subgroups of finite index.

**Solution to Exercise 4** : Let $G$ be an infinite simple group. Assume there exists $H \leq G$ with $|G : H| = n$. By the previous exercise, $G$ acts non-trivially on the set $G/H$ of cardinality $|G : H| = n$. Hence, there exists a non-trivial homomorphism $\phi : G \to Sym(G/H) \cong S_n$. Since $G$ is infinite while $S_n$ is finite, the First Isomorphism Theorem implies that $G$ has a non-trivial normal subgroup which contradicts the simplicity of $G$.

**Exercise 5:** Let $p$ be a prime. In class we proved that if $G$ is a finite non-identity $p$-group (that is $|G| = p^n$ for some $n \geq 1$), then the center of $G$, $Z(G) = \{g \in G \mid gx = xg \ \forall x \in G\} \neq \{1\}$.

1. Prove that the only simple $p$-group is a cyclic group of order $p$.
2. Prove that every group of order $p^2$ is abelian.
3. Prove that every $p$-group is solvable.

**Solution to Exercise 5** :

1. Let $G$ be a $p$-group. If $G$ is abelian, then as it is simple, we have proved in class that $G \cong C_p$. Suppose now that $G$ is non-abelian. Then $Z(G)$ is a non-trivial normal subgroup of $G$, and so $G$ is not simple.
2. Let $G$ be a group of order $p^2$. Assume that $G$ is non-abelian. Then $Z(G)$ is a non-trivial normal subgroup of $G$, and so, by Lagrange's Theorem, $|Z(G)| = p$. Consider $G/Z(G)$. By Lagrange's Theorem, $|G/Z(G)| = p$, and so by Tutorial 2 Ex 8, $G/Z(G) \cong C_p$, a cyclic group of order $p$. Let $\pi : G \to G/Z(G)$ be the natural homomorphism, $\overline{a}$ be a generator

of $G/Z(G)$ and consider $a \in G$ with $\pi(a) = \overline{a}$. Any element of $G$ can then be written as $a^n x$ for some $n \in \mathbb{Z}$ and $x \in Z(G)$. But for any $n, m \in \mathbb{Z}$ and any $x, y \in Z(G)$ :

$$(a^n x)(a^m y) = a^{m+n} xy = (a^m y)(a^n x).$$

Hence $G$ is abelian, a contradiction.

**3.** We proceed by induction on the order $p^n$ of $G$. If $n \leq 2$, then $G$ is abelian and hence solvable. For $n \geq 3$, we have shown that $Z(G)$ is not trivial. This group is abelian and thus solvable. By the inductive assumption, $G/Z(G)$ is solvable. Hence $G$ is solvable by Tutorial 5 Ex 1.

**Exercise 6:** Let $G$ be a group of order $n$.

1. Prove that $G$ is isomorphic to a subgroup of $S_n$.
2. Prove that $G$ is isomorphic to a subgroup of $\mathscr{A}_{n+2}$.
3. $\star$ Prove that $G$ is isomorphic to a subgroup of $GL_n(K)$ for any field $K$.

**Solution to Exercise 6** :

1. Did it in class. (Consider the action of $G$ on itself by left multiplication. This action is faithful, hence it induces an injective group homomorphism : $G \to Sym(G) \cong S_n$.)
2. The following map :

$$\begin{aligned} f : \mathscr{S}_n &\to \mathscr{A}_{n+2}, \\ \sigma &\mapsto \sigma((n+1)\ (n+2))^{\mathrm{sign}(\sigma)} \end{aligned}$$

is an injective group homomorphism (check!). We can now obtain the result thanks to part **1**.

3. Given a permutation $\sigma \in \mathscr{S}_n$, define the matrix $u_\sigma = (\delta_{i\sigma^{-1}(j)})_{1 \leq i,j \leq n}$. Show that the map $\sigma \mapsto u_\sigma$ is then an injective group homomorphism from $\mathscr{S}_n$ to $GL_n(K)$. We can now obtain the result thanks to part **1**.

**Exercise 7:** Let $(\mathbb{Z}/22\mathbb{Z})^\times$ be the subset of $\mathbb{Z}/22\mathbb{Z}$ given by elements of the form $[a]$ with $a$ an integer coprime to 22. Endowed with multiplication, it is a group. Now consider the action of $(\mathbb{Z}/22\mathbb{Z})^\times$ on $\mathbb{Z}/22\mathbb{Z}$ given by :

$$(a, x) \mapsto ax.$$

Is it transitive ? Is it faithful ? What are its orbits ?

**Solution to Exercise 7** : We have :

$$(\mathbb{Z}/22\mathbb{Z})^\times = \{1, 3, 5, 7, 9, 13, 15, 17, 19, 21\}.$$

The orbits of the action are :

$$\{1, 3, 5, 7, 9, 13, 15, 17, 19, 21\}, \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}, \{11\}, \{0\}.$$

In particular, the action is not transitive. One easily checks that the action is faithful.

**Exercise 8:** Define the action of the group $SL_2(\mathbb{R})$ on the Poincaré half-plane $\mathscr{H} = \{z \in \mathbb{C} \mid Im(z) > 0\}$ by :

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

Check that this is indeed a group action. Is the action faithful? Is this action transitive? What is the stabilizer of $i \in \mathscr{H}$?

**Solution to Exercise 8** :

Given $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in SL_2(\mathbb{R})$ and $z \in \mathscr{H}$, we have :

$$\mathrm{Im}\left(\frac{az+b}{cz+d}\right) = \frac{\mathrm{Im}(z)}{(c\mathrm{Re}(z)+d)^2 + (c\mathrm{Im}(z))^2} > 0,$$

$$I_2 \cdot z = z,$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \left(\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \cdot z\right) = \frac{a\frac{a'z+b'}{c'z+d'} + b}{c\frac{a'z+b'}{c'z+d'} + d} = \frac{(aa'+bc')z + ab' + bd'}{(a'c+c'd)z + b'c + dd'} = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right) \cdot z.$$

The action is not faithful since any scalar matrix acts trivially on $\mathscr{H}$.

The action is transitive since, for any $z = x + iy \in \mathscr{H}$, we have :

$$\begin{pmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} \cdot i = z.$$

The stabilizer of $i$ is given by matrices $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ with $a, b \in \mathbb{R}$ and $a^2 + b^2 = 1$ (in other words, it is the group $SO_2(\mathbb{R})$).

**Exercise 9:** Let $p$ be a prime number and let $G$ be a finite simple group whose order is divisible by $p^2$.

(a) Prove that, for each $k \in \{1, 2, 3, ..., 2p-1\}$, every group homomorphism from $G$ to $S_k$ is trivial.

(b) Deduce that any proper subgroup of $G$ has index at least $2p$.
*Hint : use a group action!*

**Solution to Exercise 9** :

(a) Let $f$ be a group homomorphism from $G$ to $\mathscr{S}_k$. Since $G$ is simple, $\ker f$ is either $G$ or $\{1\}$. Assume that $\ker f = 1$. In that case, $f$ is injective, and hence, by Lagrange's theorem, $|G|$ divides $|\mathscr{S}_k| = k!$. In particular, $p^2 | k!$, which contradicts the assumption that $k < 2p$. Hence $\ker f = G$ and $f$ is trivial.

(b) Let $H$ be a proper subgroup of $G$ and let $k$ be its index. Consider the action of $G$ on $G/H$ by left multiplication. It induces a non-trivial group homomorphism :

$$\phi : G \to Sym(G/H) \cong S_k.$$

By question (a), we deduce that $k \geq 2p$.

**Exercise 10:** ⋆ Let $K$ be a field and let $n$ be a positive integer. Endow $X := K^{n+1} \setminus \{0\}$ with the action of $G := K^\times$ defined by :

$$\forall \lambda \in K^\times, \forall v \in K^{n+1} \setminus \{0\}, \lambda \cdot v = \lambda v.$$

1. What are the orbits?

Denote by $\mathbb{P}^n(K)$ the quotient $G\backslash X$. The set $\mathbb{P}^n(K)$ is called the *$n$-th dimensional projective space over $K$*. Let $p : X \to G\backslash X$ be the quotient map.

2. Prove that one can define a transitive action of $GL_{n+1}(K)$ on $\mathbb{P}^n(K)$ by the formula :

$$A \cdot p(v) = p(Av)$$

   for $A \in GL_{n+1}(K)$ and $v \in X$. Is it faithful in general?

3. Let $PGL_{n+1}(K)$ be the quotient of $GL_{n+1}(K)$ by the subgroup given by scalar matrices. It is called the *projective linear group*. Prove that, if $\pi : GL_{n+1}(K) \to PGL_{n+1}(K)$ is the projection map, then one can define a transitive and faithful action of $PGL_{n+1}(K)$ on $\mathbb{P}^n(K)$ by the formula :

$$\pi(A) \cdot p(v) = p(Av)$$

   for $A \in GL_{n+1}(K)$ and $v \in X$.

4. Prove that $PGL_2(\mathbb{F}_3)$ is isomorphic to $\mathscr{S}_4$.

**Solution to Exercise 10** :

1. The orbits are the lines of $K^n$.
2. The fact that the formula defines a transitive action is straightforward. The action is not faithful in general because scalar matrices act trivially on the projective space.
3. The fact that the formula defines a transitive action is straightforward. The faithfulness comes from the linear algebra fact that a matrix in $GL_{n+1}(K)$ stabilizes all lines of $K^{n+1}$ if, and only if, it is scalar.
4. By the previous question, we have an injective morphism :

$$f : PGL_2(\mathbb{F}_3) \to \mathscr{S}_{|\mathbb{P}^1(\mathbb{F}_3)|} = \mathscr{S}_4.$$

   But :

$$|PGL_2(\mathbb{F}_3)| = \frac{|GL_2(\mathbb{F}_3)|}{|\mathbb{F}_3^\times|} = 24 = 4! = |\mathscr{S}_4|.$$

   Hence $f$ is an isomorphism.