Time Series Analysis

Lecture 3: Introduction to ARIMA

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Recap

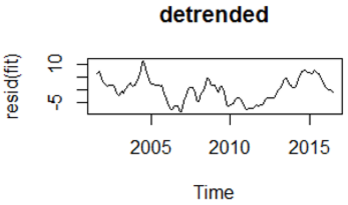
How to make data stationary?

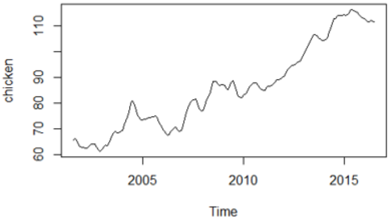
- Transformations (log, other)
- Detrending
  - Differencing
  - Linear regression
  - Kernel smoother
  - ...

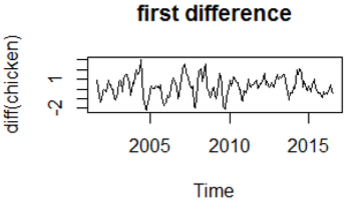
How shall we model the data after detrending and transformations (residuals)? $\rightarrow$?**ARIMA models!**

ARIMA models

- Why ARIMA models?
  - Removing trend is not sufficient







Moving average models

- **Moving average model of order q, MA(q)**

$$x_t = w_t + \theta_1 w_{t-1} + ... + \theta_q w_{t-q}$$
$$= \sum_{j=0}^{q} \theta_j w_{t-j}$$

  - $w_t \sim wn(0, \sigma_w^2)$
  - $\theta_1, ...\theta_q$ constants, $\theta_q \neq 0$ and $\theta_0 = 1$

- **Moving average operator**

$$\theta(B) = \sum_{j=0}^{q} \theta_j B^j$$

- MA(q): $x_t = \theta(B) w_t$

Linear process

x_t is a **linear process** if

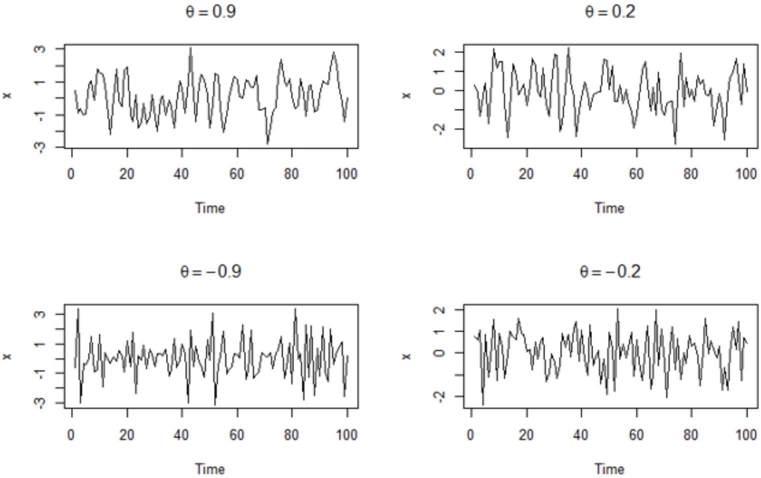
$$x_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j w_{t-j}$$

Property: It can be shown that

$$\gamma_x(h) = \sigma_w^2 \sum_{j=-\infty}^{\infty} \psi_{j+h} \psi_j$$

Example: MA(1)

$$x_t = w_t + \theta w_{t-1}$$



Example: MA(1)

$$x_t = w_t + \theta w_{t-1}$$

- Autocovariance and ACF

$$\gamma(h) = \begin{cases} (1+\theta^2)\sigma_w^2 & h = 0 \\ \theta\sigma_w^2 & h = 1 \\ 0 & h > 1 \end{cases}$$

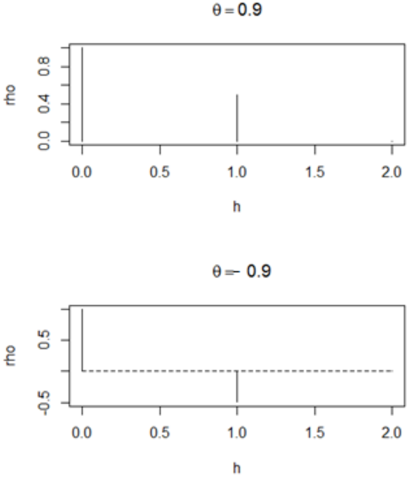
$$\rho(h) = \begin{cases} \frac{\theta}{1+\theta^2} & h = 1 \\ 0 & h > 1 \end{cases}$$

Note: $\rho(0) = 1$ is often not written as it is trivial.

- Process is stationary

Example: MA(1)

- Note: $\rho(0) = 1$ is often not shown $\rightarrow$ only 1 bar



AR models

- **Autoregressive model of order p, $AR(p)$**

$$x_t = \phi_1 x_{t-1} + ... + \phi_p x_{t-p} + w_t$$

  - x_t is stationary if x_0 is sampled from the stationary distribution
  - $w_t \sim wn(0, \sigma_w^2)$
  - $\phi_1, ...\phi_p$ constants, $\phi_p \neq 0$
  - $Ex_t = 0$

- Note: if $Ex_t = \mu \neq 0$, model $x'_t = x_t - \mu$

AR models

Another form

- **Autoregressive operator**

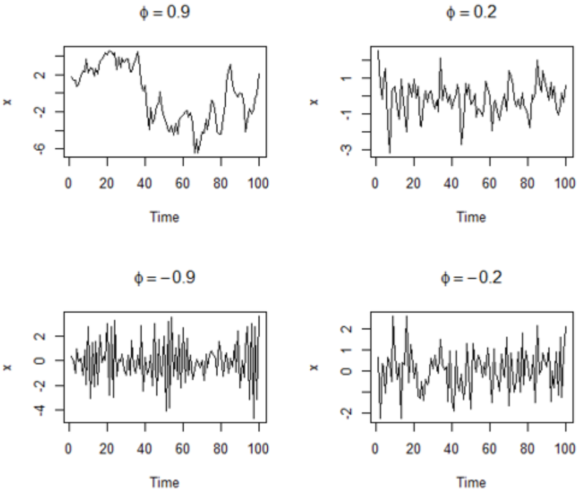
$$\phi(B) = 1 - \phi_1 B - \ldots - \phi_p B^p$$

- AR(p) model

$$\boxed{\phi(B)x_t = w_t}$$

Example: AR(1)

- How do these plots differ? $x_t = \phi x_{t-1} + w_t$



Ar(1) (read at home)

$$x_t = \phi x_{t-1} + w_t$$

Mean function:

$Ex_t = \phi Ex_{t-1} + Ew_t = \phi Ex_{t-1} = \phi(\phi Ex_{t-2}) = \cdots = \phi^t Ex_0$
for $Ex_0 = 0$, $Ex_t = 0$ for all t.

Variance $\operatorname{var}(x_t)$ when $Ex_0 = 0$ and w_t is uncorrelated with x_0 for all t:

$\operatorname{var}(x_t) = E\{(x_t - 0)^2\} = E\{\phi^2 x_{t-1}^2 + 2\phi x_{t-1} w_t + w_t^2\} =$
$\phi^2 \operatorname{var}(x_{t-1}) + 2\phi \operatorname{cov}(x_{t-1}, w_t) + \operatorname{var}(w_t) = \phi^2 \operatorname{var}(x_{t-1}) + \operatorname{var}(w_t) =$
$\phi^2 \operatorname{var}(x_{t-1}) + \sigma_w^2 = \phi^2(\phi^2 \operatorname{var}(x_{t-2}) + \sigma_w^2) + \sigma_w^2 =$
$\phi^{2t} \operatorname{var}(x_0) + \sigma_w^2 \sum_{k=0}^{t-1}(\phi^{2k}) = \phi^{2t} \operatorname{var}(x_0) + \frac{\sigma_w^2(1-\phi^{2t})}{1-\phi^2}$

When $\operatorname{var}(x_0) = \frac{\sigma_w^2}{1-\phi^2}$ then $\operatorname{var}(x_t) = \frac{\sigma_w^2}{1-\phi^2}$ and time independent.

A(1) (read at home)

$$x_t = \phi x_{t-1} + w_t$$

$$x_t = \phi(\phi x_{t-2} + w_{t-1}) + w_t = \cdots = \phi^k x_{t-k} + \sum_{j=0}^{k-1} \phi^j w_{t-j}$$

$$\gamma(x_t, x_{t-k}) = \text{cov}(x_t, x_{t-k}) = E(x_t x_{t-k}) =$$
$$E\{(\phi^k x_{t-k} + \sum_{j=0}^{k-1} \phi^j w_{t-j}) x_{t-k}\} = \phi^k \text{var}(x_{t-k}) = \frac{\phi^k \sigma_w^2}{1-\phi^2}$$

Hence,

$$\gamma(k) = \frac{\phi^k \sigma_w^2}{1 - \phi^2}$$

Also,

$$\rho(h) = \phi^h$$

AR(1) **whiteboard**

- **Property:** If $|\phi| < 1$ and $\sup \operatorname{var}(x_t) < \infty$

$$x_t = \sum_{j=0}^{\infty} \phi^j w_{t-j}$$

- Show it by
  - Substitution
  - Taylor expansion
  - Coefficient matching
- Autocovarince and ACF

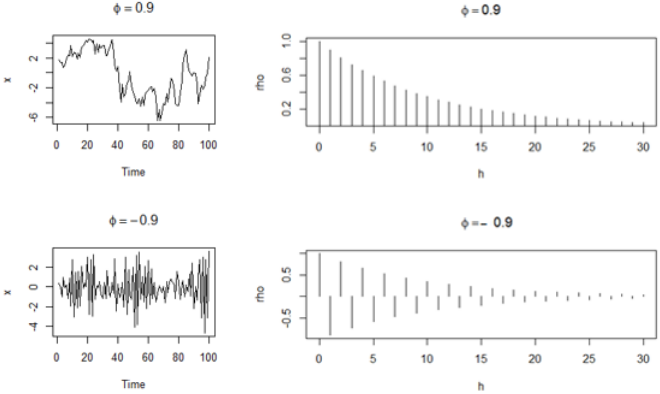
$$\gamma(h) = \frac{\sigma_w^2 \phi^h}{1-\phi^2} \quad \rho(h) = \phi^h$$

for $h \geq 0$.

Example: AR(1)

Autocovarince and ACF (for $h \geq 0$)

$$\gamma(h) = \frac{\sigma_w^2 \phi^h}{1 - \phi^2} \quad \rho(h) = \phi^h$$



Explosive AR models

- Explosive =series become arbitrarily large in magnitude
- AR(1): What if $|\phi| > 1$?
  - $x_t = \phi^p x_{t-p} + \sum_{j=0}^{p-1} \phi^p w_{t-j} \rightarrow$ grows exponentially
  - Stationary? Check variance
- Can we make it stationary?
  $x_t = \phi^{-1} x_{t+1} - \phi^{-1} w_{t+1} = \phi' x_{t+1} + w'_t$
  - Stationary, but dependent on the future
  - $w'_t \sim N(0, \phi^{-2}\sigma_w^2)$
  - $x_t = -\sum_{j=1}^{\infty} \phi_{-j} w_{t+j}$

Causal process

A stationary process is **causal** if it is only dependent on the past values of the process

Def: A linear process is nonexplosive and causal if it can be written as a one-sided sum:

$$x_t = \sum_{j=0}^{\infty} \psi_j w_{t-j} = \psi(B) w_t$$

where $\psi(B) = \sum_{j=0}^{\infty} \psi_j B^j$ and $\sum_{j=0}^{\infty} |\psi_j| < \infty$.

MA equivalence

Whiteboard

$$\rho(h) = \begin{cases} \frac{\theta}{1+\theta^2} & h = 1 \\ 0 & h > 1 \end{cases}$$

Note: MA(1) gives equivalent models for $\theta = s$ and $\theta = \frac{1}{s}$
Probabilistic expressions equivalent: ACF identical
$\rightarrow$ we can not distinguish between these models

Invertibility of MA

Def: An MA process is **invertible** if it has a causal AR representation,

$$w_t = \sum_{j=0}^{\infty} \pi_j x_{t-j}$$

Example: MA(1) with $\theta = 1/5$ is invertible, $\theta = 5$ not.

ARMA models

- **Autoregressive moving average ARMA(p,q)**

$$x_t = \phi_1 x_{t-1} + \ldots + \phi_p x_{t-p} + w_t + \theta_1 w_{t-1} + \ldots + \theta_q w_{t-q}$$

  - $\phi_p \neq 0, \theta_q \neq 0$
  - Is stationary
  - $Ex_t = 0$

- p-autoregressive order, q-moving average order
- Alternative form

$$\boxed{\phi(B)x_t = \theta(B)w_t}$$

- Note: $x_t = \phi^{-1}(B)\theta(B)w_t = \sum_{j=-\infty}^{\infty} \psi_j w_{t-j}$
  - But series might be non-convergent

Parameter redundancy

Note: we can multiply both sides with $\eta(B)$

$$\eta(B)\phi(B)x_t = \eta(B)\theta(B)w_t$$

- The resulting model looks different (higher orders)
- Underlying model is actually the same

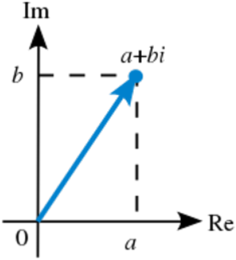
Example: $x_t = w_t$, white noise. Let $\eta(B) = 1 - 0.5B$.
We get

$$x_t - 0.5x_{t-1} = w_t - 0.5w_{t-1}$$

Looks like ARMA(1,1)!

Reminder: complex numbers

- Imaginary unit $i^2 = -1$
- Complex number z=a+ib
- Conjugate $\bar{z} = a - ib$



- Absolute value $|z|^2 = z\bar{z} = a^2 + b^2$
- Trigonometric form
  $z = r(\cos(\theta) + i\sin(\theta))$
- Eulers formula $e^{i\theta} = \cos(\theta) + i\sin(\theta))$

- Therefore

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Reminder: polynomials

- Any polynomial $P_r(x)$ of degree r can be written as

$$P_r(x) = a(x - z_1)...(x - z_r)$$

- where z_i are roots (real or complex)
- If z_i is a root, $\bar{z}_i$ is also a root

Causal ARMA

Def: Linear process is **causal** and **nonexplosive** if

- $x_t = \sum_{j=0}^{\infty} \psi_j w_{t-j}$ (depends on the past only)
- $\sum_{j=0}^{\infty} |\psi_j| < \infty$
- We set $\psi_0 = 1$ by convention.

Property: ARMA(p,q) is **causal** iff roots $\phi(z') = 0$ are outside unit circle, i.e. $|z'| > 1$

$$\boxed{\phi(B)x_t = \theta(B)w_t}$$

Causal ARMA

Example: Is the ARMA process below causal?

$$x_t = 0.4x_{t-1} + 0.3x_{t-2} + 0.2x_{t-3} + w_t - 0.1w_{t-1}$$
$$\Rightarrow \phi(B) = 1 - 0.4B - 0.3B^2 - 0.2B^3$$

```
> z=c(1, -0.4,-0.3,-0.2)
> polyroot(z)
[1]  1.060419-0.000000i -1.280210+1.753904i -1.280210-1.753904i
>
```

Invertible ARMA

Def: ARMA(p,q) is **invertible** if

- $w_t = \sum_{j=0}^{\infty} \pi_j x_{t-j}$ (depends on the past only)
- $\sum_{j=0}^{\infty} |\pi_j| < \infty$

Property: ARMA(p,q) is **invertible** iff roots $\theta(z') = 0$ are outside unit circle, i.e. $|z'| > 1$

$$\boxed{\phi(B)x_t = \theta(B)w_t}$$

Coefficient matching **whiteboard**

- $x_t = \sum_{j=0}^{\infty} \psi_j w_{t-j} \rightarrow x_t = \psi(B) w_t$
- $w_t = \sum_{j=0}^{\infty} \pi_j w_{t-j} \rightarrow w_t = \pi(B) x_t$
- How to find coefficients in ψ and π $\rightarrow$ **coefficient matching**

$$\phi(z)\psi(z) = \theta(z) \quad \pi(z)\theta(z) = \phi(z)$$

- Example: $x_t = 0.4x_{t-1} + 0.45x_{t-2} + w_t + w_{t-1} + 0.25w_{t-2}$

```
> ARMAtoMA(ar=.9,ma=0.5, 6)
[1] 1.400000 1.260000 1.134000 1.020600 0.918540 0.826686
```

Home reading

- Shumway and Stoffer, section 3.1
- R code: arima.sim, arima, polyroot, ARMAtoMA, ARMAacf
  - Check carefully arima() docs to see how ar and ma coefficients are specified in the software