The Movement of Hyperion

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December 18, 2021

Abstract

This work shows a computational simulation of Hyperion's orbit and rotation behaviour for different initial conditions. The investigation revealed that most of the initial conditions lead to an aperiodic long-term behaviour. The system's behaviour was investigated by using different Poincare-section and Fourier-Transformations. The chaotic behaviour was noticeable due to an aperiodic long-term behaviour, and that small changes in the initial conditions lead to unpredictably different spectra. The further investigation analyzed the moons of Mars, Phobos and Deimos, which showed chaotic behaviour for a smaller range of initial conditions.

1 Introduction

A complex and still widely unknown topic in physics is Chaos theory. Its definition is broad but can be simplified as a behaviour of equation in which solutions are sensitive due to small changes in initial conditions and give an aperiodic spectrum as a solution over a long time [1]. One such example is the motion of one of the eighty-two moons of Saturn, Hyperion, specifically its rotation around its axis over time. Restricting the orbit of Hyperion around Saturn at the x-y-plane, one will find with Newton's equations that there are four coupled differential equations [2]:

$$\frac{dx_H}{dt} = vx_H \tag{1}$$

$$\frac{dy_H}{dt} = vy_H \tag{2}$$

$$\frac{dvx_H}{dt} = \frac{-GMx_H}{(x_H^2 + y_H^2)^{3/2}} \tag{3}$$

$$\frac{dvy_H}{dt} = \frac{-GMy_H}{(x_H^2 + y_H^2)^{3/2}} \tag{4}$$

Here M denotes the combined mass of Hyperion and Saturn [1], G is the Gravitational constant, vi_H is the velocity in the i-direction, and x_H and y_H denotes the position of Hyperion with respect to Saturn.
The period and frequency in which Hyperion takes to orbit Saturn can be calculated as follows:

$$T_H = \sqrt{\frac{4\pi a^3}{GM}} \tag{5}$$

$$\omega_H = \frac{2\pi}{T_H} \tag{6}$$

The rotation of Hyperion seems to be Chaotic in time and therefore future prediction of its rotation are nearly impossible [3]. The equation which describes the rotation in terms of the roational axis in spherical coordinates $\theta(t)$ and circular frequency $\omega(t)$ are [2]:

$$\frac{d\theta}{dt} = \omega \tag{7}$$

$$\frac{d\omega}{dt} = -\frac{GM_{Sat}\beta^2}{2R_H^3} sin2(\theta - \phi) \tag{8}$$

R_H is a substitution of the denominator of equations (3) and (4), β is the asphericity, and ϕ is a perturbation term.
As one can see from equation 8, it has a similar shape as the equation of motion of a Harmonic oscillator [3].
In this work, the main aim is to investigate the long term chaotic behaviour of Hyperions rotation by simulating the system for different sets of initial conditions. Therefore the above-shown equations will be implemented as a Maple code [4].

2 Numerical stability

In order to perform a reasonable investigation, one will have to look at how stable the system is for numerical calculations. Therefore, we need to simulate the Orbit of the

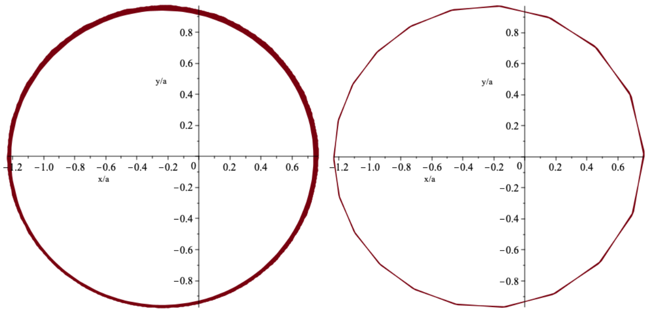
[1] M can be the total mass of any other two-body system.

system Hyperion-Saturn for many periods T_H and analyze how it changes over time and whether it is a change that will majorly affect our investigation. As every numerical solution gives different errors based, for example, on the integrators algorithm (Euler, leapfrog, Velocity Verlet.) or the stepsize, one gets different deviations of the original orbit. One wants to see how it evolves during different steps and investigate the computational time. We need a precise algorithm and a short computational time, as the simulation could get time intensive.
As an integrator we may us the Runge-Kutta-Algorithm. It is, in general, an extension of the Eulers algorithm, or even detailed; the Eulers algorithm is the 0th order Runge-Kutta algorithm. However, as the Euler method is not stable, which may lead to an over oscillation of the solutions (and therefore leads to false results), we may use the fourth-order order Rungge-Kutta- Algorithm [5].
One wants a slight deviation from the original orbit and low computational times slightly. One hundered periods were chosen to simulate. The accuracy by default for the relative error (relerr), i.e. the difference of the computed and exact solution, is *relerr* $= 10^{-6}$; therefore, we want to decrease this value until a reasonable result is obtained. The obtained values can be seen in table 1. It is visible that the Orbit moves

Option	reler	Δ r in m	time in s
1	10^{-5}	$710.63 * 10^6$	0.02
2	10^{-6}	55.7610^6	0.29
3	10^{-8}	$426 * 10^3$	0.31
4	10^{-10}	$6.610 * 10^3$	0.69
5	10^{-12}	446.08	1.29
3	10^{-13}	970.64	3.54
4	10^{-14}	349.35	3.07

Table 1: *Values of how much the orbit moves from its initial condition after 100 periods by changing the relative error. The time was measured for each run for a function that calculates the spectra over 100 periods and $(\theta(t), \omega(t))$. The default option is red highlighted, while the chosen acuuracy is highlighted in yellow*

less from its original position when increasing the accuracy of the numerical method. However, the time increases as well, and if one wants to perform the calculation over, let us say 200 times, one might get an unreasonable long time to wait for the data evaluation.
Therefore it might be interesting to see how the accuracy changes our spectra $(\theta(t), \omega(t))$. We need an accuracy that does not change our spectra. Such an evolution can be seen in figure 11. Like the first image, one would qualitatively say it is chaotic; the one with lower relative error get periodic. Therefore the conclusion is also mainly dependent on Numerical stability. As it is seen, the spectra do not



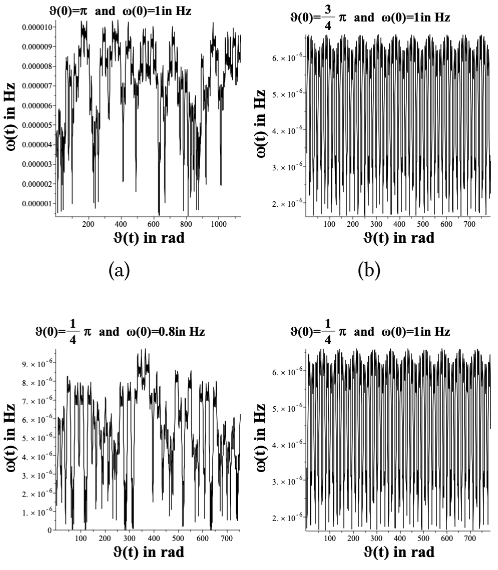
(a) (b)

Figure 1: *The plot (a) shows a Orbit calculated for the default option of Maple, while (b) shows option 5 recording to the settings of table 1.*

change regarding a change from option 3 to option 4 (see table 1). Therefore the accuracy of option three would be considered stable. However, to still be flexible for later calculations, we may choose a relative error of option 5.
To make the impact of the numerical error even more visible, an amount of $600T_H$ was used, and their orbits plotted (see figure 1).
As one can see in figure 1, the orbit changes as in table 1 over its initial position. The change is visible from the line thickening.

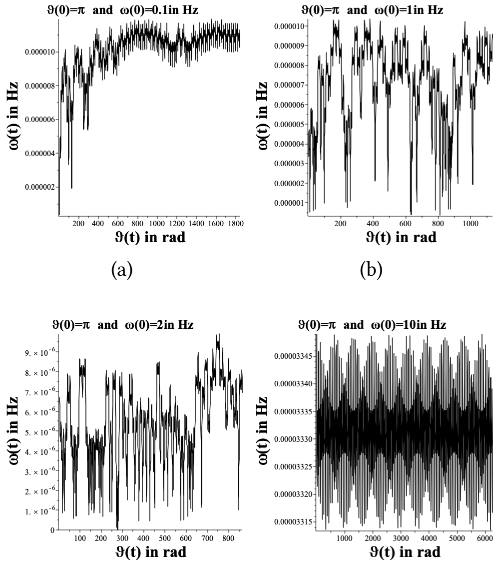
3 Initial boundary conditions

In order to investigate the chaotic behaviour of the rotation of Hyperion, one would have to look at how the system changes based on initial conditions for the rotational frequency $\omega(t)$ and the axis angle $\theta(t)$. Therefore we will try different initial Boundary conditions and investigate their impact on the spectra.
Different sets of initial conditions were chosen. The range they were plotted is $100T_H$, as one would want to investigate the long term periodic behaviour if there is one. Also, from the previous section, we can conclude that this time range should return a stable numerical spectrum.
As $\theta(t)$ is the rotational Axis defined in spherical coordinates, the initial values has to be within the range of $-\pi \leq \theta(0) \leq \pi$. The initial conditions of $\omega(0)$ are chosen for a low multitude of the circular frequency ω_H. The spectra of figure 2 a and c show no evidence of a long term periodic behaviour. On the other side, figure 2 b and d seems to show a periodic spectrum.
Another important aspect of figure 2 is that between (c) and (d), there is a small change in $\omega(0)$, which result in a completly different spectra. Even more interesting is that one leads to chaos and the other seems to be stable periodic.



(a) (b) (c) (d)

Figure 2: *Different initial confitins regarding mainly the change in circular frequency. On top of every spectra the boundary conditions as printed.* $\omega(0)$ *is always a multitude of* ω_H *from equation 5.*



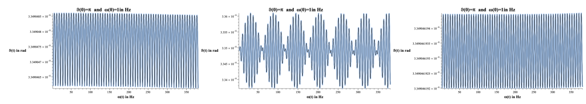
(a) (b) (c) (d)

Figure 3: *Different initial confitins regarding mainly the change in circular frequency. On top of every spectra the boundary conditions as printed.* $\omega(0)$ *is always a multitude of* ω_H *from equation 5*

This fullfills our definition of chaos and therefore we may conclude that the system of Hyperion and Saturn is chaotic. We could also conclude that both, $\theta(0)$ and $\omega(0)$ contribute to a chaotic behaviour (or periodic).

The subfigures in figure 3 show the change of the spectra regarding the change of the initial condition of the circular frequency $\omega(0)$. As the initial circular frequency of Hyperion increases, the system becomes more deterministic, i.e. periodic. However, this is not always valid. If the value is over $100 * \omega_H$, then the rotation becomes chaotic again. The same happens if ω_H is chosen as almost zero, the long periodicity is occurring again in the spectra. Therefore the periodic behaviour has windows of periodicity for $\omega(0)$ between 7 and 100.

Another important aspect of the investigation is whether the eccentricity e or the asphericity β impacts the chaotic or deterministic behaviour of the spectra. Therefore four different conditions of e and β were chosen, while the initial conditions $\theta(0)$ and $\omega(0)$ were kept constant. Therefore the change of a chaotic spectrum might be useful; consequently, the spectra of $\theta(0) = \pi$ and $\omega(0) = 1\omega_H$ were chosen. As β or e decreases, the system seems to tend to be more stable (see figure 4 and 12) and leads to quasiperiodic spectra first (see figure 12 a and 12 c) until it collapses into a periodic spectra. Even by changing the initial boundary conditions the system always tends to get more stable. Therefore one would conclude that a main contribution weither a system is chaotic or not is influenced by β and e.



(a) (b) (c)

Figure 4: *The plot (a) shows a spectra for* $\beta = \frac{1}{1000}$ *and e=0.232. (b) shows a spectra for* $\beta = 0.89$ *and* $e = \frac{1}{1000}$*. (c) shows a spectra for* $\beta = \frac{1}{1000}$ *and* $e = \frac{1}{1000}$*.*

4 Poincaré-section

As concluded in the previous section, some initial conditions yield chaotic spectra, while others are periodic.

To investigate the chaotic behaviour of the system further, one could perform a Poincaré-section, similar to [3]. To do so one records the axis θ and rotational frequency ω for every time the moon crosses the aphelion, i.e. $n * T_H$. Then all points are plotted in a scatter plot. We will generate 400 different initial conditions and record 100 orbits. The black dots represent the total amount of these 400 initial conditions.

As one can see from the obtained Poincaré-section in figure 5, some initial conditions show a closed line and some are, what seems to be, randomly distributed. At $\omega = 4$, the Poincaré-section plot shows a sinusoidal curve, indicating quasiperiodic behaviour. Besides the sea of

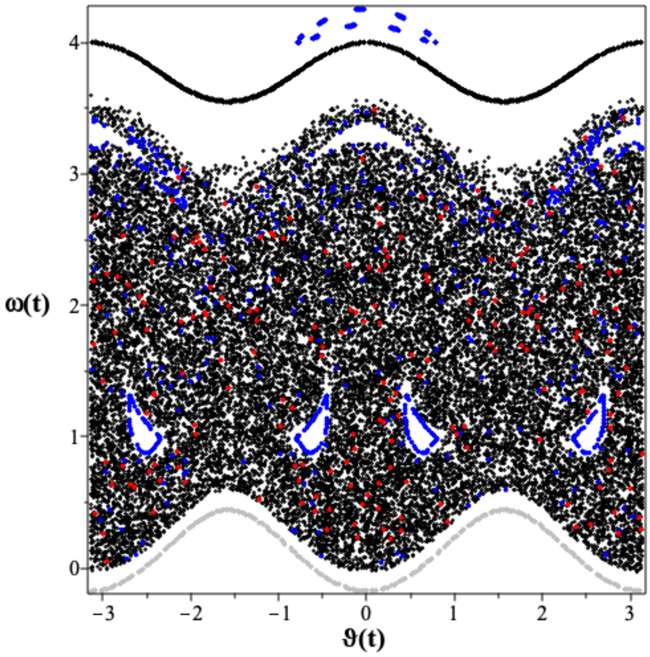


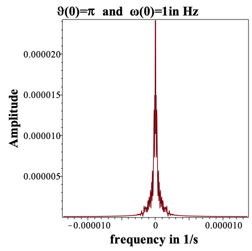
Figure 5: *This figure shows a Poincaré-section for 200 different initial conditions. for every initial condtitons 100 Orbits were recorded and therefore for every passing though the aphelion the rotational frequency were recorded and the axis orientation. Coloured dots show interesting behaviour.*

chaos and periodic behaviour, the plot shows areas not occupied by any initial conditions. Therefore we could conclude that the system avoids some areas. For the blue points, the initial condition was chosen as a loop though $\omega(0)$ while $\theta(0) = \frac{\pi}{4}$ remained constant. The grey dots have the same initial condition as the blue ones. The blue highlighted closed curve shows a periodic initial condition. It returns over some periods, always back to its starting point resulting in a closed curve. Such examples can be seen in the area around $(\theta(-1) - \theta(1), \omega(4) - \omega(4.5))$ and $(\theta(0.5) - \theta(1), \omega(1) - \omega(1.5))$. Therefore it is predictable when it will come to its original position. However, some blue points are randomly distributed and show no evidence of any periodicity. An example for a chaotic boundary condition is the red $(\theta(0) = \frac{3\pi}{4}, \omega(0) = 2)$ highlighted points, which seems to be randomly distributed with no sign of order, periodicity or any other structure.
The main part of the Poincaré-section appears to be occupied by chaos, as the little amount is periodic. Therefore we can qualitatively say that the Hyperions-system is chaotic.

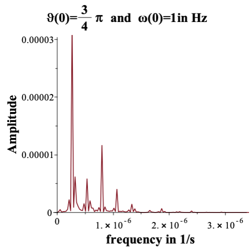
5 Fourier transformation

A different approach to investigating a continuous spectrum is the Fourier transform (FT). The FT gives information about the frequencies contributing to the spectra as it decomposes them into separate peaks. In a periodic case, it is well known that an FT would return a set of discrete and finite frequencies, which are contributing to the total spectra the FT was performed on. Now the question arises regarding the topic what kind of spectrum would the FT of a Chaotic spectrum return?
As we discussed, a chaotic spectra has no long term periodic struckture, therefore one would expect the fourier transfrom do not to result in a set of discreete peaks, but rather in a broad gaussian or lorenzian distribution or even multiple distributions, as presented in [6] or . Such an example can be seen in figure 6.

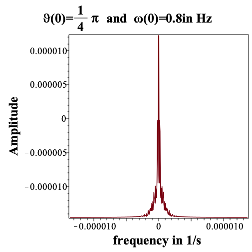
It is visible that the FT of an assumed periodic spectra



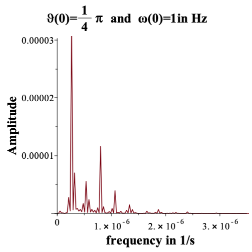
(a)



(b)



(c)



(d)

Figure 6: *Fourier Transformation spectra performed on the spectra showed in 2.*

results in discrete peaks, as expected. Periodic conditions can be seen in figure 6 b and figure 6 d. This is also a further validation that the assumed periodic system is periodic. Each peak represents a frequency contributing to the whole spectra, e.g. seen in figure 3 d.
It is noticeable that for the recorded Chaotic spectra, the shape of the FT is Lorentzian shaped, shown in 6 a and c. Also noticeable is that the FT does not yield any discrete peaks. It also seems that the curve converges towards zero and is not finite. On the other hand, the FT for periodic conditions is finite and discrete, visible through the $\delta(x)$-functional shape. As discussed in the section Initial

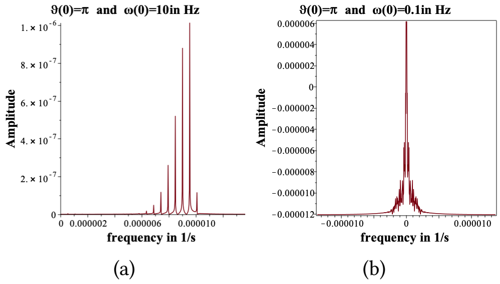


Figure 7: *An FT performed on a defenite periodic spectra from figure 3 and a Spectra that was assumed to gravitate towards a periodic spectra form figure 2*

conditions, for the initial condition of $\omega(0) = 0.1\omega_H$ and $\theta(0) = \pi$, it seems that the movement gravitates towards a periodic motion. The FT shows minor deviations within the Lorentzian shaped curve, interpreted as discrete peaks.
As the FT has always had a similar shape for a chaotic system, a standard deviation was performed.
That investigation aimed to see whether the FT of different chaotic spectra would lead to a different standard deviation and whether it would give any information or measure of the chaoticity of the system. In table 2 one can see the

$\omega(0)/\omega_H$	$\theta(0)$ in rad	standard deviation
0.1	π	$3.609 * 10^{-5}$
1	π	$3.927 * 10^{-5}$
1	$\frac{\pi}{4}$	$3.717 * 10^{-5}$
2	π	$3.105 * 10^{-5}$
3	π	$4.082 * 10^{-5}$
4 *	π	$4.088 * 10^{-7}$
10 *	π	$4.030 * 10^{-7}$

Table 2: *This table shows the standard deviation of the FT for different boundary conditins. Those marked wit * are qualitatively assumed periodic spectra.*

different standart deviations for different Initial conditions. We can see that chaotic conditions have a standartd deviation in the magnitude around 10^{-5}, while the periodic ones are around a magnitude of 10^{-7}. As one may interpreted from previous graphs that the initial contition ($\omega(0) = 0.1\omega_H$, $\theta(0) = \pi$) yields a semi periodic condition, i.e. a initial condition which is at the beginnining chaotic but gravitates towards a periodic behaviour, the value of the standart deviation for this system should be mentionably small in comparison to the other chaotic spectra, this however is not the case.
It seems like that there is no direct relation in describing the chaotic spectra with the FT nor that the Standart deviation is a usable value for determining chaos. Therefore it is no quantitative value. However, it is visible that the standard deviation is broader for chaotic systems than for periodic ones.
The FT can give a qualitative answer whether the system is chaotic or not, as the FT of a chaotic system is not discrete. The FT of a chaotic system decreases and converges towards zero, while the FT for a periodic system returns discrete and finite values/peaks.

6 Compairison to other moons

The same analysis can be carried out for other moons in the solar system to investigate if their behaviour is similar to Hyperion. Therefore the physical conditions change, e.g. Mass of the planet and moon, semi major axis and β and e.
As one can see in figure 8, the spectra shape is affected

Stellar Object	Mass in kg	β	e	a in m
Hyperion[8]	$5.5855 * 10^{18}$	0.89	0.232	$1.501 * 10^{9}$
Saturn[9]	$5.5832 * 10^{26}$	-	-	-
Phobos[10]	$1.0659 * 10^{16}$	0.82	0.0151	$9.376 * 10^{6}$
Deimos[11]	$1.4762 * 10^{15}$	0.81	0.0002	$2.346 * 10^{7}$
Mars[12]	$6.4169 * 10^{23}$	-	-	-

Table 3: *In this tbale all data of the stellar objects used in this work are presented.*

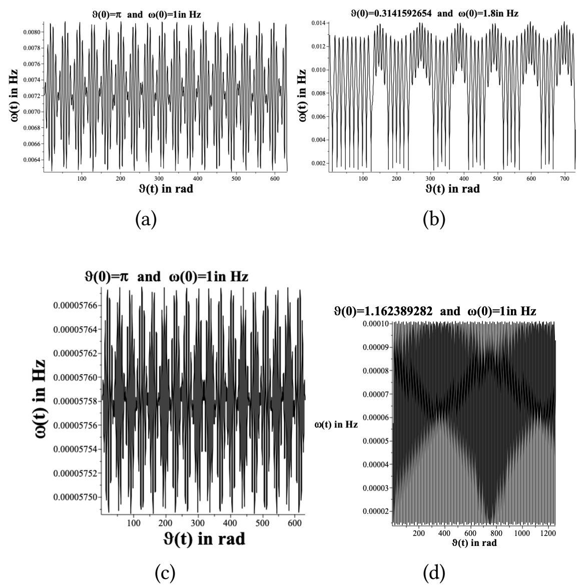
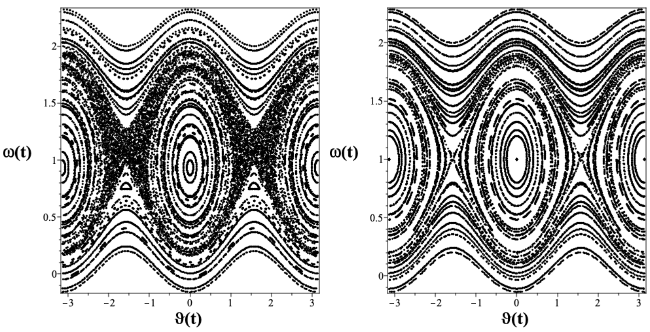


Figure 8: *Different inital conditions for the moons Phobos and Deimos. (a) and (b) are for Phobos while (c) and (d) are from Deimos.*

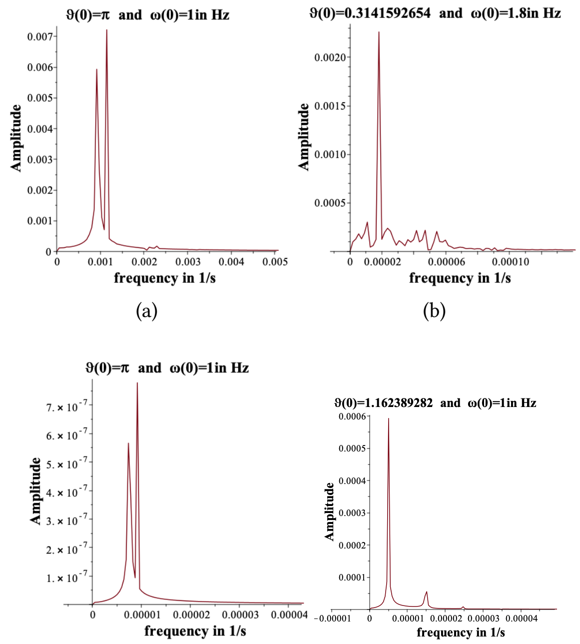
by small changes of the initial circular frequency and the

axis position. However, their motion is mostly periodic over a long time. One would say that both moons seem to be Quasichaotic.
As it is visible in figure 8 (b) for the motion of Phobos, the motion chages drastically within the interval of $\theta[100, 159]$. After the curve changing interval the Moon is in a periodic rotation. It was barely pure chaotic motion found, as for Hyperion. Since the Poincaré-section seems to be a powerful tool to investigate Chaotic behaviour, one was done as well for Phobos and Deimos with the same methology as described in the section "Poincaré-section". The Poincaré-section for both moons show less "seas of Chaos". Most Boundary conditions seems to be Quasiperiodic or periodic spectrum and therefore not chaotic. Compairing the Poincaré-section of Phobos and Deimos in figure 9 one can see that Phobos has a broader chaotic zone than Deimos. Another aspect of the Poincaré-section of Deimos is, that it has a Periodic condition at the point $(1, 0)$ .It seems like, as the Chaos is mostly caused by the eccentricity e and asphericity β. As discussed in the Fourier Transformation section, a periodic system gives sharp descrete and finite peaks, while a chaotic system is more broad and Lorenzian shaped.
In the figure 10 a, c and d it is visible that the aplication of an FT returnes dicrete and finite peaks. Therefroe one could conclude here as well periodic boundary conditions.
An interesting FT is shown in figure 10 b. It shows one sharp peak, but also a distribution which does not look finite. The broadening around the sharp peak could come from the transition within the interval $\theta[100, 159]$. Regarding the values for the stadard deviation σ from table 4 and from the conclusion from the section "Fourier Transformation" most of the initial conditions should return chaotic behaviour.



(a) (b)

Figure 9: *The plot (a) Shows the Poincaré-section for Phobos. (b) Shows the Poincaré-section for Deimos.*



(a) (b)

(c) (d)

Figure 10: *The FT perforemed on Different inital conditions for the moons Phobos and Deimos. (a) and (b) are for Phobos while (c) and (d) are from Deimos. The Fourier transformed spectra are those from 8.*

Moon	$\omega(0)/\omega_H$	$\theta(0)$ in rad	σ
Phobos	1	π	$1.386 * 10^{-5}$
Phobos	1	$0.1 * \pi$	$4.239 * 10^{-5}$
Deimos	1	π	$3.717 * 10^{-5}$
Deimos	1	0.37π	$4.709 * 10^{-8}$

Table 4: *Standard deviation of the FFT with different initial conditions. The calculated values correspond to the figure 10.*

7 Conclusion and Outlook

The investigation of Hyperion showed that the rotation of Hyperion is dominated by chaos, i.e. aperiodic long time behaviour and major changes for small changes of initial conditions. This can not only be seen by numerically simulating its circular frequency over axis orientation but also in a Poincaré-section and the FT of the Spectra.
Furthermore, the investigation showed that the periodicity is dependent on the asphericity and eccentricity. Decreasing these two values lead to a more stable and periodic system. However, this is not the only reason for Hyperions rotation behaviour, as prior researche suggest that the rotation is also a product of Hyperions tidal friction[13]. The research [13] also states that the spin-orbit coupling is not valid for

asymmetry like Hyperion. Also, it seems like Hyperion has large Chaotic zones but is dominated mainly by periodic conditions [13]. Therefore, the next step might be to investigate further and include Hyperions symmetry and verify if it fits the reasoning stated in [13].
The investigation of the moons Phobos and Deimos showed signs of chaotic behaviour through its quasiperiodic domain. Also, the Poincaré-section of these two moons showed that it is mainly dominated by quasiperiodic or even periodic leading initial conditions. The FT-analysis also showed that the Phobos and Deimos seem periodic and, therefore, not chaotic.
This motion is expected from the point of recent research [14]. It seems to be that Phobos and Deimos are at the moment locked in a Periodic, which is a long time can turn into Chaotic ones.
Chaotic motions seem as well to get damped over time. Therefore, if one waits long enough, the asphericity and eccentricity decrease over time, leading to a more periodic system [15]. Therefore, one day Hyperion may go in a Steady periodic motion.

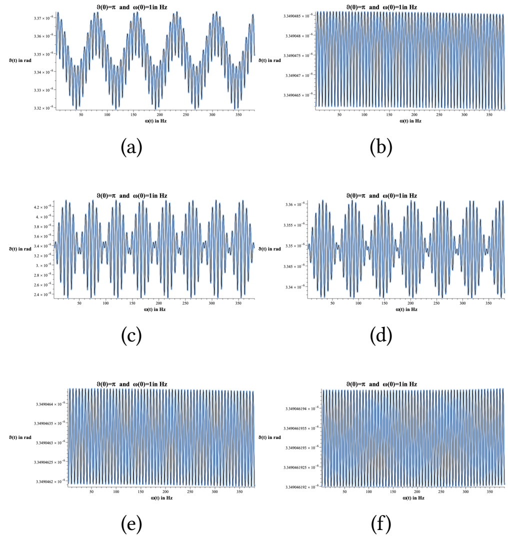


Figure 12: *The plot (a) shows a Orbit calculated for the default option of Maple, while (b) shows option 5 recording to the settings of table 1.*

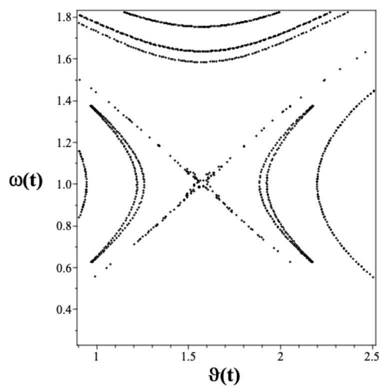


Figure 13: *The Chaotic zone of Deimos, similiar like in [13]*

Appendix

7.1 Plots and Tables

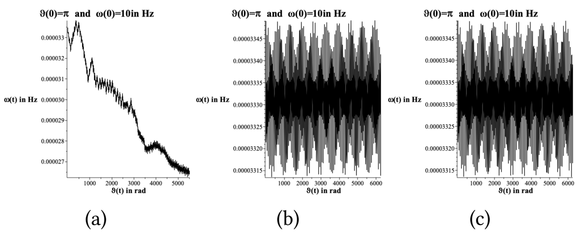


Figure 11: *These three figure present how the spectra are changing if one differs their relative error. (a) shows a relative error of (b) and (c) .*

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