Forecasting of Spatio-temporal Chaotic Dynamics with Recurrent Neural Networks: a comparative study of Reservoir Computing and Backpropagation Algorithms

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ABSTRACT

How effective are Recurrent Neural Networks (RNNs) in forecasting the spatiotemporal dynamics of chaotic systems ? We address this question through a comparative study of Reservoir Computing (RC) and backpropagation through time (BPTT) algorithms for gated network architectures on a number of benchmark problems. We quantify their relative prediction accuracy on the long-term forecasting of Lorenz-96 and the Kuramoto-Sivashinsky equation and calculation of its Lyapunov spectrum. We discuss their implementation on parallel computers and highlight advantages and limitations of each method. We find that, when the full state dynamics are available for training, RC outperforms BPTT approaches in terms of predictive performance and capturing of the long-term statistics, while at the same time requiring much less time for training. However, in the case of reduced order data, large RC models can be unstable and more likely, than the BPTT algorithms, to diverge in the long term. In contrast, RNNs trained via BPTT capture well the dynamics of these reduced order models. This study confirms that RNNs present a potent computational framework for the forecasting of complex spatio-temporal dynamics.

1. Introduction

In recent years the confluence of, advances in computing power, inception of novel algorithms, and the ample availability of data has boosted the adoption of machine learning (ML) in scientific disciplines ranging from language and speech processing to engineering and medicine. A number of techniques have been developed to handle sequential data, such as automated translation of words in a sentence and modeling of spatio-temporal dynamics of physical systems. The work of Takens and that of Sauer, Yorke and Casdagli Sauer, Yorke and Casdagli (1991) showed that the dynamics on a D-dimensional attractor of a dynamical system can generally be unfolded in an embedding of dimension greater than 2D Takens (1981). The identification of a useful embedding and the construction of a forecasting model have been the subject of life-long research effort Bradley and Kantz (2015). Here we examine and compare two of the most prominent techniques in the forecasting of dynamical systems, namely Recurrent Neural networks trained with backpropagation (RNN) and Reservoir Computing (RC). We note that our RC implementation also uses a recurrent neural network, but according to the RC paradigm, it does not train the internal network parameters. We consider the case of fully observed systems but also the case of partially observed systems, which is more relevant for real world applications where typically there is no access to all the degrees-of-freedom of the dynamical system.

On the one hand, we have RNNs which are an architecture designed to capture long-term dependencies in sequential data Pascanu, Mikolov and Bengio (2013); Bengio, Simard and Frasconi (1994); Hochreiter (1998); Goodfellow, Bengio and Courville (2016). The recent success of RNNs is largely attributed to the regularization of their training process through the Long Short-Term Memory (LSTM) gates, that learn to remember and forget information. The potential of RNNs for capturing temporal dynamics in physical systems was explored first using low dimensional Elman RNNs Elman (1990) without gates to predict unsteady boundary-layer development, separation, dynamic stall, and

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dynamic reattachment back in 1997 Faller and Schreck (1997). In the recent years, Bianchi, Maiorino, Kampffmeyer, Rizzi and Jenssen (2017) RNN architectures have been bench-marked for short-term load forecasting in networks in low dimensional time-series, while in Laptev, Yosinski, Li and Smyl (2017) they are utilized for extreme event detection in low order time-series. In Wan and Sapsis (2018) LSTM networks are used as surrogates to model the kinematics of spherical particles in fluid flows. In Vlachas, Byeon, Wan, Sapsis and Koumoutsakos (2018) RNNs with LSTM cells were utilized in conjunction with a mean stochastic model to capture the temporal dependencies and long-term statistics in the reduced order space of a dynamical system and forecast its evolution. The method demonstrated better accuracy and scaling to high-dimensions and longer sequences than Gaussian Processes (GPs). In Wan, Vlachas, Koumoutsakos and Sapsis (2018) the LSTM is deployed to model the residual dynamics in an imperfect Galerkin-based reduced order model derived from the system equations. RNNs are practical and efficient data-driven approximators of chaotic dynamical systems, due to their **(1)** universal approximation ability Schäfer and Zimmermann (2006); Siegelmann and Sontag (1995) and **(2)** ability to capture temporal dependencies and implicitly identify the required embedding for forecasting.

On the other hand, we have Reservoir Computing (RC) which has shown significant success in modeling the full-order space dynamics of high dimensional chaotic systems. In Pathak, Lu, Hunt, Girvan and Ott (2017) RC is utilized to build surrogate models for chaotic systems and compute their Lyapunov exponents based solely on data. The data-driven RC model is coupled with a knowledge based model based on equations in Pathak, Wikner, Fussell, Chandra, Hunt, Girvan and Ott (2018b) to enhance the forecasting accuracy. A scalable approach to high-dimensional systems with local interactions is proposed in Pathak, Hunt, Girvan, Lu and Ott (2018a). In this case, an ensemble of RC networks is used in parallel. Each ensemble member is forecasting the evolution of a group of modes while all other modes interacting with this group is fed at the input of the network. The model takes advantage of the local interactions in the state-space to decouple the forecasting of each mode group and improve the scalability.

Despite the rich literature on both methods, there are very few side-by-side comparisons of the two frameworks. Therefore, the primary scope of this work is to compare the accuracy, performance, and computationally efficiency of the two methods on the full-order and reduced-order modeling of two prototype chaotic dynamical systems. We will also examine the modeling capabilities of the two approaches on reproducing correct Lyapunov Exponents and frequency spectra.

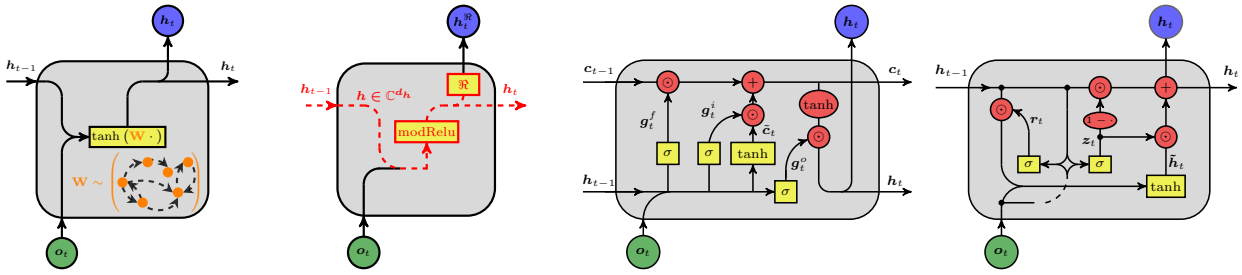
Motivated by the fact that some more recent RNN architectures, like Unitary Arjovsky, Shah and Bengio (2016); Jing, Shen, Dubcek, Peurifoy, Skirlo, LeCun, Tegmark and Soljacic (2017) and Gated Recurrent Units (GRUs) Chung, Gulcehre, Cho and Bengio (2014); Cho, van Merrienboer, Gulcehre, Bahdanau, Bougares, Schwenk and Bengio (2014), have shown superior performance than LSTMs in a wide variety of language, speech signal and polyphonic music modeling tasks, we will include these variants in our comparison studies.

We are interested in model-agnostic treatment of chaotic dynamical systems, where the time evolution of the full state or some observable is available, but we do not possess any knowledge about the underlying equations. In the latter case, we examine which method is more suitable for modeling temporal dependencies in the reduced order space (observable) of dynamical systems. Furthermore, we evaluate the efficiency of an ensemble of RNNs in predicting the full state dynamics of a high-dimensional dynamical system in parallel and compare it with that of RC. Last but not least, we discuss the advantages, implementation aspects and limitations of each model.

The structure of the paper is as follows. Section 2 provides an introduction to the sequence modeling tasks and an outline of the architectures and training methods used in this work. Section 3 introduces the measures used to compare the efficiency of the models. In section 4 the networks are compared in forecasting reduced order dynamics in the Lorenz-96 system. In section 5, a parallel architecture leveraging local interactions in the state space is introduced and utilized to forecast the dynamics of the Lorenz-96 system and the Kuramoto-Sivashinsky equation. In section 6 the GRU and RC networks are utilized to reproduce the Lyapunov spectrum of the Kuramoto-Sivashinsky equation, while section 7 concludes the paper.

2. Methods - Sequence Modeling

We consider machine learning algorithms for time-series forecasting. The models are trained on time-series of an observable $\boldsymbol{o} \in \mathbb{R}^{d_o}$ sampled at a fixed rate $1/\Delta t$, $\{\boldsymbol{o}_1, \dots, \boldsymbol{o}_T\}$, where we eliminate Δt from the notation for simplicity. The models posses an internal high-dimensional hidden state denoted by $\boldsymbol{h}_t \in \mathbb{R}^{d_h}$ that enables the encoding of temporal dependencies on past state history. Given the current observable $\boldsymbol{o}_t$, the output of each model is a forecast $\hat{\boldsymbol{o}}_{t+1}$ for the observable at the next time instant $\boldsymbol{o}_{t+1}$. This forecast is a function of the hidden state. As a consequence, the general



(a) RC Cell **(b)** Unit Cell **(c)** LSTM Cell **(d)** GRU Cell

Figure 1: The information flow for a Reservoir Computing (RC) cell, a complex Unitary cell (Unit), a Long Short-Term Memory (LSTM) cell and a Gated Recurrent Unit (GRU) cell. The cells were conceptualized to tackle the vanishing gradients problem of Elman-RNNs. The cell used in RC is the standard architecture of the Elman-RNN. However, the weights of the recurrent connections are randomly picked to satisfy the echo state property and create a large reservoir of rich dynamics. Only the output weights are trained (e.g., with ridge regression). The Unitary RNN utilizes a complex unitary matrix to ensure that the gradients are not vanishing. LSTM and GRU cells employ gating mechanisms that allow forgetting and storing of information in the processing of the hidden state. Ellipses and circles denote entry-wise operations, while rectangles denote layer operations. The information flow of the complex hidden state in the Unitary RNN is illustrated with dashed red color, while the untrained randomly picked weights of the RC with orange.

functional form of the models is given by

$$\boldsymbol{h}_t = f_h^h(\boldsymbol{o}_t, \boldsymbol{h}_{t-1}), \quad \hat{\boldsymbol{o}}_{t+1} = f_h^o(\boldsymbol{h}_t), \tag{1}$$

where f_h^h is the hidden-to-hidden mapping and f_h^o is the hidden-to-output mapping. All recurrent models analyzed in this work share this common architecture. They differ in the realizations of f_h^o and f_h^h and in the way the parameters or **weights** of these functions are learned from data, i.e., **trained**, to forecast the dynamics.

2.1. Long Short-Term Memory

In Elman RNNs Elman (1990), the vanishing or exploding gradients problem stems from the fact that the gradient is multiplied repeatedly during back-propagation through time Werbos (1988) with a recurrent weight matrix. As a consequence, when the spectral radius of the weight matrix is positive (negative), the gradients are prone to explode (shrink). The LSTM Hochreiter and Schmidhuber (1997) was introduced in order to regularize the training of RNNs and alleviate their vanishing gradient problem Hochreiter (1998). The equations that implicitly define the recurrent mapping f_h^h of the LSTM are given by

$$\begin{aligned} \boldsymbol{g}_t^f &= \sigma_f\big(\boldsymbol{W}_f[\boldsymbol{h}_{t-1}, \boldsymbol{o}_t] + \boldsymbol{b}_f\big) & \boldsymbol{g}_t^i &= \sigma_i\big(\boldsymbol{W}_i[\boldsymbol{h}_{t-1}, \boldsymbol{o}_t] + \boldsymbol{b}_i\big) \\ \tilde{\boldsymbol{c}}_t &= \tanh\big(\boldsymbol{W}_c[\boldsymbol{h}_{t-1}, \boldsymbol{o}_t] + \boldsymbol{b}_c\big) & \boldsymbol{c}_t &= \boldsymbol{g}_t^f \odot \boldsymbol{c}_{t-1} + \boldsymbol{g}_t^i \odot \tilde{\boldsymbol{c}}_t \\ \boldsymbol{g}_t^o &= \sigma_h\big(\boldsymbol{W}_h[\boldsymbol{h}_{t-1}, \boldsymbol{o}_t] + \boldsymbol{b}_h\big) & \boldsymbol{h}_t &= \boldsymbol{g}_t^o \odot \tanh(\boldsymbol{c}_t), \end{aligned} \tag{2}$$

where $\boldsymbol{g}_t^f, \boldsymbol{g}_t^i, \boldsymbol{g}_t^o \in \mathbb{R}^{d_h}$, are the gate vector signals (forget, input and output gates), $\boldsymbol{o}_t \in \mathbb{R}^{d_o}$ is the observable input at time t, $\boldsymbol{h}_t \in \mathbb{R}^{d_h}$ is the hidden state, $\boldsymbol{c}_t \in \mathbb{R}^{d_h}$ is the cell state, while $\boldsymbol{W}_f, \boldsymbol{W}_i, \boldsymbol{W}_c, \boldsymbol{W}_h \in \mathbb{R}^{d_h \times (d_h + d_o)}$, are weight matrices and $\boldsymbol{b}_f, \boldsymbol{b}_i, \boldsymbol{b}_c, \boldsymbol{b}_h \in \mathbb{R}^{d_h}$ biases. The symbol $\odot$ denotes the entry-wise product. The activation functions σ_f, σ_i and σ_h are sigmoids. For a more detailed explanation of the LSTM architecture refer to Hochreiter and Schmidhuber (1997). The dimension of the hidden state d_h (number of hidden units) controls the capability of the cell to encode history information. The hidden-to-output functional form f_h^o is given by a linear layer

$$\hat{\boldsymbol{o}}_{t+1} = \boldsymbol{W}_o \boldsymbol{h}_t, \tag{3}$$

where $\boldsymbol{W}_o \in \mathbb{R}^{d_o \times d_h}$. The forget gate bias is initialized to one according to Jozefowicz, Zaremba and Sutskever (2015) to accelerate training. An illustration of the information flow in a LSTM Cell is given in Fig. 1c.

2.2. Gated Recurrent Unit

The Gated Recurrent Unit (GRU) Cho et al. (2014) was proposed as a variation of LSTM utilizing a similar gating mechanism. Even though GRU lacks an output gate and thus has fewer parameters, it achieves comparable performance with LSTM in polyphonic music and speech signal datasets Chung et al. (2014). The GRU equations are given by

$$\begin{aligned} \boldsymbol{z}_t &= \sigma_g\big(\boldsymbol{W}_z[\boldsymbol{h}_{t-1}, \boldsymbol{o}_t] + \boldsymbol{b}_z\big) \\ \boldsymbol{r}_t &= \sigma_g\big(\boldsymbol{W}_r[\boldsymbol{h}_{t-1}, \boldsymbol{o}_t] + \boldsymbol{b}_r\big) \\ \tilde{\boldsymbol{h}}_t &= \tanh\big(\boldsymbol{W}_h\big[\boldsymbol{r}_t \odot \boldsymbol{h}_{t-1}, \boldsymbol{o}_t\big] + \boldsymbol{b}_h\big) \\ \boldsymbol{h}_t &= (1 - \boldsymbol{z}_t) \odot \boldsymbol{h}_{t-1} + \boldsymbol{z}_t \odot \tilde{\boldsymbol{h}}_t, \end{aligned} \tag{4}$$

where $\boldsymbol{o}_t \in \mathbb{R}^{d_o}$ is the observable at the input at time t, $\boldsymbol{z}_t \in \mathbb{R}^{d_h}$ is the update gate vector, $\boldsymbol{r}_t \in \mathbb{R}^{d_h}$ is the reset gate vector, $\tilde{\boldsymbol{h}}_t \in \mathbb{R}^{d_h}$, $\boldsymbol{h}_t \in \mathbb{R}^{d_h}$ is the hidden state, $\boldsymbol{W}_z$, $\boldsymbol{W}_r$, $\boldsymbol{W}_h \in \mathbb{R}^{d_h \times (d_h + d_o)}$, are weight matrices and $\boldsymbol{b}_z, \boldsymbol{b}_r, \boldsymbol{b}_h \in \mathbb{R}^{d_h}$ biases. The gating activation σ_g is a sigmoid. The output $\hat{\boldsymbol{o}}_{t+1}$ is given by the linear layer:

$$\hat{\boldsymbol{o}}_{t+1} = \boldsymbol{W}_o \boldsymbol{h}_t, \tag{5}$$

where $\boldsymbol{W}_o \in \mathbb{R}^{d_o \times d_h}$. An illustration of the information flow in a GRU Cell is given in Fig. 1d.

2.3. Unitary Evolution

Unitary RNNs Arjovsky et al. (2016); Jing et al. (2017), similar to LSTMs and GRUs, aim to alleviate the vanishing gradients problem of plain RNNs. Here, instead of employing sophisticated gating mechanisms, the effort is focused on the identification of a re-parametrization of the recurrent weight matrix, such that its spectral radius is a-priori set to one. This is achieved by optimizing the weights on the subspace of complex unitary matrices. The architecture of the Unitary RNN is given by

$$\begin{aligned} \boldsymbol{h}_t &= \text{modReLU}\left(\boldsymbol{W}_h \boldsymbol{h}_{t-1} + \boldsymbol{W}_o \boldsymbol{o}_t\right) \\ \hat{\boldsymbol{o}}_{t+1} &= \boldsymbol{W}_o\, \Re\big(\boldsymbol{h}_t\big), \end{aligned} \tag{6}$$

where $\boldsymbol{W}_h \in \mathbb{C}^{d_h \times d_h}$ is the complex unitary **recurrent** weight matrix, $\boldsymbol{W}_o \in \mathbb{C}^{d_h \times d_o}$ is the complex **input** weight matrix, $\boldsymbol{h}_t \in \mathbb{C}^{d_h}$ is the complex state vector, $\Re(\cdot)$ denotes the real part of a complex number, $\boldsymbol{W}_o \in \mathbb{R}^{d_h \times d_h}$ is the real output matrix, and the modified ReLU non-linearity modReLU is given by

$$\Big(\text{modReLU}(\boldsymbol{z})\Big)_i = \frac{z_i}{|z_i|} \odot \text{ReLU}(|z_i| + b_i), \tag{7}$$

where $|z_i|$ is the norm of the complex number z_i. The complex unitary matrix $\boldsymbol{W}_h$ is parametrized as a product of a diagonal matrix and multiple rotational matrices. The reparametrization used in this work is the one proposed in Jing et al. (2017). The complex input weight matrix $\boldsymbol{W}_o \in \mathbb{C}^{d_h \times d_o}$ is initialized with $\boldsymbol{W}_o^{re} + j\, \boldsymbol{W}_o^{im}$, with real matrices $\boldsymbol{W}_o^{re}$, $\boldsymbol{W}_o^{im} \in \mathbb{R}^{d_h \times d_o}$ whose values are drawn from a random uniform distribution $\mathcal{U}[-0.01, 0.01]$ according to Jing et al. (2017). An illustration of the information flow in a Unitary RNN cell is given in Fig. 1b.

In the original paper of Jing et al. (2017) the architecture was evaluated on a speech spectrum prediction task, a copying memory task and a pixel permuted MNIST task demonstrating superior performance to LSTM either in terms of final testing accuracy or wall-clock training speed.

2.4. Back-Propagation Through Time

Backpropagation dates back to the work of Dreyfus (1962); Linnainmaa (1976); Rumelhart, Hinton and Williams (1986), while its extension to recurrent neural networks termed Backpropagation through time (BPTT) was presented in Werbos (1988, 1990).

A forward pass of the network is required to compute its output and compare it against the label (or target) from the train data based on an error metric (e.g. mean squared loss). Backpropagation amounts to the computation of the partial derivatives of this loss with respect to the network parameters by iteratively applying the chain rule, transversing backwards the network. These derivatives are computed analytically with automatic differentiation. Based on these

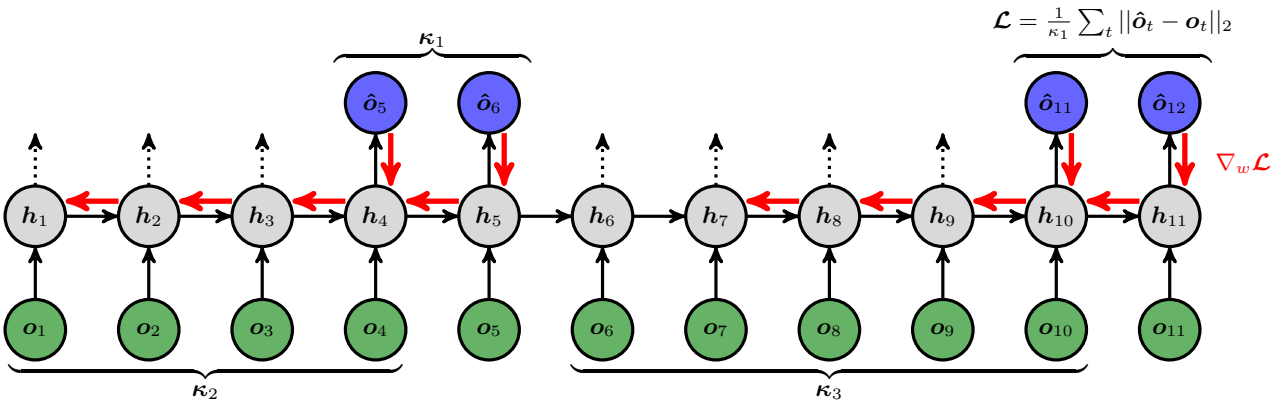


Figure 2: Illustration of an unfolded RNN. Time-series data $\boldsymbol{o}$ are provided at the input of the RNN. The RNN is forecasting the evolution of the observable at its outputs $\hat{\boldsymbol{o}}$. The difference (mean square error) of the RNN output $\hat{\boldsymbol{o}}$ and the target $\boldsymbol{o}$ from the time-series data is computed every κ_3 steps for κ_1 iterative time-steps summing up the error. The gradient of this quantity, illustrated with red arrows, is back-propagated through time (BPTT) for κ_2 previous temporal time-steps, computing the gradients of the network parameters that are shared at each "time" layer. The output of intermediate steps illustrated with dashed lines is ignored. Stateless models initialize the hidden state before every sample update with zero (in this case $\boldsymbol{h}_6 \hat{=} 0$) and cannot capture dependencies longer than κ_2. In this way, temporally adjacent samples do not have to be processed iteratively and are independent. In stateful models, the hidden state is never set to zero and there is dependency between the samples. Skipping updates by picking $\kappa_3 \geq \kappa_2$ alleviates the problem. In our study we pick $\kappa_3 = \kappa_2 + \kappa_1 - 1$ as illustrated in the figure.

partial derivative the network parameters are updated using a first-order optimization method, e.g. stochastic gradient descent.

The power of BBTT lies in the fact that it can be employed to learn the partial derivatives of the weights of any network architecture with differentiable activation functions, utilizing state-of-the-art automatic differentiation software, while (as the data are processed in small pieces called batches) it scales to large datasets and networks, and can be accelerated by employing Graphics Processing Units (GPUs). These factors made backpropagation the workhorse of state-of-the-art deep learning methods Goodfellow et al. (2016).

In our study, we utilize BBTT to train the LSTM (section 2.1), GRU (section 2.2) and Unitary (section 2.3) RNNs. There are three key parameters of this training method that can be tuned. The first parameter is the number of forward-pass timesteps performed to accumulate the error for back-propagation. The second parameter is the number of time steps in the past for the back-propagation of the gradient κ_2. This is also denoted as truncation length, or sequence length. This parameter has to be large enough to capture the temporal dependencies in the data. However, as κ_2 becomes larger, training becomes much slower, and may cause the gradient vanishing problem. In the following, we characterize as stateless, models whose hidden state before κ_2 is hard-coded to zero, i.e., $\boldsymbol{h}_{-\kappa_2} = 0$. Stateless models cannot learn dependencies that expand in a time horizon larger that κ_2. However, in many practical cases stateless models are widely employed assuming that only short-term temporal dependencies exist. In contrast, stateful models propagate the hidden state $\boldsymbol{h}_{-\kappa_2} \neq 0$ between temporally consecutive batches. In our study, we consider only **stateful** networks.

A problem of stateful models is that the hidden state $\boldsymbol{h}_{-\kappa_2}$ has to be available from a previous batch. This introduces correlations between weight updates. For this reason, between two consecutive weight updates, the network is teacher-forced (providing correct values at the input and performing forward passing without any back-propagation) for κ_3 time-steps. In this way, the hidden state is available for the next batch update and the problem of correlated training samples is alleviated. This parameter, has an influence on the training speed, as it determines how often the weights are updated. We pick $\kappa_3 = \kappa_2 + \kappa_1 - 1$ as illustrated on Fig. 2.

We utilize a stochastic optimization method with adaptive learning rate called Adam Kingma and Ba (2015) to update the weights. We add Zoneout Krueger, Maharaj, Kramár, Pezeshki, Ballas, Ke, Goyal, Bengio, Courville and Pal (2017) regularization in the recurrent weights and variational dropout Gal and Ghahramani (2016) regularization at the output weights (with the same keep probability) to both GRU and LSTM networks to alleviate over-fitting. Furthermore, following Vlachas et al. (2018) we add Gaussian noise sampled from $\mathcal{N}(0, \kappa_n \sigma)$ to the training data, where σ is the

standard deviation of the data. The noise level κ_n is tuned. Moreover, we also vary the number of RNN layers by stacking **residual** layers He, Zhang, Ren and Sun (2016) on top of each other. These deeper architectures may improve forecasting efficiency by learning more informative embedding at the cost of higher computational times.

As an additional over-fitting counter-measure we use validation based early-stopping, where 90% of the data is used for training and the rest 10% for validation. When the validation error stops decreasing, the training round is over. We train the network for $N_{rounds} = 10$ rounds decreasing the learning rate geometrically by dividing with a factor of ten at each round to avoid tuning the learning rate.

2.5. Reservoir Computing

Reservoir Computing (RC) aims to alleviate the difficulty in learning the recurrent connections of RNNs and reduce their training time Lukoševičius and Jaeger (2009); Lukoševičius (2012). RC relies on **randomly selecting** the recurrent weights such that the hidden state captures the history of the evolution of the observable $\boldsymbol{o}_t$ and train the hidden-to-output weights. The evolution of the hidden state depends on the random initialization of the recurrent matrix and is driven by the input signal. The hidden state is termed reservoir state to denote the fact that it captures temporal features of the observed state history. This technique has been proposed in the context of Echo-State-Networks (ESNs) Jaeger and Haas (2004) and Liquid State Machines with spiking neurons (LSM) Maass, Natschläger and Markram (2002).

In this work, we consider reservoir computers with f_h^h given by the functional form

$$\boldsymbol{h}_t = \tanh\left(\boldsymbol{W}_{h,i}\boldsymbol{o}_t + \boldsymbol{W}_{h,h}\boldsymbol{h}_{t-1}\right), \tag{8}$$

where $\boldsymbol{W}_{h,i} \in \mathbb{R}^{d_h \times d_o}$, and $\boldsymbol{W}_{h,h} \in \mathbb{R}^{d_h \times d_h}$. Other choices of RC architectures are possible, including Larger, Soriano, Brunner, Appeltant, Gutierrez, Pesquera, Mirasso and Fischer (2012); Larger, Baylón-Fuentes, Martinenghi, Udaltsov, Chembo and Jacquot (2017); Haynes, Soriano, Rosin, Fischer and Gauthier (2015); Antonik, Haelterman and Massar (2017) Following Jaeger and Haas (2004), the entries of $\boldsymbol{W}_{h,i}$ are uniformly sampled from $[-\omega, \omega]$, where ω is a hyperparameter. The reservoir matrix $\boldsymbol{W}_{h,h}$ has to be selected in a way such that the network satisfies the "echo state property". This property requires all of the conditional Lyapunov exponents of the evolution of $\boldsymbol{h}_t$ conditioned on the input (observations $\boldsymbol{o}_t$) to be negative so that, for large t, the reservoir state $\boldsymbol{h}_t$ does not depend on initial conditions. For this purpose, $\boldsymbol{W}_{h,h}$ is set to a large low-degree matrix, scaled appropriately to posses a largest eigenvalue ρ whose value is a hyperparameter adjusted so that the echo state property holds[1]. Following Pathak et al. (2018a) the output coupling f_h^o is set to

$$\hat{\boldsymbol{o}}_{t+1} = \boldsymbol{W}_{o,h}\widetilde{\boldsymbol{h}}_t, \tag{9}$$

where the augmented hidden state $\widetilde{\boldsymbol{h}}_t$ is a d_h dimensional vector such that the i^{th} component of $\widetilde{\boldsymbol{h}}_t$ is $\widetilde{h}_t^i = h_t^i$ for half of the reservoir nodes and $\widetilde{h}_t^i = (h_t^i)^2$ for the other half, enriching the dynamics with the square of the hidden state in half of the nodes. This was empirically shown to improve forecasting efficiency of RCs in the context of dynamical systems Pathak et al. (2018a). The matrix $\boldsymbol{W}_{o,h} \in \mathbb{R}^{d_o \times d_h}$ is trained with regularized least-squares regression with Tikhonov regularization to alleviate overfitting Tikhonov and Arsenin (1977); Yan and Su (2009) following the same recipe as in Pathak et al. (2018a). The Tikhonov regularization η is optimized as a hyper-parameter. Moreover, we further regularize the training procedure of RC by adding Gaussian noise in the training data. This was shown to be beneficial for both short-term performance and stabilizing the RC in long-term forecasting. For this reason, we add noise sampled from $\mathcal{N}(0, \kappa_n\sigma)$ to the training data, where σ is the standard deviation of the data and the noise level κ_n a tuned hyper-parameter.

3. Comparison Framework

In order to set up an unbiased comparison framework, the training time of all models is **limited to 24 hours**. For each model we perform an extensive grid search of optimal hyperparameters as reported in the Appendix. All model evaluations are mapped to a single Nvidia Tesla P100 GPU and are executed on the XC50 compute nodes of the Piz Daint supercomputer at the Swiss national supercomputing centre (CSCS). In the following we quantify the prediction

[1] Because of the nonlinearity of the tanh function, $\rho < 1$ is not necessarily required for the echo state property to hold true.

accuracy of the methods in terms of the normalized root mean square error, given by

$$\text{NRMSE}(\hat{\boldsymbol{o}}) = \sqrt{\left\langle \frac{(\hat{\boldsymbol{o}} - \boldsymbol{o})^2}{\boldsymbol{\sigma}^2} \right\rangle}, \tag{10}$$

where $\hat{\boldsymbol{o}} \in \mathbb{R}^{d_o}$ is the forecast at a single time-step, $\boldsymbol{o} \in \mathbb{R}^{d_o}$ is the target value, and $\boldsymbol{\sigma} \in \mathbb{R}^{d_o}$ is the standard deviation in time of each state component. In expression (10), the notation $\langle \cdot \rangle$ denotes the state space average (average of all elements of a vector). To alleviate the dependency on the initial condition, we report the evolution of the NRMSE over time averaged over 100 initial conditions randomly sampled from the attractor. Moreover, in order to obtain a single metric of the predictive performance of the models we compute the valid prediction time (VPT) in terms of the largest Lyapunov exponent of the system Λ_1 as

$$\text{VPT} = \frac{1}{\Lambda_1} \underset{t_f}{\text{argmax}} \{ t_f \mid \text{NRMSE}(\boldsymbol{o}_t) < \epsilon, \forall t \leq t_f \} \tag{11}$$

which is the largest time t_f the model forecasts the dynamics with a NRMSE error smaller than ϵ normalized with respect to the largest LE Λ_1. In the following, we set $\epsilon = 0.5$.

4. Forecasting Reduced Order Observable Dynamics in the Lorenz-96

The accurate long-term forecasting of the state of a deterministic chaotic dynamical system is challenging as even a minor initial error can be propagated exponentially in time due to the system dynamics even if the model predictions are perfect. A characteristic time-scale of this propagation is the largest Lyapunov exponent of the system. In practice, we are often interested in forecasting the evolution of an observable (that we can measure and obtain data from), which does not contain the full state information of the system. The observable dynamics are more irregular and challenging to model and forecast because of the additional loss of information.

Classical approaches to forecast the observable dynamics based on Takens seminal work Takens (1981), rely on reconstructing the full dynamics in a high-dimensional **phase space**. The state of the phase space is constructed by stacking delayed versions of the observed state. Assume that the state of the dynamical system is $\boldsymbol{x}_t$, but we only have access to the less informative observable $\boldsymbol{o}_t$. The phase space state, i.e., the **embedding state**, is given by $\boldsymbol{z}_t = [\boldsymbol{o}_t, \boldsymbol{o}_{t-\tau}, \dots, \boldsymbol{o}_{t-(d-1)\tau}]$, where the time-lag τ and the embedding dimension d are the embedding parameters. For d large enough, and in the case of deterministic nonlinear dynamical chaotic systems, there is generally a one-to-one mapping between a point in the phase space and the full state of the system and vice versa. This implies that the dynamics of the system are deterministically reconstructed in the phase space Kantz and Schreiber (1997) and that there exists a phase space forecasting rule $\boldsymbol{z}_{t+1} = \mathcal{F}^{\boldsymbol{z}}(\boldsymbol{z}_t)$, and thus an observable forecasting rule $\hat{\boldsymbol{o}}_{t+1} = \mathcal{F}^{\boldsymbol{o}}(\boldsymbol{o}_t, \boldsymbol{o}_{t-\tau}, \dots, \boldsymbol{o}_{t-(d-1)\tau})$.

The recurrent architectures presented in Section 2 fit to this framework, as the embedding state information can be captured in the high-dimensional hidden state $\boldsymbol{h}_t$ of the networks by processing the observable time series $\boldsymbol{o}_t$, without having to tune the embedding parameters τ and d.

In the following, we introduce a high-dimensional dynamical system, the Lorenz-96 model and evaluate the efficiency of the methods to forecast the evolution of a reduced order observable of the state of this system. Here the observable is not the full state of the system, and the networks need to capture temporal dependencies to efficiently forecast the dynamics.

4.1. Lorenz-96 Model

The Lorenz-96 model was introduced by Edward Lorenz Lorenz (1995) to model the large-scale behavior of the mid-latitude atmosphere. The model describes the time evolution of an atmospheric variable that is discretized spatially over a single latitude circle modelled in the high-dimensional state $\boldsymbol{x} = [\boldsymbol{x}_1, \dots, \boldsymbol{x}_J]$, and is defined by the equations

$$\frac{d\boldsymbol{x}_j}{dt} = (\boldsymbol{x}_{j+1} - \boldsymbol{x}_{j-2})\boldsymbol{x}_{j-1} - \boldsymbol{x}_j + F, \tag{12}$$

for $j \in \{0, 1, \dots, J-1\}$, where we assume periodic boundary conditions $\boldsymbol{x}_{-1} = \boldsymbol{x}_{J-1}$, $\boldsymbol{x}_{-2} = \boldsymbol{x}_{J-2}$. In the following we consider a grid-size $J = 40$ and two different forcing regimes, $F = 8$ and $F = 10$.

We solve equation (12) starting from a random initial condition with a Fourth Order Runge-Kutta scheme and a time-step of $\delta t = 0.01$. We run the solver up to $T = 2000$ after ensuring that transient effects are discarded ($T_{trans} = 1000$).

The first half 10^5 samples are used for training and the rest for testing. For the forecasting test in the reduced order space, we construct observables of dimension $\boldsymbol{d}_o \in \{35, 40\}$ by performing Singular Value Decomposition (SVD) and keeping the most energetic d_o components. The complete procedure is described in the Appendix. The 35 most energetic modes taken into account in the reduced order observable, explain approximately 98% of the total energy of the system in both $F \in \{8, 10\}$.

As a reference timescale that characterizes the chaoticity of the system we use the Lyapunov time, which is the inverse of the largest Lyapunov Exponent, i.e., $T^{\Lambda_1} = 1/\Lambda_1$. The Lyapunov spectrum of the Lorenz-96 system is calculated using a standard technique based on QR decomposition Abarbanel (2012). This leads to $\Lambda_1 \approx 1.68$ for $F = 8$ and $\Lambda_1 \approx 2.27$ for $F = 10$.

4.2. Results on the Lorenz-96 Model

The evolution of the NRMSE of the model with the largest VPT (11) of each architecture for $F \in \{8, 10\}$ is plotted in Fig. 3 for two values of the dimension of the observable $d_o \in \{35, 40\}$, where $d_o = 40$ corresponds to full state information. Note that the observable is given by first transforming the state to its SVD modes and then keeping the d_o most energetic ones. As indicated by the slopes of the curves, models predicting the observable containing full state information ($d_o = 40$) exhibit a slightly slower NRMSE increase compared to models predicting in the reduced order state, as expected.

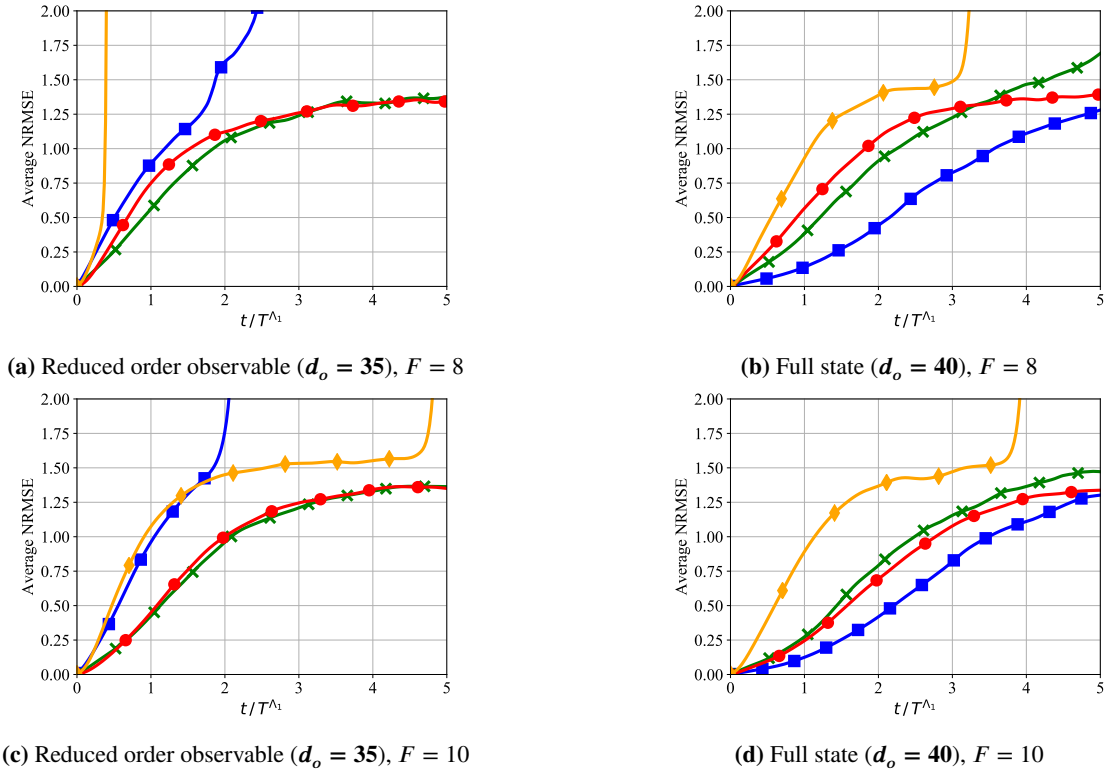
When the full state of the system is observed, the predictive performance of RC is superior to that of all other models. Unitary networks diverge from the attractor in both reduced order and full space in both forcing regimes $F \in \{8, 10\}$. This divergence stems from the iterative propagation of the forecasting error. The issue has been also demonstrated in previous studies in both RC Pathak et al. (2018b); Lu, Hunt and Ott (2018) and RNNs Vlachas et al. (2018) and as discussed in Ref. Lu et al. (2018) can be attributed to a spurious Lyapunov exponent transverse to the state space set of the machine learning prediction system on which the desired prediction dynamics are reproduced. Empirically, such divergences can also often can be attributed to insufficient network size and training, or testing samples not sufficiently represented in the training dataset. In these scenarios we use 10^5 samples to densely capture the attractor. Still, RC suffers from the iterative propagation of errors leading to divergence especially in the reduced order forecasting scenario. In order to alleviate the problem, a parallel scheme for RC is proposed in Pathak et al. (2018b) that enables training of many reservoirs locally forecasting the state. However, this method is limited to systems with local interactions in their state space. In the case we discuss here the observable obtained by singular value decomposition does not fulfill this assumption. In many systems the assumption of local interaction may not hold. GRU and LSTM show superior forecasting performance in this reduced order scenario setting in Lorenz-96 as depicted in Figs. 3a-3c. Especially in the case of $F = 10$, the LSTM and GRU models are able to predict up to 2 Lyapunov time ahead before reaching an NRMSE of $\epsilon = 1$, compared to RC and Unitary RNNs that reach this error threshold in 1 Lyapunov time. However, it should be noted that the predictive utility of all models (considering an error threshold of $\epsilon = 0.5$) is limited to one Lyapunov time when applied to reduced order data and up to two Lyapunov times in the full state.

In Fig. 4a, we plot the VPT for $d_o = 35$ and $d_o = 45$ for 20% of the hyperparameter sets with the highest VPT of each architecture for $F = 8$. Quantitative results for both $F \in \{8, 10\}$ are provided on Table 1. In Fig. 4a, the marker denotes the mean value, while the errorbars denote the minimum and maximum VPT. In the full state scenario, RC shows a remarkable performance with a maximum VPT ≈ 2.31 and a mean VPT≈ 1.95, while GRU exhibits a max VPT of 1.34 mean VPT of ≈ 1.02. The LSTM has a mean VPT of ≈ 0.77, while Unitary RNNs show the lowest forecasting ability with a mean VPT of ≈ 0.43.

In contrast, in the case of $d_o = 35$, GRU is superior to all other models with a maximum VPT ≈ 0.98 and a mean of VPT≈ 0.56 compared to LSTM showing a max VPT ≈ 0.74 and an average VPT ≈ 0.52. LSTM shows inferior performance to GRU which we speculate may be due to insufficient hyperparameter optimization. RC shows inferior performance compared to both GRU and LSTM networks with max VPT≈ 0.55 and mean VPT≈ 0.5. Last but not least, we observe that Unitary RNNs show a low forecasting ability with a mean VPT ≈ 0.39.

However, when picking the 20% of the models with the highest VPT, there is no guarantee that the long-term statistics of the dynamical system are captured. In almost all scenarios and all cases considered in this work, forecasts of Unitary RNN networks fail to remain close to the attractor and diverge. For this reason, we omit the results on these networks.

We quantify the long-term behavior in terms of the power spectrum of the predicted dynamics and its difference with the true spectrum of the testing data. Indeed, the RC networks with the highest VPT diverge from the attractor in both the reduced order and the full state scenario. This is illustrated in Figs. 4b-4c, where the spectrum of the



(a) Reduced order observable ($\boldsymbol{d_o = 35}$), $F = 8$

(b) Full state ($\boldsymbol{d_o = 40}$), $F = 8$

(c) Reduced order observable ($\boldsymbol{d_o = 35}$), $F = 10$

(d) Full state ($\boldsymbol{d_o = 40}$), $F = 10$

Figure 3: The evolution of the NRMSE error (average over 100 initial conditions) of model with the highest VPT of each architecture in the Lorenz-96 with $F \in \{8, 10\}$ and $d_o \in \{35, 40\}$. Reservoir computers show remarkable predictive capabilities when the full state is observed, surpassing all other models. Predictions of Unitary networks diverge from the attractor in all scenarios, while iterative forecasts of RC suffer from instabilities when only partial information of a reduced order observable is available. In contrast, GRU and LSTM show stable behavior and superior performance in the reduced order scenario. In the forcing regime $F = 10$ we observe faster growing rates of the iterative forecasting error for all models as the largest Lyapunov exponent is higher.
RC ; GRU ; LSTM ; Unit ;

predicted dynamics from each model is compared against the original spectrum (dashed black line) for an observable with $d_o \in \{30, 40\}$ respectively. Although the LSTM models with the highest VPT are able to match the original spectrum, RC and GRU models in the reduced order space do not seem to follow the long-term statistics, even though their short-term forecasting performance is relatively good.

In order to examine if this is a general observation, we pick the models with the lowest frequency error instead of the highest VPT. This is an a posteriori search for non-divergent models. In this case, all models match the statistics in the full order space as depicted in Fig. 4f, without influencing the forecasting effectiveness plotted in Fig. 4d. This demonstrates that RC is a powerful predictive tool in the full order state scenario. However, in the case of a reduced order observable, the RC cannot match the statistics. In contrast, GRU and LSTM networks achieve superior forecasting performance while matching the long-term statistics, even at this challenging setting of a chaotic system with reduced order information.

An important aspect of RNNs is their scalability to high-dimensional systems. In Figs. 5a and 5d, we present a Pareto front of the VPT with respect to the CPU RAM memory utilized to train the models with the highest VPT for each architecture for an input dimensions of $d_o = 35$ (reduced order) and $d_o = 40$ (full dimension) respectively. Figs. 5b and 5e, show the corresponding Pareto fronts of the VPT with respect to the training time. In case of the full state space ($d_o = 40$), the RC is able to achieve superior VPT with smaller memory usage and vastly smaller training time than the other methods. However in the case of reduced order information ($d_o = 35$), the BPTT algorithms (GRU and LSTM) are superior to the RC even when the latter is provided with one order of magnitude more memory.

We remark that memory requirement for RNNs trained with BBPT scale linearly with network size. At the same

Table 1
Maximum and average Valid Prediction Time (VPT) of the 20% hyperparameter sets with the highest VPT averaged over 100 initial conditions sampled from the testing data for each model.

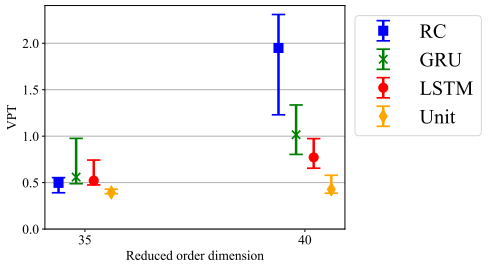
SCENARIO / MODEL	$F = 8$				$F = 10$			
	$d_o = 35$		$d_o = 40$		$d_o = 35$		$d_o = 40$	
	MAX	AVG	MAX	AVG	MAX	AVG	MAX	AVG
Unit	0.43	0.40	0.58	0.43	0.49	0.46	0.63	0.49
LSTM	0.74	0.52	0.97	0.77	**1.17**	**0.66**	1.73	1.07
GRU	**0.98**	**0.56**	1.34	1.02	**1.22**	**0.67**	1.59	1.27
RC	0.55	0.50	**2.31**	**1.95**	0.60	0.53	**2.35**	**1.98**

time, in RC the maximum reservoir size (imposed by computer memory limitations) may not be sufficient to capture the dynamics of high dimensional systems with non-local interactions.

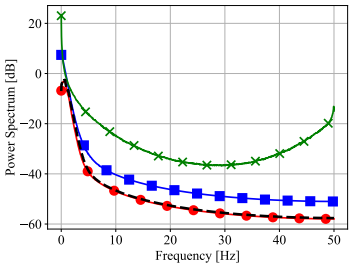
Training time for RC scales quadratically with the reservoir size as it requires computation of the inverse of a matrix with dimensions $d_h \times d_h$. In contrast, the training time of an RNN is very difficult to estimate a priori, as convergence of the training method depends on initialization and various other hyperparameters and are not necessarily dependent on the size. That is why we observe a greater variation of the training time of RNN models. The training procedure of RNNs may not converge even after the threshold of 24 hours. This is the case for the LSTM models on the the right side of the plots 5b and 5e. Additional training time or fine-tuning might further increase their predictive performance Vlachas et al. (2018). Similar results are obtained for $F = 10$, the interested reader is referred to the appendix.

In the following, we evaluate to which extend the trained models overfit to the training data. For this reason, we measure the VPT in the training dataset and plot it against the VPT in the test dataset for every model we trained. The results are shown in Figs. 5c, and 5f for $d_o = 35$ and $d_o = 40$. Ideally a model architecture that guards effectively against overfitting should be represented by a point in this plot that is close to the identity mapping. As the expressive power of a model increases, the model may fit better to the training data, but bigger models are more prone to memorizing the training dataset and overfitting. In the reduced order scenario, GRU and LSTM models lie closer to the identity mapping curve than RC models. This is due to the validation based training procedure utilized in the RNNs that guards effectively against overfitting. In contrast, alleviating overfitting is more challenging in RC as it amounts to the tuning of the Tikhonov regularization parameter. However, in the full-order scenario, the RC models achieve superior forecasting accuracy and generalization ability as clearly depicted in Fig. 5f. Especially the additional regularization of the training procedure introduced by adding Gaussian noise in the data was decisive to achieve this result.

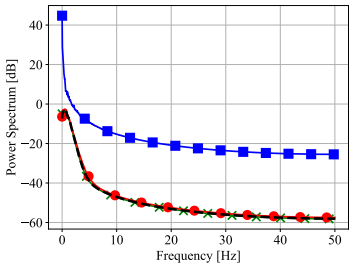
An example of an iterative forecast in the test dataset, is illustrated in Fig. 6 for $F = 8$ and $d_o \in \{35, 40\}$.



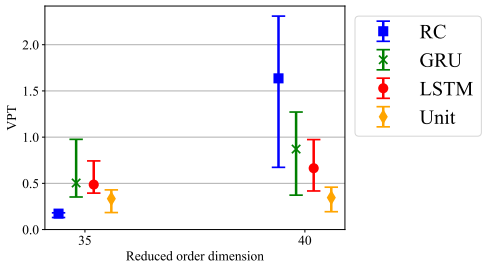
(a) VPT for observable dimension $d_o = 35$ and $d_o = 40$. Best 20% of all hyperparameter sets selected according to the **maximum VPT**.



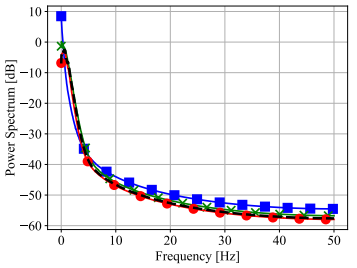
(b) Power spectrum for $d_o = 35$ of the model with **highest VPT** for each architecture.



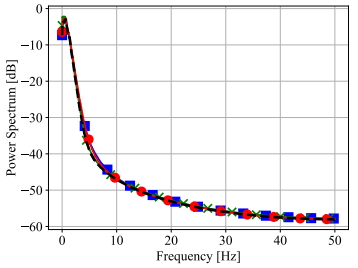
(c) Power spectrum for $d_o = 40$ of the model with **highest VPT** for each architecture.



(d) VPT for observable dimension $d_o = 35$ and $d_o = 40$. Best 20% of all hyperparameter sets selected according to the **minimum power spectrum error**.

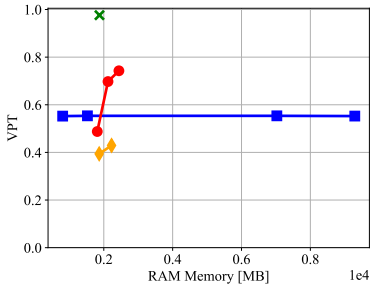


(e) Power spectrum for $d_o = 35$ of the model with **minimum power spectrum error** for each architecture.

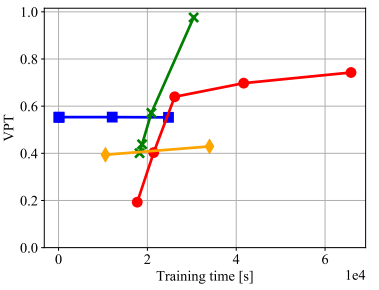


(f) Power spectrum for $d_o = 40$ of the model with **minimum power spectrum error** for each architecture.

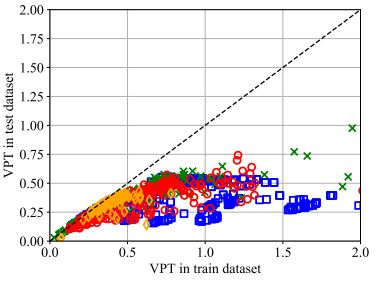
Figure 4: Results of RC, GRU, LSTM and Unitary networks forecasting the dynamics of an observable consisting of the SVD modes of the Lorenz-96 system with $F = 8$ and state dimension 40. The observable consists of the $d_o = 35$ most energetic modes or the full state information ($d_o = 40$). **(a)-(d)** Mean valid prediction time (VPT) (marker) of the 20% of the hyperparameter sets of each model with highest VPT **(a)** or lowest frequency error **(b)**, along with maximum and minimum (errorbars) plotted w.r.t. the observable dimension d_o. **(b)-(e)** Reconstructed power spectrum of the signal for $d_o = 35$. **(c)-(f)** Reconstructed power spectrum of the signal for $d_o = 40$.
RC ; GRU ; LSTM ; Unit ; Groundtruth ;



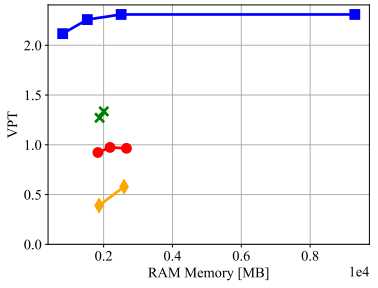
(a) VPT w.r.t. RAM memory for $d_o = 35$.



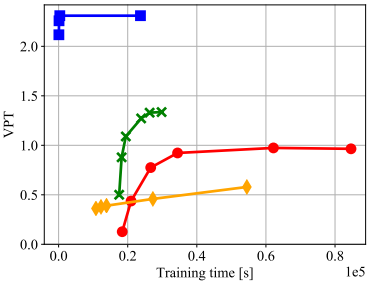
(b) VPT w.r.t. the total training time for $d_o = 35$.



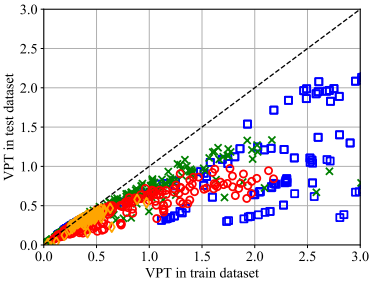
(c) VPT in test data w.r.t. VPT in the train data for $d_o = 35$.



(d) VPT w.r.t. RAM memory for $d_o = 40$.

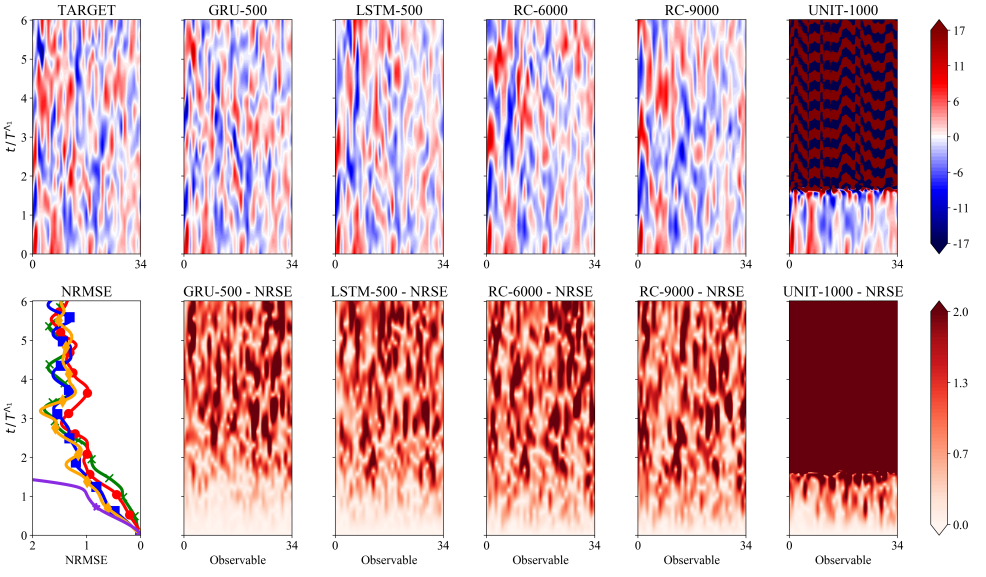


(e) VPT w.r.t. the total training time for $d_o = 40$.

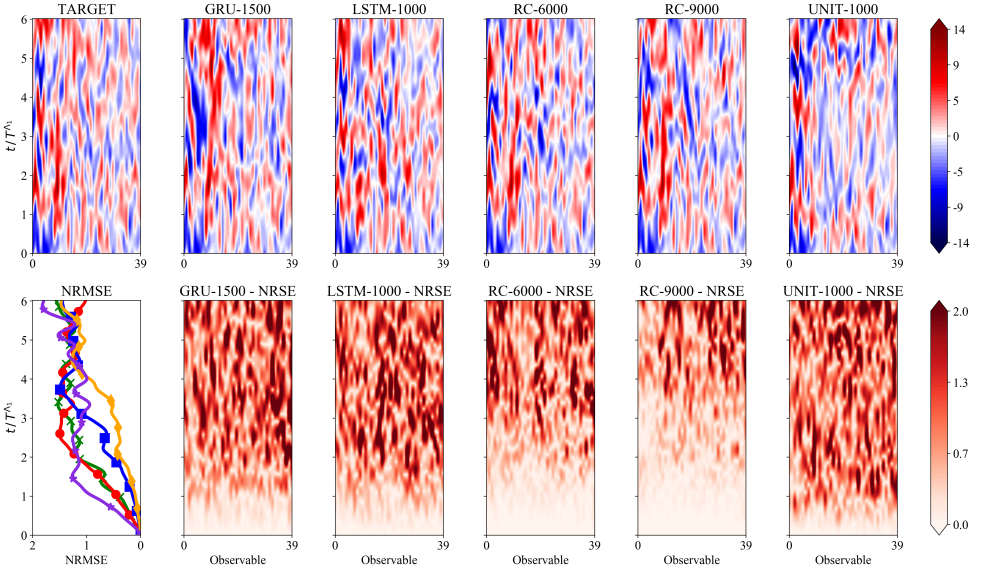


(f) VPT in test data w.r.t. VPT in the train data for $d_o = 40$.

Figure 5: Forecasting results on the dynamics of an observable consisting of the SVD modes of the Lorenz-96 system with $F = 8$ and state dimension 40. The observable consists of the $d_o \in \{35, 40\}$ most energetic modes. (a), (d) Valid prediction time (VPT) plotted w.r.t. the required RAM memory for dimension $d_o \in \{35, 40\}$. (b), (e) VPT plotted w.r.t. total training time for dimension $d_o \in \{35, 40\}$. (c), (f) VPT measured from 100 initial conditions sampled from the **test** data plotted against the VPT from 100 initial conditions sampled from the **training** data for each model for $d_o \in \{35, 40\}$. In the reduced order space ($d_o = 35$) RCs tend to overfit easier compared to GRUs/LSTMs that utilize a validation based training routine. In the full order space ($d_o = 40$) , RCs demonstrate excellent generalization ability and high forecasting accuracy.
RC (or) ; GRU (or) ; LSTM (or) ; Unit (or) ; Ideal ;



(a) Input dimension (dimension of the observable in the reduced order space) $\boldsymbol{d_o = 35}$



(b) Input dimension (dimension of the observable in the reduced order space) $\boldsymbol{d_o = 40}$ (full state)

Figure 6: Contour plots of a spatio-temporal forecast on the SVD modes of the Lorenz-96 system with $F = 8$ in the testing dataset with GRU, LSTM, RC and a Unitary network along with the true (target) evolution and the associated NRSE contours for the reduced order observable **(a)** $d_o = 35$ and the full state **(b)** $d_o = 40$. The evolution of the component average NRSE (NMRSE) is plotted to facilitate comparison. Unitary networks suffer from propagation of forecasting error and eventually their forecasts diverge from the attractor. Forecasts in the case of an observable dimension $d_o = 40$ diverge slower as the dynamics are deterministic. In contrast, forecasting the observable with $d_o = 35$ is challenging due to both **(1)** sensitivity to initial condition and **(2)** incomplete state information that requires the capturing of temporal dependencies. In the full-state setting, RC models achieve superior forecasting accuracy compared to all other models. In the challenging reduced order scenario, LSTM and GRU networks demonstrate a stable behavior in iterative prediction and reproduce the long-term statistics of the attractor. In contrast, in the reduced order scenario RC suffer from frequent divergence (refer to the appendix).
GRU ; LSTM ; RC-6000 ; RC-9000 ; Unit ;

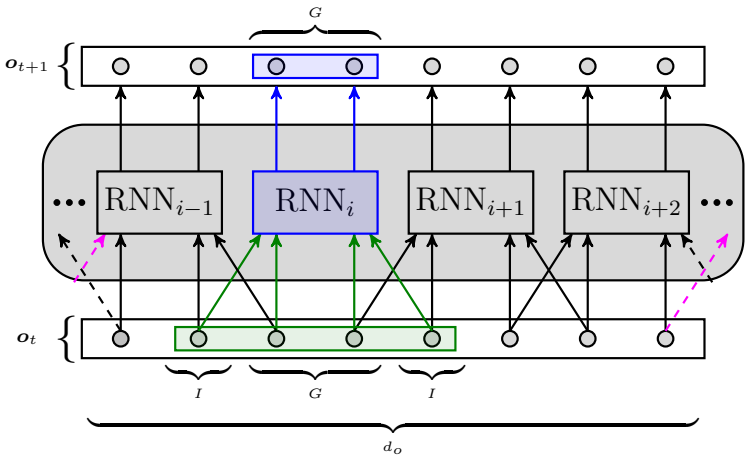


Figure 7: Illustration of the parallel architecture for a group size of $G = 2$ and an interaction length of $I = 1$. The network consists of multiple RNNs with different parameters. Each RNN is trained to forecast the evolution of G elements of the observable. Additional information of I elements from each neighboring network (left and right) are provided as additional input to capture local correlations.

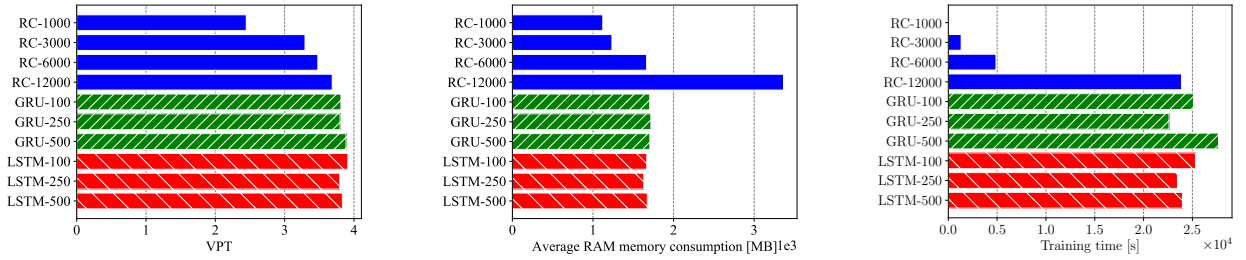
5. Parallel Forecasting Leveraging Local Interactions

In spatially extended dynamical systems the state space (e.g., vorticity, velocity field, etc.) is high-dimensional (or even infinite dimensional), since an adequately fine grid is needed to resolve the relevant spatio-temporal scales of the dynamics. Even though RC and RNNs can be utilized for modeling and forecasting of these systems in the short-term, the RC and RNN methods described in Section 2 do not scale efficiently with the input dimension, i.e., as the dimensionality of the observable $\boldsymbol{o}_t \in \mathbb{R}^{d_o}$ increases. Two limiting factor are the required time and RAM memory to train the model. As d_o increases, the size d_h of the reservoir network required to predict the system using only a single reservoir rises. This implies higher training times and more computational resources (RAM memory), which render the problem intractable for large values of d_o. The same applies for RNNs. More limiting factors arise by taking the process of identification of optimal model hyperparameters into account, since loading, storing and processing a very large number of large models can be computationally infeasible. However, these scaling problems for large systems can be alleviated in case the system is characterized by local state interactions or translationally invariant dynamics. In the first case, as shown in Fig. 7 the modeling and forecasting task can be parallelized by employing multiple individually trained networks forecasting locally in parallel exploiting the local interactions, while, if translation invariance also applies, the individual parallel networks can be identical and training of only one will be sufficient. This parallelization concept is utilized in RC in Pathak et al. (2018a); Parlitz and Merkwirth (2000). The idea dates back to local delay coordinates Parlitz and Merkwirth (2000). The model shares ideas from convolutional RNN architectures Sainath, Vinyals, Senior and Sak (2015); Shi, Chen, Wang, Yeung, Wong and chun Woo (2015) designed to capture local features that are translationally invariant in image and video processing tasks. In this section, we extend this parallelization scheme to RNNs and compare the efficiency of parallel RNNs and RCs in forecasting the state dynamics of the Lorenz-96 model and Kuramoto-Sivashinsky equation discretized in a fine grid.

5.1. Parallel Architecture

Assume that the observable is $\boldsymbol{o}_t \in \mathbb{R}^{d_o}$ and each element of the observable is denoted by $o_t^i \in \mathbb{R}, \forall i \in \{1, \dots, d_o\}$. In case of local interactions, the evolution of each element is affected by its spatially neighboring grid points. The elements $\boldsymbol{o}^i$ are split into N_g groups, each of which consisting of G spatially neighboring elements such that $d_o = GN_g$. The parallel model employs N_g RNNs, each of which is utilized to predict a spatially local region of the system observable indicated by the G group elements $\boldsymbol{o}^i$. Each of the N_g RNNs receives G inputs $\boldsymbol{o}^i$ from the elements i it forecasts in addition to I inputs from neighboring elements on the left and on the right, where I is the interaction length. An example with $G = 2$ and $I = 1$ is illustrated in Fig. 7.

During the training process, the networks can be trained independently. However, for long-term forecasting, a communication protocol has to be utilized as each network requires the predictions of neighboring networks to infer. In



(a) Valid prediction time in the test dataset

(b) Average RAM memory requirement

(c) Training time

Figure 8: **(a)** Valid prediction time (VPT), **(b)** CPU memory utilization and **(c)** total training time of RNN parallel architectures with group size $G = 2$ and an interaction length $I = 4$ forecasting the dynamics of Lorenz-96 with state dimension $d_o = 40$ (full state). GRU and LSTM results do not depend significantly on network size. RC with 3000 or 6000 nodes have slightly lower VPT, but require much less training time. Increasing RC size to more than 12000 nodes was not feasible due to memory requirements.

the case of a homogeneous system, where the dynamics are translation invariant, the training process can be drastically reduced by utilizing one single RNN and training it on data from all groups. The weights of this RNN are then copied to all other members of the network. In the following we assume that our data is not homogeneous. The elements of the parallel architecture are trained independently, while the MPI Dalcin, Paz, Kler and Cosimo (2011); Dalcín, Paz, Storti and DâĂŹElía (2008); Walker and Dongarra (1996) communication protocol is utilized to communicate the elements of the interaction for long-term forecasting.

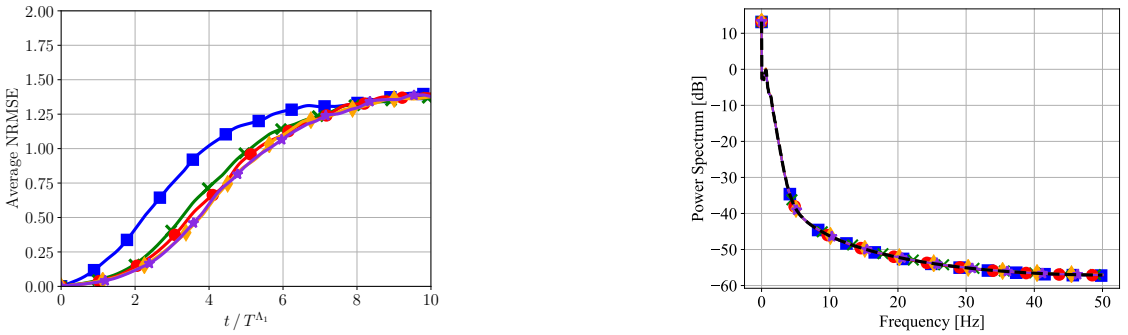
5.2. Results on the Lorenz-96

In this section, we employ the parallel architecture to forecast the state dynamics of the Lorenz-96 system explained in Section 4.1 with a state dimension of $d_o = 40$. Note that in contrast to section 4.2, we do not construct an observable and then forecast the reduced order dynamics. Instead, we leverage the local interactions in the state space and employ an ensemble of networks forecasting the local dynamics.

The group size of the parallel models is set to $G = 2$, while the interaction length is $I = 4$. Each group of the total $N_g = 20$ is forecasting the evolution of 2 state components, receiving 10 state components at the input. The size of the hidden state in RC is $d_h \in \{1000, 3000, 6000, 12000\}$. Smaller networks of size $d_h \in \{100, 250, 500\}$ are selected for GRU and LSTM. The rest of the hyperparameters are given in the appendix. Results for Unitary networks are omitted, as the identification of hyperparameters leading to stable iterative forecasting was computationally heavy and all trained models led to unstable systems that diverged after a few iterations.

In Fig. 8a, we plot the VPT time of the RC and the BPTT networks. We find that RNN trained by BPTT achieve comparable predictions with RC, albeit using much smaller number hidden nodes (between 100 and 500 for BPTT vs 6000 to 12000 for RC). We remark that RC with 3000 and 6000 nodes have slightly lower VPT than GRU and LSTM but require significantly lower training times as shown in Fig. 8c. At the same time, using 12000 nodes for RC implies high RAM sizes, more than 3 GB per rank, as depicted in Fig. 8b.

As elaborated in Section 4.2 and depicted in Fig. 3a, the VPT reached by large nonparallelized models that are forecasting the 40 SVD modes of the system is approximately 1.4. We also verified that the nonparallelized models of section 4.1 when forecasting the 40 dimensional state containing local interactions instead of the 40 modes of SVD, reach the same predictive performance. Consequently, as expected the VPT remains the same whether we are forecasting the state or the SVD modes as the system is deterministic. By exploiting the local interactions and employing the parallel networks, the VPT is increased from ≈ 1.4 to ≈ 3.9 as shown in Fig. 8a. The NRMSE error of the best performing hyperparameters is given in Fig. 9a. All models are able to reproduce the climate as the reconstructed power spectrum plotted in Fig. 9b matches the true one. An example of an iterative prediction with LSTM, GRU and RC models starting from an initial condition in the test dataset is provided in Fig. 10.



(a) NRMSE error evolution

(b) Power spectrum

Figure 9: (a) The evolution of the NRMSE error (averaged over 100 initial conditions) of different parallel models in the Lorenz-96 with state dimension $d_o = 40$. **(b)** The reconstructed power spectrum. All models accurately capture the power spectrum. RCs with $d_h \in \{6000, 12000\}$ nodes are needed to match the predictive performance of an LSTM with 100 nodes. RC-1000 ; RC-6000 ; RC-12000 ; GRU-500 ; LSTM-100 ; Groundtruth ;

5.3. Kuramoto-Sivashinsky

The Kuramoto-Sivashinsky (KS) equation is a nonlinear partial differential equation of fourth order that is used as a turbulence model for various phenomena. It was derived by Kuramoto in Kuramoto (1978) to model the chaotic behavior of the phase gradient of a slowly varying amplitude in a reaction-diffusion type medium with negative viscosity coefficient. Moreover, Sivashinsky Sivashinsky (1977) derived the same equations when studying the instantaneous instabilities in a laminar flame front. For our study, we restrict ourselves to the one dimensional K-S equation

$$\frac{\partial u}{\partial t} = -\nu \frac{\partial^4 u}{\partial x^4} - \frac{\partial^2 u}{\partial x^2} - u\frac{\partial u}{\partial x}, \tag{13}$$

on the domain $\Omega = [0, L]$ with periodic boundary conditions $u(0, t) = u(L, t)$. The dimensionless boundary size L directly affects the dimensionality of the attractor. For large values of L, the attractor dimension scales linearly with L Manneville (1984).

In order to spatially discretize (13) we select a grid size Δx with $D = L/\Delta x + 1$ the number of nodes. Further, we denote with $u_i = u(i\Delta x)$ the value of u at node $i \in \{0, \dots, D-1\}$. In the following, we select $\nu = 1, L = 200$, $\delta t = 0.25$ and a grid of $d_o = 512$ nodes. We discretize the equations (13) and solve them using the fourth-order method for stiff PDEs introduced in Kassam and Trefethen (2005) up to $T = 6 \cdot 10^4$. This corresponds to $24 \cdot 10^4$ samples. The first $4 \cdot 10^4$ samples are truncated to avoid initial transients. The remaining data are divided to a training and a testing dataset of 10^5 samples each. The observable is considered to be the $d_o = 512$ dimensional state. The largest Lyapunov Exponent Λ_1 of the system is utilized as a reference timescale. We approximate it with the method of Pathak Pathak et al. (2018a) for $L = 200$ and it is found to be $\Lambda_1 \approx 0.094$.

5.4. Results on the Kuramoto-Sivashinsky Equation

In this section, we present the results of the parallel models in the Kuramoto-Sivashinsky equation. The full system state is used as an observable, i.e., $d_o = 512$. The group-size of the parallel models is set to $G = 8$, while the interaction length is $I = 8$. The total number of groups is $N_g = 64$. Each member forecasts the evolution of 8 state components, receiving at the input 24 components in total. The size of the reservoir in RC is $d_h \in \{500, 1000, 3000\}$. For GRU and LSTM networks we vary $d_h \in \{100, 250, 500\}$. The rest of the hyperparameters are given in the appendix. Results on Unitary networks are omitted, as the configurations tried in this work led to unstable models diverging after a few time-steps in the iterative forecasting procedure.

The results are summed up in the bar-plots in Fig. 11. In Fig. 11a, we plot the VPT time of the models. LSTM models reach VPTs of ≈ 4, while GRU show an inferior predictive performance with VPTs of ≈ 3.5. An RC with $d_h = 500$ reaches a VPT of ≈ 3.2, and an RC with 1000 modes reaches the VPT of LSTM models with a VPT of ≈ 3.9. Increasing the reservoir capacity of the RC to $d_h = 3000$ leads to a model exhibiting a VPT of ≈ 4.8. In this case, the large RC model shows slightly superior performance to GRU/LSTM. The low performance of GRU models can be attributed to the fact that in the parallel setting the probability that any RNN may converge to bad local minima rises,

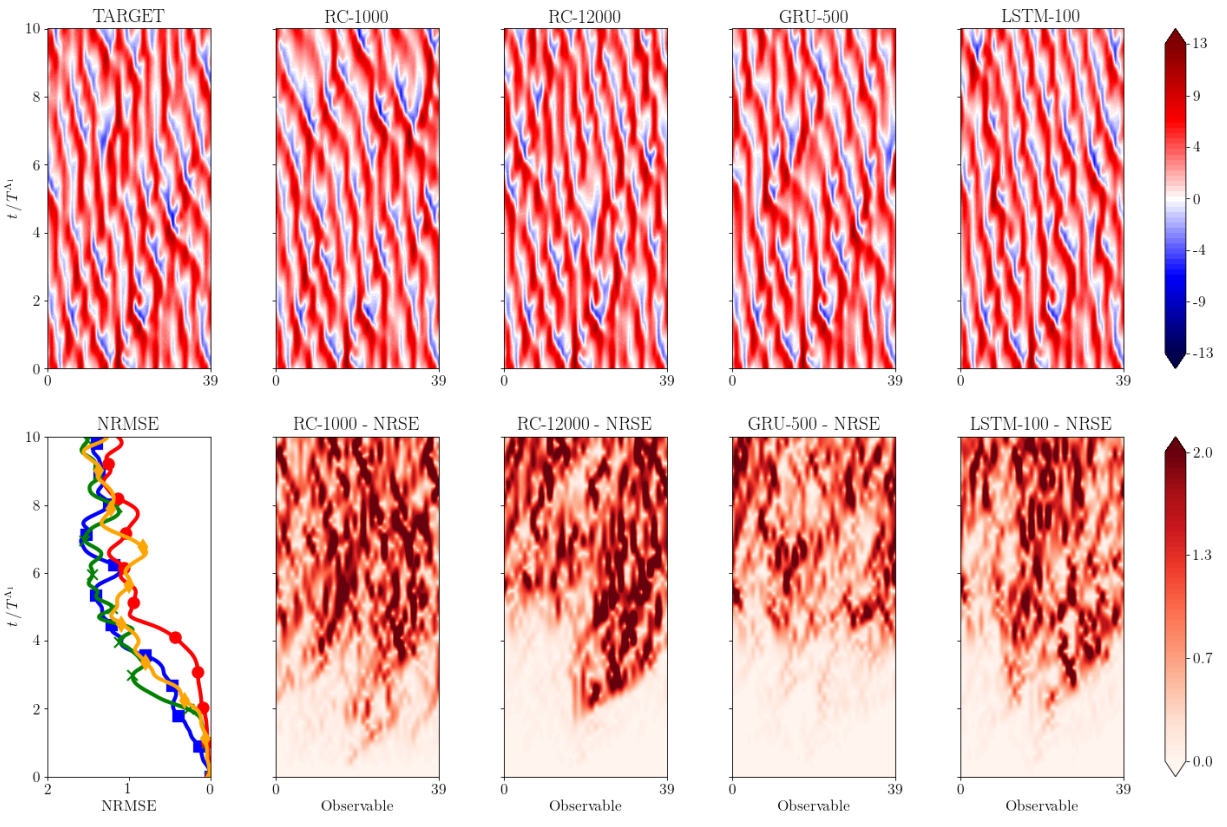
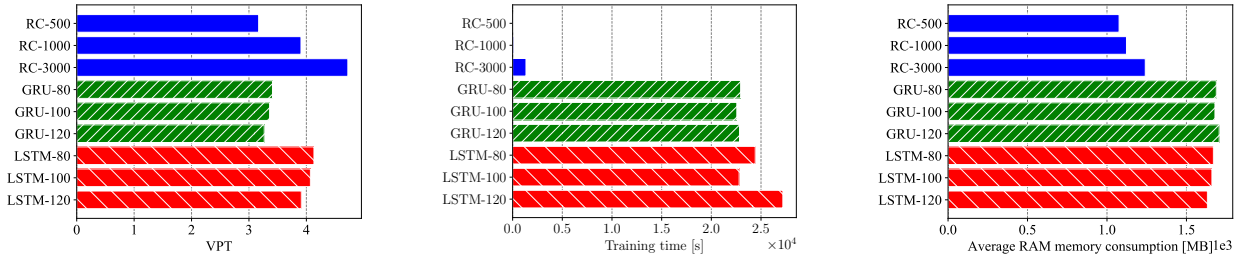


Figure 10: Contour plots of a spatio-temporal forecast in the testing dataset with parallel GRU, LSTM, and RC networks along with the true (target) evolution and the associated NRSE contours in the Lorenz-96 system with the full state as an observable $d_o = 40$. The evolution of the component average NRSE (NMRSE) is plotted to facilitate comparison. RC-1000 ; RC-12000 ; GRU-500 ; LSTM-100 ;



(a) Valid prediction time in the test dataset

(b) Training time

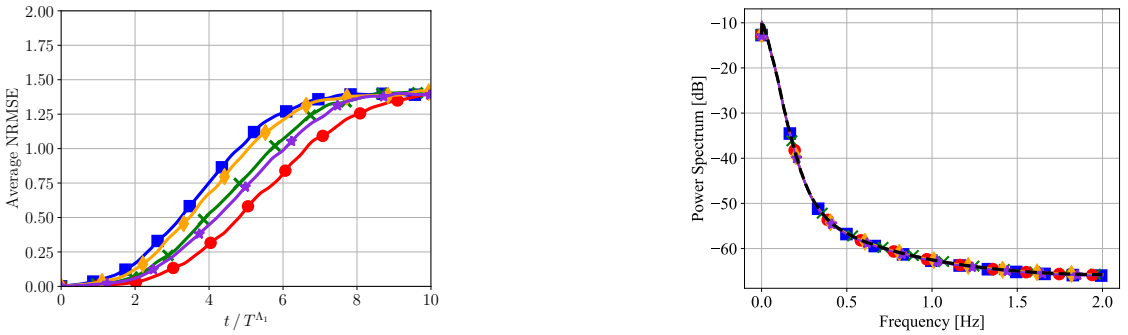
(c) Average RAM memory requirement

Figure 11: **(a)** Valid prediction time (VPT), **(b)** total training time, and **(c)** CPU memory utilization of parallel RNN architectures with group size $G = 8$ and an interaction length $I = 8$ forecasting the dynamics of Kuramoto-Sivashinsky equation with state dimension $d_o = 512$.

with a detrimental effect on the total predictive performance of the parallel ensemble. In case of spatially translational invariant systems, we could alleviate this problem by using one single network. Still, training the single network to data from all spatial locations would be expensive.

As depicted in Fig. 11, the reservoir size of 3000 is enough for RC to reach and surpass the predictive performance of RNNs utilizing a similar amount of RAM memory and a much lower amount of training time as illustrated in Fig. 11b.

The evolution of the NRMSE is given in Fig. 12a. The predictive performance of a small LSTM network with



(a) Average number of accurate test-set predictions

(b) CPU memory utilization

Figure 12: (a) The evolution of the NRMSE error (averaged over 100 initial conditions) of different parallel models in the Kuramoto-Sivashinsky equation with state dimension $d_o = 512$. **(b)** The power spectrum. All models capture the statistics of the system.
RC-500 ; RC-1000 ; RC-3000 ; GRU-80 ; LSTM-80 ; Groundtruth ;

80 hidden units, matches that of a large RC with 1000 hidden units. In Fig. 12b, the power spectrum of the predicted state dynamics of each model is plotted along with the true spectrum of the equations. The three models captured successfully the statistics of the system, as we observe a very good match. An example of an iterative prediction with LSTM, GRU and RC models starting from an initial condition in the test dataset is provided in Fig. 13.

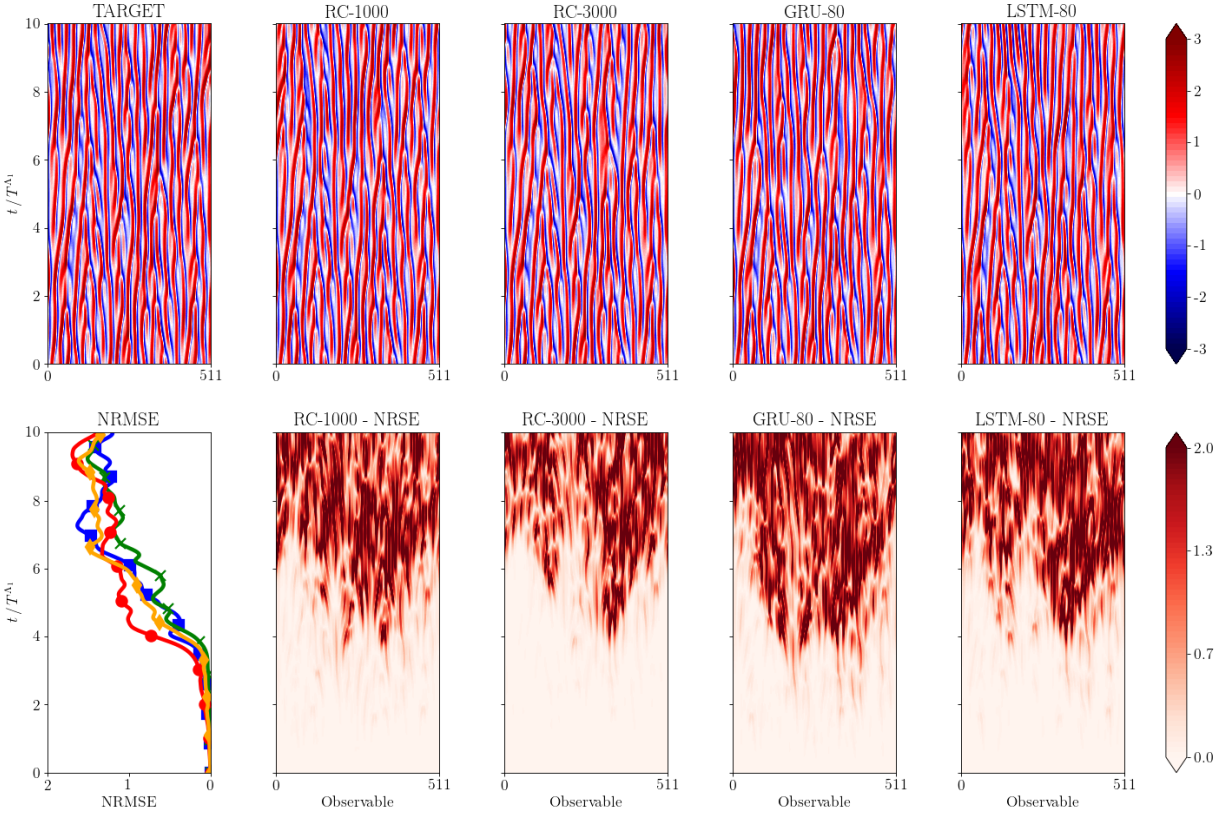


Figure 13: Contour plots of a spatio-temporal forecast on the Kuramoto-Sivashinsky equation in the testing dataset with parallel GRU, LSTM, and RC networks along with the true (target) evolution and the associated NRSE contours in the Kuramoto-Sivashinsky system with the full state as an observable $d_o = 512$. The evolution of the component average NRSE (NMRSE) is plotted to facilitate comparison.
RC-1000 ; RC-3000 ; GRU-80 ; LSTM-80 ; Groundtruth ;

6. Calculation of Lyapunov Exponents in the Kuramoto-Sivashinsky Equation

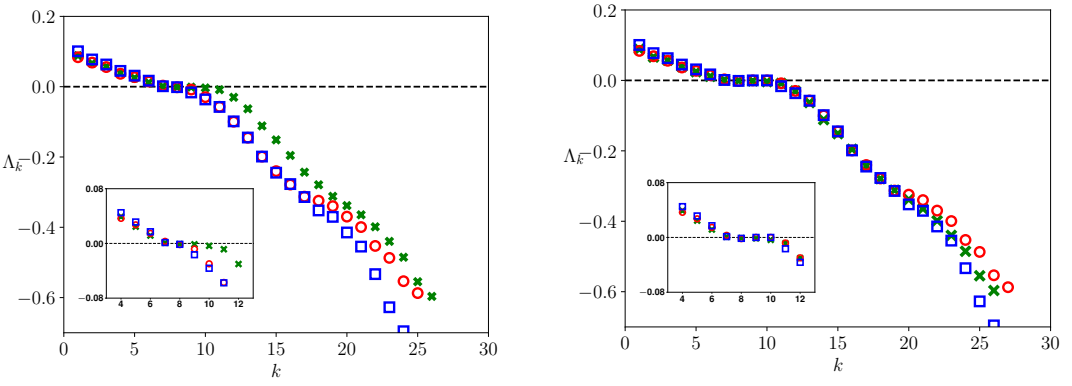
The recurrent models utilized in this study can be used as surrogate models to calculate the Lyapunov exponents (LEs) of a dynamical system relying only on experimental time-series data. The LEs characterize the rate of separation if positive (or convergence if negative) of trajectories that are initialized infinitesimally close in the phase space. They can provide an estimate of the attractor dimension according to the Kaplan-Yorke formula Kaplan and Yorke (1979). Early efforts to solve the challenging problem of data-driven LE identification led to local approaches Wolf, B. Swift, Swinney and A. Vastano (1985); Sano and Sawada (1985) that are computationally inexpensive at the cost of requiring a large amount of data. Other approaches fit a global model to the data Maus and Sprott (2013) and calculate the LE spectrum using the Jacobian algorithm. These approaches were applied to low-order systems.

A recent machine learning approach utilizes deep convolutional neural networks for LE and chaos identification, without estimation of the dynamics Makarenko (2018). An RC-RNN approach capable of uncovering the whole LE spetrum in high-dimensional dynamical systems is proposed in Pathak et al. (2018a). The method is based on the training of a surrogate RC model to forecast the evolution of the state dynamics, and the calculation of the Lyapunov spectrum of the hidden state of this surrogate model. The RC method demonstrates excellent agreement for all positive Lyapunov exponents and many of the negative exponents for the KS equation with $L = 60$ Pathak et al. (2018a), alleviating the problem of spurious Lyapunov exponents of delay coordinate embeddings Dechert and Gençay (1996). We build on top of this work and demonstrate that a GRU trained with BPTT can reconstruct the Lyapunov spectrum accurately with lower error for all positive Lyapunov exponents at the cost of higher training times.

The Lyapunov spectrum of the KS equation is computed by solving the KS equations in the Fourier space with a fourth order time-stepping method called ETDRK4 Kassam and Trefethen (2005) and utilizing a QR decomposition approach as in Pathak et al. (2018a). The Lyapunov spectrum of the RNN and RC surrogate models is computed based on the Jacobian of the hidden state dynamics along a reference trajectory, while Gram-Schmidt orthonormalization is utilized to alleviate numerical divergence. We employ a GRU-RNN over LSTM-RNN, due to the fact that the latter has two coupled hidden states, rendering the computation of the Lyapunov spectrum mathematically more involved and computationally more expensive. The interested reader can refer to the Appendix for the details of the method. The identified maximum LE is $\Lambda_1 \approx 0.08844$. In this work, a large RC with $d_h = 9000$ nodes is employed for LS calculation in the Kuramoto-Sivashinsky equation with parameter $L = 60$ and $D = 128$ grid points as in Pathak et al. (2018a). The largest LE identified in this case is $\Lambda_1 \approx 0.08378$ leading to a relative error of 5.3%. In order to evaluate the efficiency of RNNs, we utilize a large GRU with $d_h = 2000$ hidden units. An iterative RNN roll-out of $N = 10^4$ total time-steps was needed to achieve convergence of the spectrum. The largest Lyapunov exponent identified by the GRU is $\Lambda_1 \approx 0.0849$ reducing the error to $\approx 4\%$. Both surrogate models identify the correct Kaplan-Yorke dimension $\mathrm{KY} \approx 15$, which is the largest LE such that $\sum_i \Lambda_i > 0$.

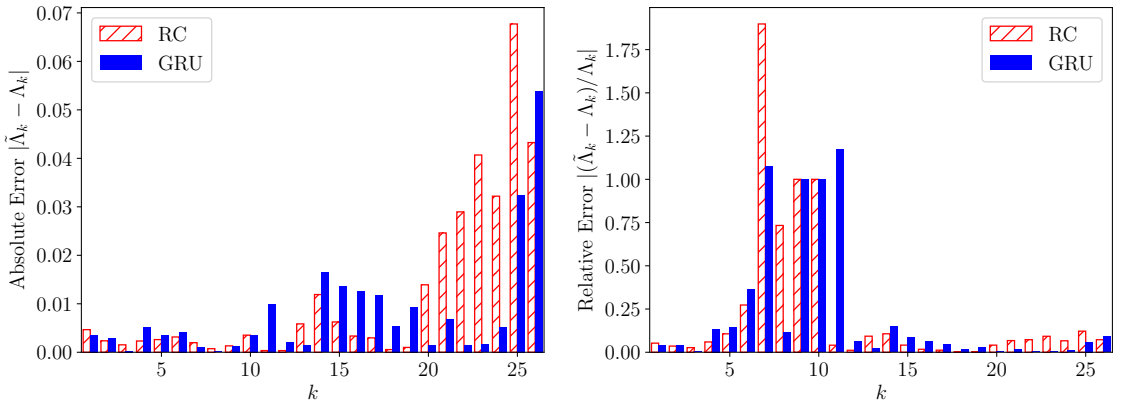
The first 26 Lyapunov exponents computed the GRU, RC as well as using the true equations of the Kuramoto-Sivashinsky are plotted in Fig. 14. We observe a good match between the positive Lyapunov exponents by both GRU and RC surrogates. The positive Lyapunov exponents are characteristic of chaotic behavior. However, the zero Lyapunov exponents Λ_7 and Λ_8 cannot be captured neither with RC nor with RNN surrogates. This is also observed in RC in Pathak et al. (2018a), and apparently the GRU surrogate employed in this work do not alleviate the problem. In Fig. 14b, we augment the RC and the GRU spectrum with these two additional exponents to illustrate that there is an excellent agreement between the true LE and the augmented LS identified by the surrogate models.

The relative and absolute errors in the spectrum calculation is illustrated in Fig. 15. After augmenting with these zero LE, we get a mean absolute error of 0.012 for RC and 0.008 for GRU. The mean relative error is 0.23 for RC, and 0.22 for GRU. As a conclusion, GRU in par with RC networks can be used to replicate the chaotic behavior of a reference system and calculate the Lyapunov spectrum accurately.



(a) Lyapunov spectrum. **(b)** Lyapunov spectrum augmented with Λ_7, Λ_8 set to zero.

Figure 14: (a) Estimated Lyapunov exponents Λ_k of the KS equation with $L = 60$. The true Lyapunov exponents are illustrated with green crosses, red circles are calculated with the RC surrogate, while the blue rectangles with GRU. In **(b)** we augment the computed spectrums with the two zero Lyapunov exponents Λ_7, Λ_8. Inset plots zoom in the zero crossing regions.
True ✖; RC ○; GRU □;



(a) Absolute error **(b)** Relative error

Figure 15: (a) Absolute and **(b)** Relative error of the LE spectrum of the KS equation with $L = 60$. The LE spectrum identified using the GRU shows a better agreement with the spectrum identified by the Kuramoto-Sivashinsky equations.

7. Conclusions

In this work, we employed several variants of recurrent neural networks and reservoir computing to forecast the dynamics of chaotic systems. We present a comparative study based on their efficiency in capturing temporal dependencies, evaluate how they scale to high-dimensional systems,and how to guard against overfitting. We highlight the advantages and limitations of these methods and elucidate their applicability to forecasting spatiotemporal dynamics.

We considered three different types of RNN cells that alleviate the well-known vanishing and exploding gradient problem in Back-propagation through time training (BPTT), namely LSTM, GRU and Unitary cells. We benchmarked these networks against reservoir computers with random hidden to hidden connection weights, whose training procedure amounts to least square regression on the output weights.

The efficiency of the models in capturing temporal dependencies in the reduced order state space is evaluated on the Lorenz-96 system in two different forcing regimes $F = \{8, 10\}$, by constructing a reduced order observable using Singular Value Decomposition (SVD) and keeping the most energetic modes. Even though this forecasting task is challenging due to (1) chaotic dynamics and (2) reduced order information, LSTM and GRU show superior forecasting ability to RC utilizing similar amounts of memory at the cost of higher training times. GRU and LSTM models demonstrate stable behavior in the iterative forecasting procedure in the sense that the forecasting error usually does not diverge, in stark contrast to RC and Unitary forecasts. Large RC models tend to overfit easier than LSTM/GRU models, as the latter are utilizing a validation based routine and regularization techniques (e.g., Zoneout, Dropout) that guard against overfitting which are not directly applicable to RC. Validation in RC amounts to tuning the Tikhonov regularization parameter, which may be computationally costly. However, RC shows excellent forecasting efficiency when the full state of the system is observed, outperforming all other models by a wide margin, while also reproducing the frequency spectrum of the underlying dynamics.

RNNs and RC both suffer from scalability problems in high-dimensional systems, as the required hidden state size d_h to capture the high-dimensional dynamics might be prohibitively large. In order to scale the models to high-dimensional systems we employ a parallelization scheme that exploits the local interactions in the state of a dynamical system. As a reference, we consider the Lorenz-96 system and the Kuramoto-Sivashinsky equation, and we train parallel RC, GRU, and LSTM models of various sizes. Iterative forecasting with parallel Unitary models diverged after a few timesteps in both systems. Parallel GRU, LSTM and RC networks reproduced the long-term attractor climate, as well as the power spectrum of the state of the Lorenz-96 and the Kuramoto-Sivashinsky equation matched with the predicted ones.

In the Lorenz-96 and the Kuramoto-Sivashinsky equation, the parallel LSTM and GRU models exhibited similar predictive performance compared to the parallel RC. The memory requirements of the models are comparable. RC networks require large reservoirs with $1000 - 6000$ nodes per member to reach the predictive performance of parallel GRU/LSTM with a few hundred nodes, but their training time is significantly lower.

Last but not least, we evaluated and compared the efficiency of GRU and RC networks in capturing the Lyapunov spectrum of the KS equation. The positive Lyapunov exponents are captured accurately by both RC and GRU. Both networks cannot reproduce two zero LEs Λ_7 and Λ_8. When these two are discarded from the spectrum, GRU and RC networks show comparable accuracy in terms of relative and absolute error of the Lyapunov spectrum.

Further investigation on the underlying reasons why the RNNs and RC cannot capture the zero Lyapunov exponents is a matter of ongoing work. Another interesting direction could include studying the memory capacity of the networks. This could offer more insight into which architecture and training method is appropriate for tasks with long-term dependencies. Moreover, we plan to investigate a coupling of the two training approaches to further improve their predictive performance, for example a network can utilize both RC and LSTM computers to identify the input to output mapping. While the weights of the RC are initialized randomly to satisfy the echo state property, the output weights alongside with the LSTM weights can be optimized by back-propagation. This approach, although more costly, might achieve higher efficiency, as the LSTM is used as a residual model correcting the error that a plain RC would have.

Although we considered a batched version of RC training to reduce the memory requirements, further research is needed to alleviate the memory burden associated with the matrix inversion (see Appendix A, Eq. (14)) and the numerical problems associated with the eigenvalue decomposition of the sparse weight matrix.

Further directions could be the initialization of RNN weights with RC based heuristics based on the echo state property and fine-tuning with BPTT. Another promising direction is to evaluate the models in terms of the amount of data needed to learn the system dynamics. This is possible for the plain cell RNN, where the heuristics are directly applicable. However, in more complex architectures like the LSTM or the GRU, more sophisticated initialization schemes that ensure some form of echo state property have to be investigated. The computational cost of training

networks of the size of RC with back-propagation is also challenging. This hybrid training method is an interesting future direction.

In conclusion, recurrent neural networks for data-driven surrogate modeling and forecasting of chaotic systems can efficiently be used to model high-dimensional dynamical systems, can be parallelized alleviating scaling problems and constitute a promising research subject that requires further analysis.

8. Data and Code

The code and data will be available upon publication in the following link https://github.com/pvlachas/RNN to assist reproducibility of the results. The software was written in Python utilizing Tensorflow Abadi, Barham, Chen, Chen, Davis, Dean, Devin, Ghemawat, Irving, Isard, Kudlur, Levenberg, Monga, Moore, Murray, Steiner, Tucker, Vasudevan, Warden, Wicke, Yu and Zheng (2016) and Pytorch Paszke, Gross, Chintala, Chanan, Yang, DeVito, Lin, Desmaison, Antiga and Lerer (2017) for automatic differentiation and the design of the neural network architectures.

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CRediT authorship contribution statement

PR Vlachas: Methodology, software development, data curation, analysis of the results, original draft preparation. **J Pathak:** Methodology, data curation, analysis of the results, contribution to the manuscript. **BR Hunt:** Methodology, consultation, analysis of the results, proofreading. **TP Sapsis:** Conceptualization of the study, consultation, analysis of the results, proofreading. **M Girvan:** Conceptualization of the study. **E Ott:** Conceptualization of the study, consultation, analysis of the results. **P Koumoutsakos:** Conceptualization of the study, consultation, analysis of the results.

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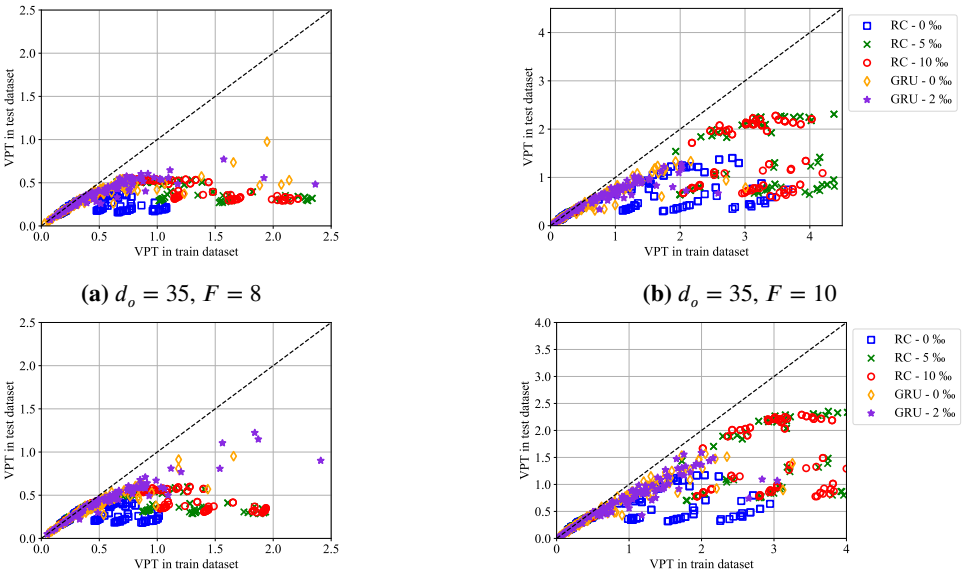


(a) $d_o = 35$, $F = 8$ **(b)** $d_o = 35$, $F = 10$

(c) $d_o = 40$, $F = 8$ **(d)** $d_o = 40$, $F = 10$

Figure 16: VPT in the testing data plotted against VPT in the training data for RC and GRU models trained with added noise of different levels in the data. Noise only slightly varies the forecasting efficiency. In contrast, the effectiveness of RC in forecasting the full-order system is increased as depicted in plots **(b)** and **(d)**.

A. Memory Efficient Implementation of RC Training

In order to alleviate the RAM requirement for the computation of the RC weights we resort to a batched approach. Assuming the hidden reservoir size is given by $\boldsymbol{h} \in \mathbb{R}^{d_h}$, by teacher forcing the RC network with true data from the system for d_N time-steps and stacking all data in a single matrix we end up with matrix $\mathbf{H} \in \mathbb{R}^{d_N \times d_h}$. Moreover, by stacking the target values, which are the input data shifted by one time-step, we end up in the target matrix $\mathbf{Y} \in \mathbb{R}^{d_N \times d_o}$, where d_o is the dimension of the observable we are predicting. The weights are computed using

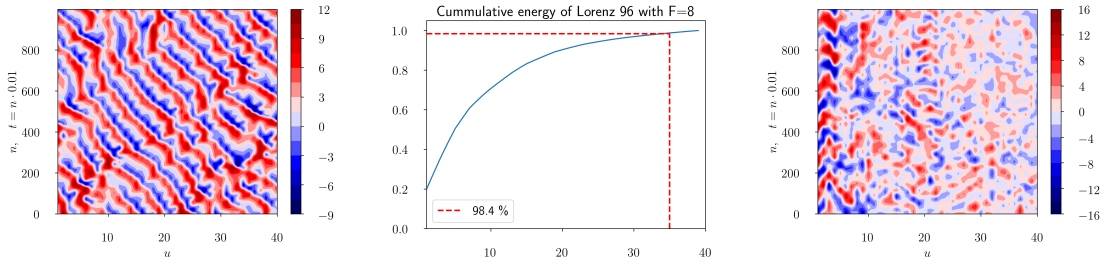
$$W_{out} = \underbrace{\mathbf{Y}^T\mathbf{H}}_{\overline{\mathbf{Y}}} \Big(\underbrace{\mathbf{H}^T\mathbf{H}}_{\overline{\mathbf{H}}} + \eta\mathbf{I} \Big)^{-1} \tag{14}$$

where η is the Tikhonov regularization parameter and $\mathbf{I}$ the unit matrix. In our case d_N is of the order of 10^5 and $d_N >> d_h$. To reduce the memory requirements of the training method, we compute the matrices $\overline{\mathbf{H}} = \mathbf{H}^T\mathbf{H} \in \mathbb{R}^{d_h \times d_h}$ and $\overline{\mathbf{Y}} = \mathbf{Y}^T\mathbf{H} \in \mathbb{R}^{d_o \times d_h}$ in a time-batched schedule.

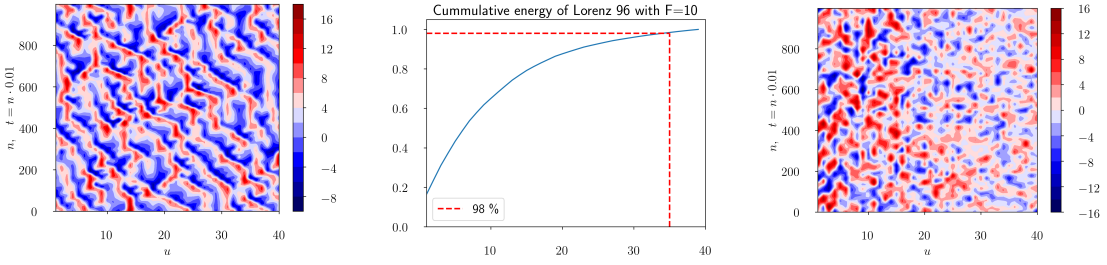
Specifically, we initialize $\overline{\mathbf{Y}} = \mathbf{0}$ and $\overline{\mathbf{H}} = \mathbf{0}$. Then every d_n time-steps with $d_n << d_N$, we compute the batch matrix $\overline{\mathbf{H}_b} = \mathbf{H}_b^T\mathbf{H}_b \in \mathbb{R}^{d_h \times d_h}$, where $\mathbf{H}_b \in \mathbb{R}^{d_n \times d_h}$ is formed by the stacking the hidden state only for the last d_n time-steps. In the same way, we compute $\overline{\mathbf{Y}_b} = \mathbf{Y}_b^T\mathbf{H}_b \in \mathbb{R}^{d_o \times d_h}$, where $\mathbf{Y}_b \in \mathbb{R}^{d_n \times d_o}$ is formed by the stacking of the target data for the last d_n time-steps. After every batch computation we update our beliefs with $\overline{\mathbf{H}} \leftarrow \overline{\mathbf{H}} + \overline{\mathbf{H}_b}$ and $\overline{\mathbf{Y}} \leftarrow \overline{\mathbf{Y}} + \overline{\mathbf{Y}_b}$.

B. Regularizing Training with Noise

In our study, we investigate the effect of noise to the training data. In Fig. 16, we plot the Valid Prediction Time (VPT) in the testing data with respect to the VPT that each model achieves in the training data. We find out that RC models trained with additional noise of $5 - 10$ ‰not only achieve better generalization, but their forecasting efficiency improves in both training and testing dataset. Moreover, the effect of divergent predictions by iterative forecasts is alleviated significantly. In contrast, adding noise does not seem to have an important impact on the performance of GRU models.



(a) Energy spectrum $F = 8$ **(b)** State evolution $F = 8$ **(c)** SVD mode evolution $F = 8$



(d) Energy spectrum $F = 10$ **(e)** State evolution $F = 10$ **(f)** SVD mode evolution $F = 10$

Figure 17: Energy spectrum of Lorenz-96

C. Dimensionality Reduction with Singular Value Decomposition

Singular Value Decomposition (SVD) can be utilized to perform dimensionality reduction in a dataset by identifying the modes that capture the highest variance in the data and then performing a projection on these modes. Assuming that a data matrix is given by stacking the time-evolution of a state $\mathbf{u} \in \mathbb{D}$ as $U = [\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_N]$, where the index N is the number of data samples. By subtracting the temporal mean $\overline{\mathbf{u}}$ and stacking the data, we end up with the data matrix $\mathbf{U} \in \mathbb{R}^{T \times D}$. Performing SVD on $\mathbf{U}$ leads to

$$\mathbf{U} = \mathbf{M}\mathbf{\Sigma}\mathbf{V}^T, \quad \mathbf{M} \in \mathbb{R}^{N \times N}, \quad \mathbf{\Sigma} \in \mathbb{R}^{N \times D}, \quad \mathbf{V} \in \mathbb{R}^{D \times D}, \tag{15}$$

with Σ diagonal, with descending diagonal elements. The columns of matrix V are considered the modes of the SVD, while the square D singular values of $\mathbf{\Sigma}$ correspond to the data variance explained by these modes. This variance is also referred to as energy. In order to calculate the percentage of the total energy the square of the singular value of each mode has to be divided by the sum of squares of the singular values of all modes. In order to reduce the dimensionality of the dataset, we first have to decide on the reduced order dimension $r_{dim} < D$. Then we identify the eigenvectors corresponding to the most high-energetic eigenmodes. These are given by the first columns $\mathbf{V_r}$ of $\mathbf{V}$, i.e., $\mathbf{V} = [\mathbf{V_r}, \mathbf{V_{-r}}]$. We discard the low-energetic modes $\mathbf{V_{-r}}$. The dimension of the truncated eigenvector matrix is $\mathbf{V_r} \in \mathbb{R}^{D \times r_{dim}}$. In order to reduce the dimensionality of the dataset, each vector $\mathbf{u} \in \mathbb{D}$ is projected to $\mathbf{u}_r \in r_{dim}$ by

$$\mathbf{c} = \mathbf{V_r}^T \mathbf{u}, \quad \mathbf{c} \in \mathbb{R}^{r_{dim}}. \tag{16}$$

In the Lorenz-96 system, we construct a reduced order observable with $d_o = 35$ modes of the system. The cumulative energy distribution along with a contour plot of the state and the mode evolution is illustrated in Fig. 17.

D. Calculation of Lyapunov Spectrum

The true Lyapunov exponents of the KS equation are computed as in Pathak et al. (2018a) by solving the KS equations in the Fourier space with a fourth order time-stepping method called ETDRK4 Kassam and Trefethen (2005) and utilizing a QR decomposition approach. The trained RNN model with GRU cell is used as a surrogate to compute the full Lyapunov spectrum of the Kuramoto-Sivashinsky system. Recall that the RNN dynamics are given by

$$\begin{aligned} \boldsymbol{h}_t &= f_h^h(\boldsymbol{o}_t, \boldsymbol{h}_{t-1}) \\ \boldsymbol{o}_{t+1} &= f_h^o(\boldsymbol{h}_t), \end{aligned} \tag{17}$$

where f_h^h is the hidden-to-hidden and f_h^o is the hidden-to-output mapping, $\boldsymbol{o} \in \mathbb{R}^{d_o}$ is an observable of the state, and $\boldsymbol{h}_t \in \mathbb{R}^{d_h}$ is the hidden state of the RNN. All models utilized in this work share this common architecture. They only differ in the forms of f_h^o and f_h^h. More importantly, the output mapping is linear, i.e.,

$$\boldsymbol{o}_{t+1} = f_h^o(\boldsymbol{h}_t) = \boldsymbol{W}_o \boldsymbol{h}_t. \tag{18}$$

The LEs are calculated based on the Jacobian $\boldsymbol{J} = \frac{d\boldsymbol{h}_t}{d\boldsymbol{h}_{t-1}}$ of the hidden state dynamics along the trajectory. In the following we compute the Jacobian using equations (17). By writing down the equations for two consecutive time-steps, we get

$$\text{Timestep } t-1: \; h_{t-1} = f_h^h(o_{t-1}, h_{t-2}) \tag{19}$$

$$o_t = f_h^o(h_{t-1}) = W_o h_{t-1} \tag{20}$$

$$\text{Timestep } t: \; h_t = f_h^h(o_t, h_{t-1}). \tag{21}$$

The partial Jacobians needed to compute the total Jacobian are:

$$\frac{\partial f_h^h}{\partial o} = \boldsymbol{J}_o^{hh} \in \mathbb{R}^{d_h \times d_o} \tag{22}$$

$$\frac{\partial f_h^h}{\partial h} = \boldsymbol{J}_h^{hh} \in \mathbb{R}^{d_h \times d_h} \tag{23}$$

$$\frac{\partial f_h^o}{\partial h} = \boldsymbol{J}_h^{oh} \in \mathbb{R}^{d_o \times d_h}. \tag{24}$$

In total we can write:

$$\frac{dh_t}{dh_{t-1}} = \frac{df_h^h(o_t, h_{t-1})}{dh_{t-1}} = \frac{\partial f_h^h(o_t, h_{t-1})}{\partial o_t}\frac{\partial o_t}{\partial h_{t-1}} + \frac{\partial f_h^h(o_t, h_{t-1})}{\partial h_{t-1}} \implies \tag{25}$$

$$\frac{dh_t}{dh_{t-1}} = \frac{\partial f_h^h(o_t, h_{t-1})}{\partial o_t}\frac{\partial f_h^o(h_{t-1})}{\partial h_{t-1}} + \frac{\partial f_h^h(o_t, h_{t-1})}{\partial h_{t-1}} \implies \tag{26}$$

$$\frac{dh_t}{dh_{t-1}} = \underbrace{\boldsymbol{J}_o^{hh}\Big|_{(o_t, h_{t-1})}}_{\text{evaluated at t}} \cdot \underbrace{\boldsymbol{J}_h^{oh}\Big|_{h_{t-1}}}_{\text{evaluated at t-1}} + \underbrace{\boldsymbol{J}_h^{hh}\Big|_{(o_t, h_{t-1})}}_{\text{evaluated at t}} \tag{27}$$

A product of this Jacobian along the orbit δ is developed and iteratively orthonormalized every T_n steps using the Gram-Schmidt method to avoid numerical divergence and keep the columns of the matrix R independent. We check the convergence criterion by tracking the estimated LE values every T_c time-steps. The input provided to the algorithm is a short time-series of length T_w to initialize the RNN and warm-up the hidden state $\widetilde{\boldsymbol{o}}_{1:T_w+1}$ (where the tilde denotes experimental or simulation data), the length of this warm-up time-series T_w, the number of the LE to calculate N, the maximum time to unroll the RNN T, a normalization time T_n and an additional threshold ϵ used as an additional termination criterion. The function ColumnSum$(\cdot)$ computes the sum of each column of a matrix, i.e., sum$(\cdot, \text{axis} = 1)$. This method can be applied directly to RNNs with one hidden state like RC or GRUs. An adaptation to the LSTM is left for future research. The pseudocode of the algorithm to calculate the Lyapunov exponents of the RNN is given in Algorithm 1.

Algorithm 1 Algorithm to calculate Lyapunov Exponents of a trained surrogate RNN model

procedure LE_RNN($\widetilde{\boldsymbol{o}}_{1:T_w+1}, T_w, N, T, T_n, \epsilon$)
- Initialize $\boldsymbol{h}_0 \leftarrow 0$.
- **for** $t = 1 : T_w$ **do** ▷ Warming-up the hidden state of the RNN based on true data
  - $\boldsymbol{h}_t \leftarrow f_h^h(\widetilde{\boldsymbol{o}}_t, \boldsymbol{h}_{t-1})$
- **end for**
- $\boldsymbol{h}_0 \leftarrow \boldsymbol{h}_{T_w}$
- $\boldsymbol{o}_1 \leftarrow \widetilde{\boldsymbol{o}}_{T_w+1}$
- Pick a random orthonormal matrix $\delta \in \mathbb{R}^{d_h \times N_{LE}}$. ▷ Initializing N_{LE} deviation vectors
- $\widetilde{T} \leftarrow T/T_n$
- Initialize $\widetilde{R} \leftarrow \mathbf{0} \in \mathbb{R}^{N \times \widetilde{T}}$.
- $l_{prev}, l \leftarrow \mathbf{0} \in \mathbb{R}^N$ ▷ Initializing the N LE to zero.
- $\boldsymbol{J}_0 \leftarrow \nabla_{\boldsymbol{h}} f_h^o(\boldsymbol{h}_0)$.
- **for** $t = 1 : T$ **do** ▷ Evolve the RNN dynamics
  - $\boldsymbol{h}_t \leftarrow f_h^h(\boldsymbol{o}_t, \boldsymbol{h}_{t-1})$
  - $\boldsymbol{o}_{t+1} \leftarrow f_h^o(\boldsymbol{h}_t)$
  - $\boldsymbol{J}_1 \leftarrow \nabla_{\boldsymbol{h}} f_h^h(\boldsymbol{o}_{t+1}, \boldsymbol{h}_t)$. ▷ Calculating the partial Jacobians
  - $\boldsymbol{J}_2 \leftarrow \nabla_{\boldsymbol{o}} f_h^h(\boldsymbol{o}_{t+1}, \boldsymbol{h}_t)$.
  - $\boldsymbol{J} \leftarrow \boldsymbol{J}_1 + \boldsymbol{J}_2 \cdot \boldsymbol{J}_0$. ▷ Calculating the total Jacobian
  - $\delta \leftarrow \boldsymbol{J} \cdot \delta$ ▷ Evolving the deviation vectors δ
  - **if** $\mod(t, T_{norm}) = 0$ **then** ▷ Re-orthonormalizing with QR-decomposition
    - $Q, R \leftarrow QR(\delta)$
    - $\delta \leftarrow Q[:, :N]$ ▷ Replacing the deviation vectors with the columns of Ԛ^ʌ…
    - $\widetilde{R}[:, t/T_{norm}] \leftarrow \log(\mathrm{diag}(R[:N, :N]))$
    - **if** $\mod(t, T_c) = 0$ **then** ▷ Checking the convergence criterion
      - $l \leftarrow \mathrm{Real}(\mathrm{ColumnSum}(\widetilde{R}))/(t * \delta t)$ ▷ Divide with the total timespan
      - $l \leftarrow \mathrm{sort}(l)$
      - $d \leftarrow |l - l_{prev}|_2$
      - **if** $d < \epsilon$ **then**
        - **break**
      - **end if**
    - **end if**
  - **end if**
  - $\boldsymbol{J}_0 \leftarrow \nabla_{\boldsymbol{h}} f_h^o(\boldsymbol{h}_t)$.
- **end for**
- **return** l ▷ Returning the estimated Lyapunov Exponents

end procedure

Table 2
Hyperparameters of **RC** for Lorenz-96

Hyperparameter	Explanation	Values
D_r	reservoir size	{6000, 9000, 12000, 18000}
N	training data samples	10^5
d	degree of $W_{h,h}$	{3, 8}
ρ	radius of $W_{h,h}$	{0.4, 0.8, 0.9, 0.99}
ω	input scaling	{0.1, 0.5, 1.0, 1.5, 2.0}
η	regularization	$\{10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}\}$
d_o	observed state dimension	{35, 40}
n_w	warm-up steps (testing)	2000
κ_n	noise level in data	{0, 0.5%, 1%}

Table 3
Hyperparameters of **GRU/LSTM** for Lorenz-96

Hyperparameter	Explanation	Values
d_h	hidden state size	{1, 2, 3} layers of {500, 1000, 1500}
N	training data samples	10^5
κ_1	BPTT forward time steps	{1, 8}
κ_2	BPTT truncated backprop. length	{8, 16}
κ_3	BPTT skip gradient parameter	$= \kappa_2 + \kappa_1 - 1$
η	initial learning rate	10^{-3}
p	zoneout probability	{0.99, 0.995 1.0}
d_o	observed state dimension	{35, 40}
n_w	warm-up steps (testing)	2000
κ_n	noise level in data	{0, 0.2%}

Table 4
Hyperparameters of **Unitary Evolution** networks for Lorenz-96

Hyperparameter	Explanation	Values
d_h	hidden state size	{1, 2, 3} layers of {500, 1000, 1500}
N	training data samples	10^5
κ_1	BPTT forward time steps	{1, 8}
κ_2	BPTT truncated backprop. length	{8, 16}
κ_3	BPTT skip gradient parameter	$= \kappa_2 + \kappa_1 - 1$
η	initial learning rate	10^{-3}
p	zoneout probability	1.0
d_o	observed state dimension	{35, 40}
n_w	warm-up steps (testing)	2000
κ_n	noise level in data	{0, 0.2%}

E. Model Hyperparameters

For the Lorenz-96 system space with $d_o \in \{35, 40\}$ (in the PCA mode), we used the hyperparameters reported on Table 2 for RC and 3 for GRU/LSTM models. For the parallel architectures in the state space of Lorenz-96 the hyperparameters are reported on Tables 5 and 6 for the parallel RC and GRU/LSTM models respectively. For the parallel architectures in the state space of the Kuramoto-Sivashinsky architecture the hyperparameters are reported on Tables 7 and 8 for the parallel RC and GRU/LSTM models respectively. We note here that in all RNN methods, the optimizer used to update the network can also be optimized. To alleviate the computational burden we stick to Adam.

Table 5
Hyperparameters of **Parallel RC** for Lorenz-96

Hyperparameter	Explanation	Values
D_r	reservoir size	{1000, 3000, 6000, 12000}
N_g	number of groups	20
G	group size	2
I	interaction length	4
N	training data samples	10^5
d	degree of $W_{h,h}$	10
ρ	radius of $W_{h,h}$	0.6
ω	input scaling	0.5
η	regularization	10^{-6}
d_o	observed state dimension	40
n_w	warm-up steps (testing)	2000

Table 6
Hyperparameters of **Parallel GRU/LSTM** for Lorenz-96

Hyperparameter	Explanation	Values
d_h	hidden state size	{100, 250, 500}
N_g	number of groups	20
G	group size	2
I	interaction length	4
N	training data samples	10^5
κ_1	BPTT forward time steps	4
κ_2	BPTT truncated backprop. length	4
κ_3	BPTT skip gradient parameter	4
η	initial learning rate	10^{-3}
p	zoneout probability	{0.998, 1.0}
d_o	observed state dimension	40
n_w	warm-up steps (testing)	2000

Table 7
Hyperparameters of **Parallel RC** for Kuramoto-Sivashinsky

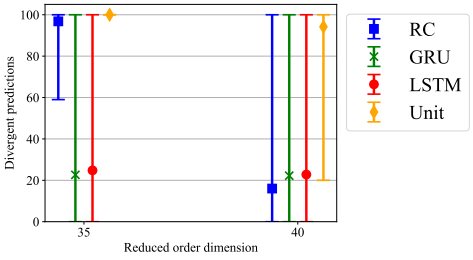
Hyperparameter	Explanation	Values
D_r	reservoir size	{500, 1000, 3000, 6000, 12000}
N_g	number of groups	64
G	group size	8
I	interaction length	8
N	training data samples	10^5
d	degree of $W_{h,h}$	10
ρ	radius of $W_{h,h}$	0.6
ω	input scaling	1.0
η	regularization	10^{-5}
d_o	observed state dimension	512
n_w	warm-up steps (testing)	2000

Table 8
Hyperparameters of **Parallel GRU/LSTM** for Kuramoto-Sivashinsky

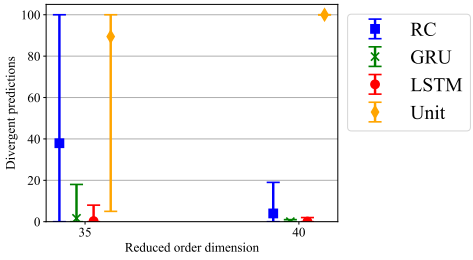
Hyperparameter	Explanation	Values
d_h	hidden state size	{80, 100, 120}
N_g	number of groups	64
G	group size	8
I	interaction length	8
N	training data samples	10^5
κ_1	BPTT forward time steps	4
κ_2	BPTT truncated backprop. length	4
κ_3	BPTT skip gradient parameter	4
η	initial learning rate	10^{-3}
p	zoneout probability	{0.998, 1.0}
d_o	observed state dimension	512
n_w	warm-up steps (testing)	2000

Table 9
Hyperparameters of **Parallel Unitary Evolution** networks for Kuramoto-Sivashinsky

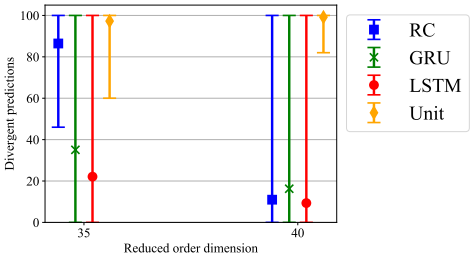
Hyperparameter	Explanation	Values
d_h	hidden state size	{100, 200, 400}
N_g	number of groups	64
G	group size	8
I	interaction length	8
N	training data samples	10^5
κ_1	BPTT forward time steps	4
κ_2	BPTT truncated backprop. length	4
κ_3	BPTT skip gradient parameter	4
η	initial learning rate	10^{-2}
p	zoneout probability	1.0
d_o	observed state dimension	512
n_w	warm-up steps (testing)	2000



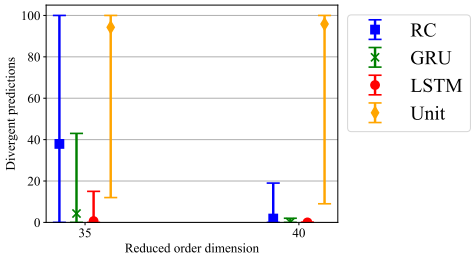
(a) $F = 8$, models selected according to the highest VPT in the testing dataset.



(b) $F = 8$, models selected according to the lowest error in the frequency spectrum.



(c) $F = 10$, models selected according to the highest VPT in the testing dataset.



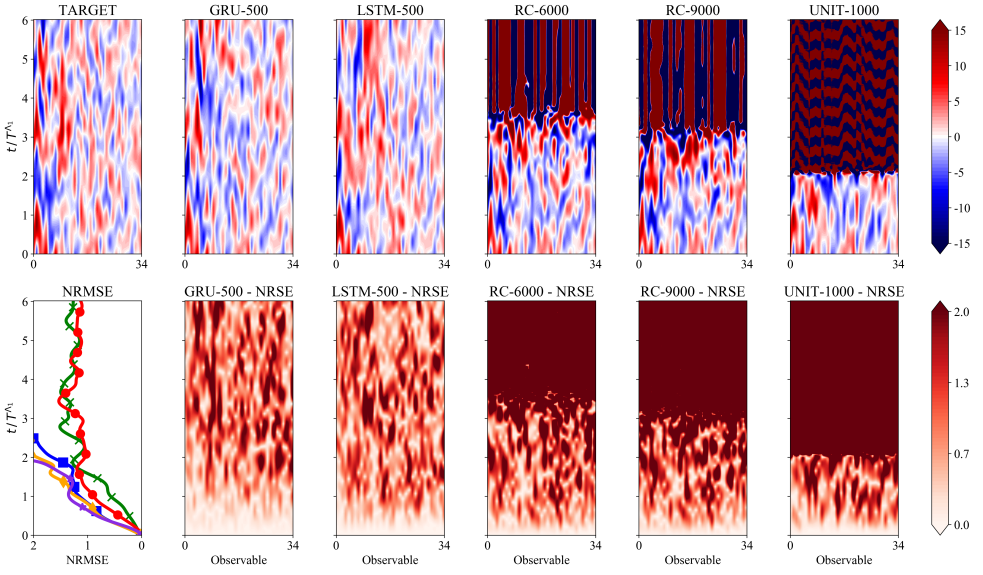
(d) $F = 10$, models selected according to the lowest error in the frequency spectrum.

Figure 18: Average number of initial conditions that lead to divergent iterative prediction from the 20% of the hyperparameter model runs with either the highest VPT (plots **(a)** and **(c)**) or lowest frequency error (plots **(b)** and **(d)**) for $F \in \{8, 10\}$. LSTM and GRU show more stable iterative forecasting behavior, while the effect of divergence seems to be more prominent in RC networks. Unitary networks seem to be completely diverging, presumably due to the inability to identify proper hyperparameters that lead to more stable behavior.

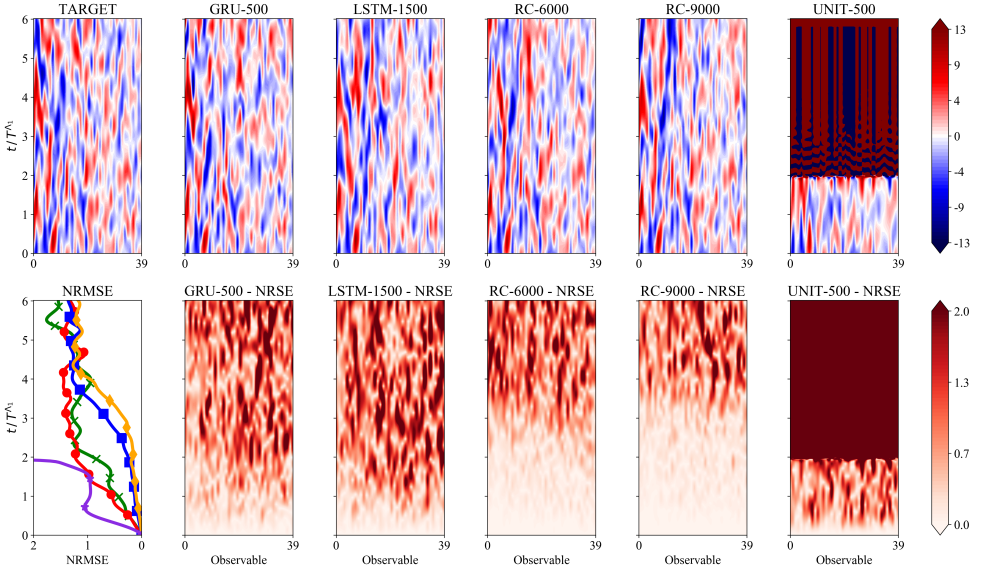
F. Additional Results - Lorenz 96 - Divergence of Unitary and RC RNNs

In this section, we try to quantify the divergence effect due to the accumulation of the forecasting error in the iterative prediction. As illustrated in Fig. 19 predictions of the models may diverge from the attractor. This unstable behavior seems to be more prominent in RC and Unitary networks. In order to quantify this phenomenon, we plot the average number of diverging trajectories from the 20% hyperparameter runs of each model with the highest VPT in Figs. 18a for $F = 8$ and 18c for $F = 10$. The errorbars denote minimum and maximum number of divergences over the 100 initial conditions. On average the divergence effect is much more prominent in the RC networks in the reduced order space, while divergence of GRUs and LSTMs occurs slightly more frequently in the full-oder space. However, these models are not selected according to their fit to the long-term behavior of their forecasts, and comparison might be unfair, as models with slightly smaller VPT but much better stability properties might be ignored in the selection of the 20% of the best model.

For this reason, we plot the same quantity, but instead of selecting the models with the highest VPT, we consider the 20% models with the lowest frequency error in 18b for $F = 8$ and 18d for $F = 10$. Indeed, we observe that in the full order space the situation changes, as all networks on average have close to zero divergent initial conditions, exempt from Unitary RNNs whose divergence seems to stem from the difficulty to identify proper hyperparameters. However, in the reduced order space, even when the models with the lowest error in the frequency spectrum are considered, the RC networks seem to suffer much more from divergences, while the LSTM seems to have close to zero divergent initial conditions on average. One example of this divergence in an initial condition from the **test** dataset is illustrated in Fig. 19 . The RC and the Unitary networks diverge in the reduced order state predictions after approximately two Lyapunov times.

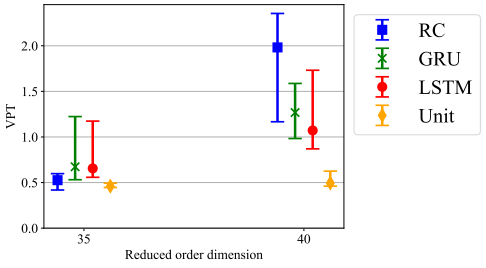


(a) Reduced order state ($\boldsymbol{d_o = 35}$), $F = 8$

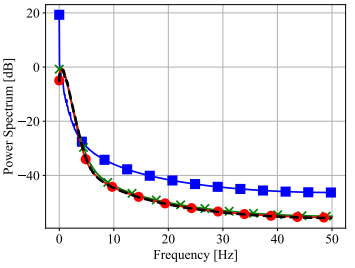


(b) Full order state ($\boldsymbol{d_o = 40}$), $F = 8$

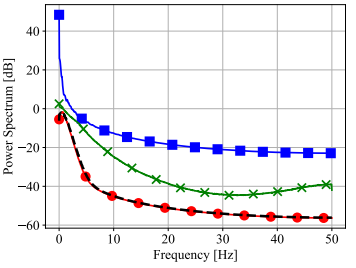
Figure 19: Contour plots of a spatio-temporal forecast on the SVD modes of the Lorenz-96 system with $F = 8$ in the testing dataset with GRU, LSTM, RC and a Unitary network along with the true (target) evolution and the associated NRSE contours for the reduced order observable **(a)** $d_o = 35$ and the full state **(b)** $d_o = 40$. The evolution of the component average NRSE (NMRSE) is plotted to facilitate comparison. Unitary networks suffer from propagation of forecasting error and eventually their forecasts diverge from the attractor. Forecasts in the case of an observable dimension $d_o = 40$ diverge slower as the dynamics are deterministic. In contrast, forecasting the observable with $d_o = 35$ is challenging due to both **(1)** sensitivity to initial condition and **(2)** incomplete state information that requires the capturing of temporal dependencies. In the full-state setting, RC models achieve superior forecasting accuracy compared to all other models. In the challenging reduced order scenario, LSTM and GRU networks demonstrate a stable behavior in iterative prediction and reproduce the long-term statistics of the attractor. In contrast, in the reduced order scenario RC suffer from frequent divergence. The divergence effect is illustrated in this chosen initial condition.
GRU ; LSTM ; RC-6000 ; RC-9000 ; Unit ;



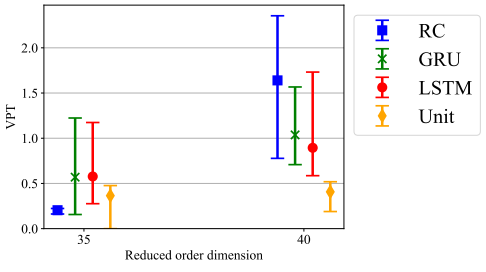
(a) VPT for observable dimension $d_o = 35$ and $d_o = 40$. Best 20% of all hyperparameter sets selected according to the **maximum VPT**.



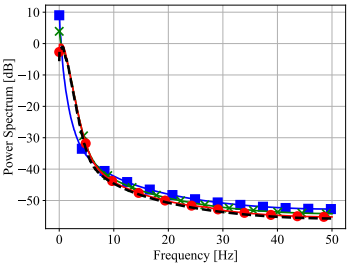
(b) Power spectrum for $d_o = 35$ of the model with **highest VPT** for each architecture.



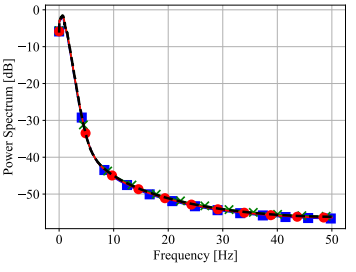
(c) Power spectrum for $d_o = 40$ of the model with **highest VPT** for each architecture.



(d) VPT for observable dimension $d_o = 35$ and $d_o = 40$. Best 20% of all hyperparameter sets selected according to the **minimum power spectrum error**.



(e) Power spectrum for $d_o = 35$ of the model with **minimum power spectrum error** for each architecture.

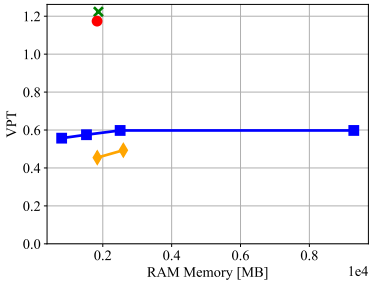


(f) Power spectrum for $d_o = 40$ of the model with **minimum power spectrum error** for each architecture.

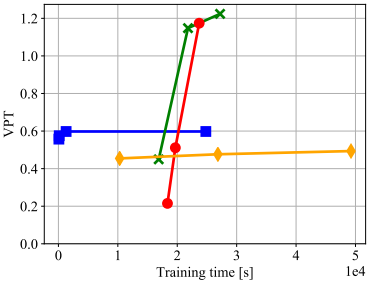
Figure 20: Results of RC, GRU, LSTM and Unitary networks forecasting the dynamics of an observable consisting of the SVD modes of the Lorenz-96 system with $F = 10$ and state dimension 40. The observable consists of the $d_o = 35$ most energetic modes or the full state information ($d_o = 40$). **(a)-(d)** Mean valid prediction time (VPT) (marker) of the 20% of the hyperparameter sets of each model with highest VPT **(a)** or lowest frequency error **(b)**, along with maximum and minimum (errorbars) plotted w.r.t. the observable dimension d_o. **(b)-(e)** Reconstructed power spectrum of the signal for $d_o = 35$. **(c)-(f)** Reconstructed power spectrum of the signal for $d_o = 40$.
RC ; GRU ; LSTM ; Unit ; Groundtruth ;

G. Additional Results - Lorenz 96 $F = 10$

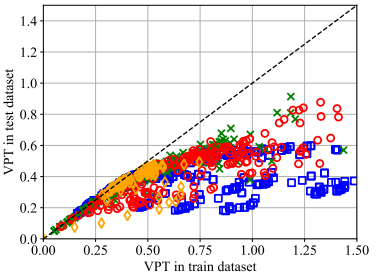
In this section, we provide additional results (Figs. 20,21) for the forcing regime $F = 10$ that are in agreement with the main conclusions drawn in the main manuscript for the forcing regime $F = 8$. An example of a single forecast of the models starting from an initial condition in the **test** dataset is given in Fig. 22



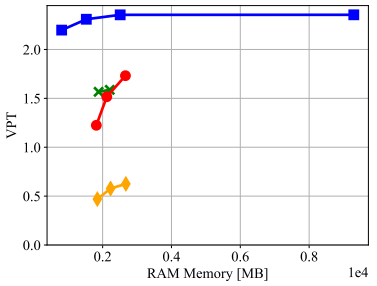
(a) VPT w.r.t. RAM memory for $d_o = 35$.



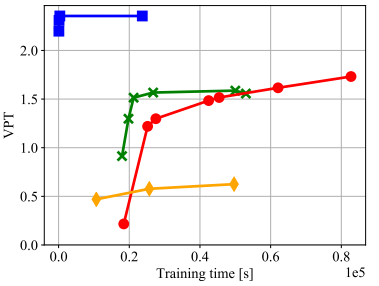
(b) VPT w.r.t. the training time for $d_o = 35$.



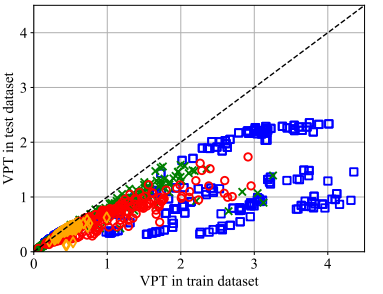
(c) VPT in **test** data w.r.t. VPT in the **training** data for $d_o = 35$.



(d) VPT w.r.t. RAM memory for $d_o = 40$.

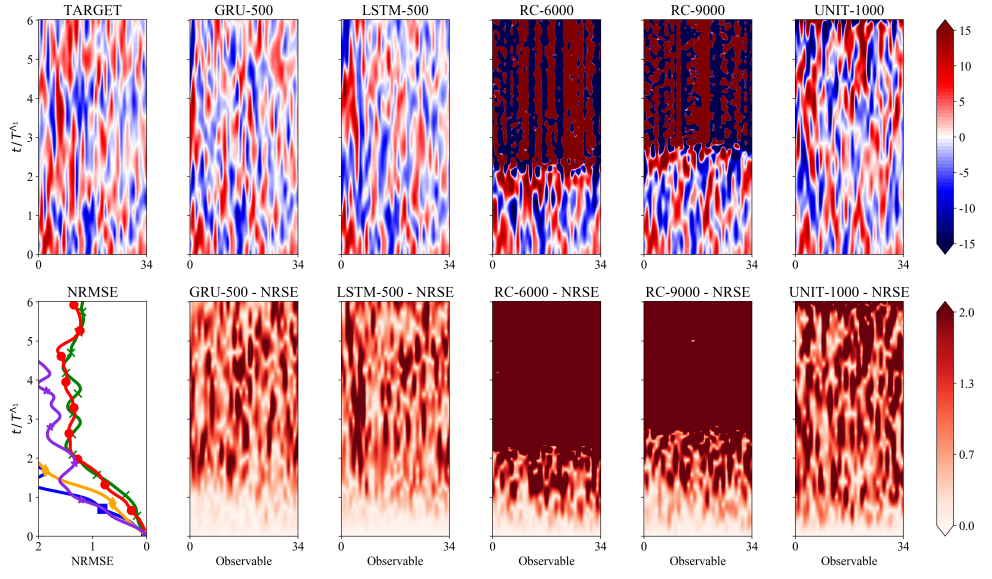


(e) VPT w.r.t. the training time for $d_o = 40$.

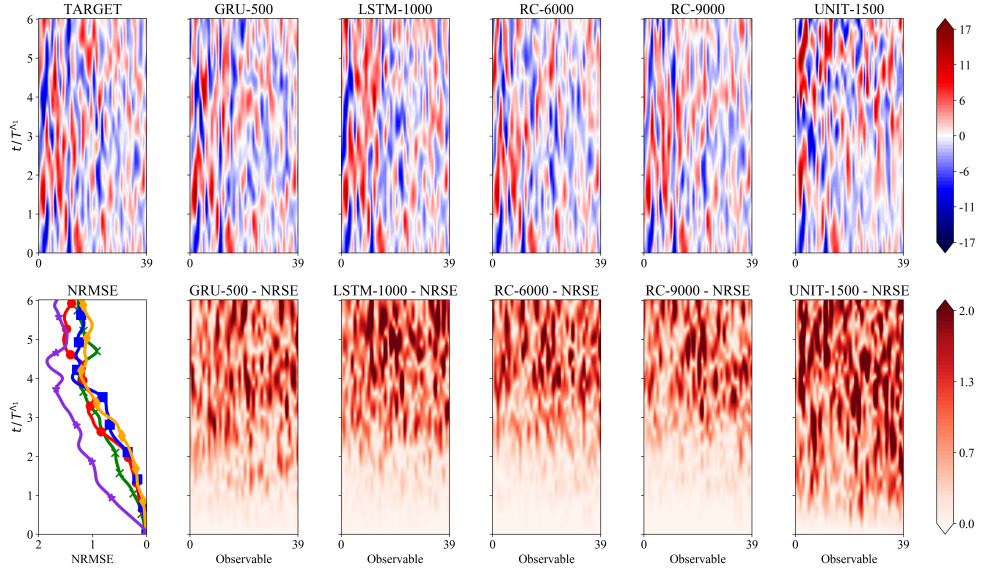


(f) VPT in **test** data w.r.t. VPT in the **training** data for $d_o = 40$.

Figure 21: Forecasting results on the dynamics of an observable consisting of the SVD modes of the Lorenz-96 system with $F = 10$ and state dimension 40. The observable consists of the $d_o \in \{35, 40\}$ most energetic modes. (a), (d) Valid prediction time (VPT) plotted w.r.t. the required RAM memory for dimension $d_o \in \{35, 40\}$. (b), (e) VPT plotted w.r.t. training time for dimension $d_o \in \{35, 40\}$. (c), (f) VPT measured from 100 initial conditions sampled from the test data plotted w.r.t. VPT from 100 initial conditions sampled from the training data for each model for $d_o \in \{35, 40\}$. RCs tend to overfit easier compared to GRUs/LSTMs that utilize a validation based training routine.
RC ▬■▬; GRU ✱; LSTM ●; Unit ◆; Ideal ▪▪▪;

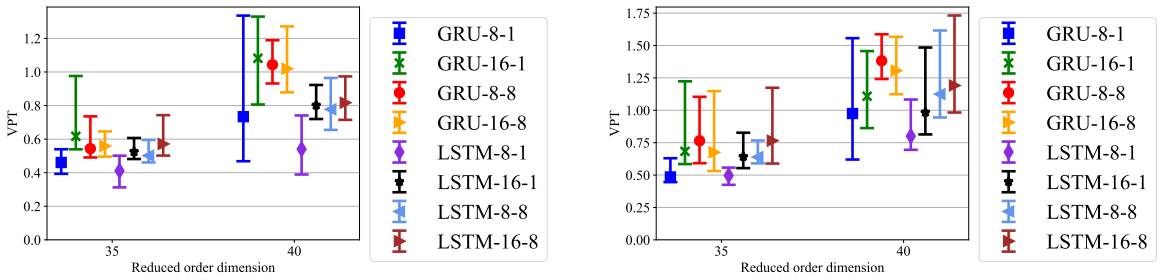


(a) Reduced order state ($\boldsymbol{d_o = 35}$), $F = 10$



(b) Full state ($\boldsymbol{d_o = 40}$), $F = 10$

Figure 22: Contour plots of a spatio-temporal forecast on the SVD modes of the Lorenz-96 system with $F = 10$ in the testing dataset with GRU, LSTM, RC and a Unitary network along with the true (target) evolution and the associated NRSE contours for the reduced order observable **(a)** $d_o = 35$ and the full state **(b)** $d_o = 40$. The evolution of the component average NRSE (NMRSE) is plotted to facilitate comparison. Forecasts in the case of an observable dimension $d_o = 40$ diverge slower as the dynamics are deterministic. In contrast, forecasting the observable with $d_o = 35$ is challenging due to both **(1)** sensitivity to initial condition and **(2)** incomplete state information that requires the capturing of temporal dependencies. Even in this challenging scenario, LSTM and GRU networks demonstrate a stable behavior in iterative prediction and reproduce the long-term statistics of the attractor. Accurate short-term predictions can be achieved with very large RC networks ($d_h = 9000$) at the cost of high memory requirements. However, even in this case, RC models may diverge from the attractor and do not reproduce the attractor climate.
GRU ✱; LSTM ●; RC-6000 ■; RC-9000 ◆; Unit ★;



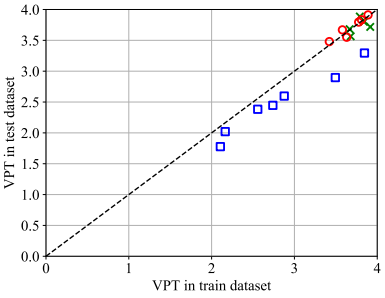
(a) $F = 8$ **(b)** $F = 10$

Figure 23: VPT in the testing data for stateful LSTM and GRU models trained with different truncated Backpropagation through time parameters κ_1 and κ_2 in the (reduced) SVD mode observable of the Lorenz 96 system. The legend of each plot reports the models along with their $\kappa_1 - \kappa_2$ parameters used to train them. Each marker reports the mean VPT, while the errorbars report the minimum and maximum VPT. We observe that especially in the reduced order observable scenario ($d_0 = 35$), having a large truncated back-propagation parameter κ_1 (also referred to as sequence length) is vital to capture the temporal dependencies in the data and achieve high forecasting efficiency. In contrast in the full-state scenario ($d_0 = 40$) a model with a small back-propagation horizon suffices.

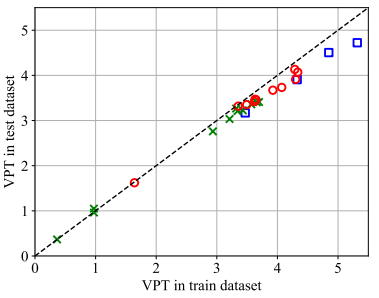
H. Temporal Dependencies and Backpropagation

In our study, in order to train the GRU and LSTM models with back-propagation through time (BPTT), we need to tune the parameters κ_1 and κ_2. The first one denotes the truncated back-propagation length (also referred to to as sequence length) and the second the number of future time-steps used to compute the loss and backpropagate the gradient during training at each batch. In the hyperparameter study considered in this work, we varied $\kappa_1 \in \{8, 16\}$ and $\kappa_2 \in \{1, 8\}$.

In Fig. 23 we plot the forecasting efficiency of LSTM and GRU models trained with different parameters in terms of the Valid Prediction Time (VPT) in the test dataset (averaged over 100 initial conditions) on the Lorenz-96 system for $F \in \{8, 10\}$. In the reduced order scenario case, we observe that models with a large sequence length κ_1 and a large prediction length κ_2 are pivotal in order to achieve a high forecasting efficiency. This implies that there are temporal correlations in the data that cannot be easily captured by other models with smaller horizons. In contrast, in the full order scenario, models with smaller κ_1 perform reasonably well, demonstrating that the need of capturing temporal correlations in the data in order to forecast the evolution is less prominent, since the full information of the state of the system is available.



(a) VPT in test w.r.t. VPT in train



(b) VPT in test w.r.t. VPT in train

Figure 24: The average VPT measured from 100 initial conditions sampled from the **test** dataset is plotted against the average VPT measured from 100 initial conditions sampled from the **training** dataset for parallel models forecasting the dynamics of (a) the Lorenz-96 system with state dimension $d_o = 40$ and (b) the Kuramoto Sivashinsky equation with state dimension $d_o = 512$.
RC □; GRU ✕; LSTM ○;

I. Over-fitting of Parallel Models

In this section, we provide results on the overfitting of the models in the parallel setting in the Lorenz-96 model in Fig. 24a and the Kuramoto-Sivashinsky equation in Fig. 24b. In both cases we do not observe overfitting issues as the predictive performance in terms of the VPT of the models in the test dataset is very close to that in the training dataset, emphasizing that the generalization ability of the models is good.