CSCI 3104- Calculus Review

Michael Levet

September 1, 2019

Contents

1 Motivation- Asymptotics

Recall the definitions of Big-O, Big-Omega, and Big-Theta.

Definition 1 (Big-O)**.** Let $f, g : \mathbb{N} \to \mathbb{N}$ be functions. We say that $f(n) \in \mathcal{O}(g(n))$ if there exist constants $c, k \in \mathbb{Z}^+$ such that $f(n) \leq c \cdot g(n)$ for all $n \geq k$.

Example 1. You may recall from class that $n^2 \in \mathcal{O}(n^3)$, for example. Similarly, $2^n \in \mathcal{O}(n!)$. However, $n! \not\in \mathcal{O}(2^n)$.

In some sense, we think of Big-O as an *asymptotic upper bound*. That is, if $f(n) \in \mathcal{O}(g(n))$, we say that $g(n)$ grows at least as quickly as $f(n)$ in the long run (that is, for all $n \geq k$). In contrast, Big-Omega is our *asymptotic lower bound*, and it is defined analogously as Big-O.

Definition 2 (Big-Omega)**.** Let $f, g : \mathbb{N} \to \mathbb{N}$ be functions. We say that $f(n) \in \Omega(g(n))$ if there exist constants $c, k \in \mathbb{Z}^+$ such that $f(n) \geq c \cdot g(n)$ for all $n \geq k$.

Remark: We note that if $f(n) \in \mathcal{O}(g(n))$, then $g(n) \in \Omega(f(n))$. In particular, $n^3 \in \Omega(n^2)$ and $n! \in \Omega(2^n)$. However, $2^n \not\in \Omega(n!)$.

We next discuss Big-Theta. Informally, $f(n) \in \Theta(g(n))$ provided that $f(n)$ and $g(n)$ grow at the same asymptotic rate. This is formalized as follows.

Definition 3. Let $f, g : \mathbb{N} \to \mathbb{N}$ be functions. We say that $f(n) \in \Theta(g(n))$ provided that $f(n) \in \mathcal{O}(g(n))$ and $f(n) \in \Omega(g(n))$.

Example 2. We have that the function $f(n) = 3n^2 + 5n + 7 \in \Theta(n^2)$. Similarly, $n! \in \Theta(2 \cdot n!)$.

Example 3. Recall that $n^2 \in \mathcal{O}(n^3)$. However, $n^3 \not\in \mathcal{O}(n^2)$. So $n^2 \not\in \Theta(n^3)$.

A key tool in establishing these asymptotic relations is the Limit Comparison Test.

Theorem 1.1 (Limit Comparison Test)**.** *Let $f, g : \mathbb{N} \to \mathbb{N}$ be functions. Suppose that the following limit exists:*

$$L := \lim_{n \to \infty} \frac{f(n)}{g(n)}. \tag{1}$$

- *If $0 < L < \infty$, then $f(n) \in \Theta(g(n))$.*
- *If $L = 0$, then $f(n) \in \mathcal{O}(g(n))$. However, $f(n) \not\in \Theta(g(n))$.*
- *If $L = \infty$, then $f(n) \in \Omega(g(n))$. However, $f(n) \not\in \Theta(g(n))$.*

We illustrate using the Limit Comparison Test with the following example.

Example 4. Take $f(n) = 3n^2 + 5n + 7$, and take $g(n) = n^2$. We have that:

$$\begin{aligned}
&\lim_{n\to\infty} \frac{3n^2 + 5n + 7}{n^2} \\
&= \lim_{n\to\infty} \frac{3n^2}{n^2} + \lim_{n\to\infty} \frac{5n}{n^2} + \lim_{n\to\infty} \frac{7}{n^2} \\
&= \lim_{n\to\infty} 3 + \lim_{n\to\infty} \frac{5}{n} + \lim_{n\to\infty} \frac{7}{n^2} \\
&= 3 + 0 + 0 \\
&= 3.
\end{aligned}$$

As $0 < 3 < \infty$, $3n^2 + 5n + 7 \in \Theta(n^2)$.

In practice, $f(n)$ and $g(n)$ are often not (both) polynomials, and so the limit computations are not always as straight-forward. Other techniques are needed to evaluate (1). To this end, we recall L'Hopital's Rule and some useful convergence tests from the first year calculus sequence.

2 L'Hopital's Rule

Frequently, when computing (1), we will obtain the indeterminate forms $\pm\frac{\infty}{\infty}$ or $\frac{0}{0}$. L'Hopital's Rule provides a means to evaluate (1), operating under certain modest assumptions.

Theorem 2.1 (L'Hopital's Rule)**.** *Suppose $f(x)$ and $g(x)$ are differentiable functions, where either:*

(a) $\lim_{x\to\infty} f(x) = 0$ *and* $\lim_{x\to\infty} g(x) = 0$*; or*

(b) $\lim_{x\to\infty} f(x) = \pm\infty$ *and* $\lim_{x\to\infty} g(x) = \pm\infty$.

If $\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ *exists, then*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

We now consider an example of applying L'Hopital's Rule.

Example 5. Let $f(n) = n$, and let $g(n) = 2^n$. Consider:

$$\lim_{n\to\infty} \frac{n}{2^n},$$

which has the indeterminate form $\frac{\infty}{\infty}$. Note that both $f(n) = n$ and $g(n) = 2^n$ are differentiable. So we can apply L'Hopital's rule. We have that:

$$\lim_{n\to\infty} \frac{n}{2^n} = \lim_{n\to\infty} \frac{1}{\ln(2) \cdot 2^n} = 0.$$

As $\lim_{n\to\infty} \frac{n}{2^n} = 0$, the Limit Comparison Test provides that $f(n) = n \in \mathcal{O}(2^n)$.

Remark: In certain instances, L'Hopital's Rule may need to be applied multiple times. We provide an example to illustrate.

Example 6. Let $f(n) = n^2 + 3n + 5$, and let $g(n) = 3^n$. The limit

$$\lim_{n\to\infty} \frac{n^2 + 3n + 5}{3^n}$$

has the indeterminate form $\frac{\infty}{\infty}$. Furthermore, $f(n) = n^2 + 3n + 5$ and $g(n) = 3^n$ are both differentiable. In order to apply L'Hopital's rule, we must first establish that the following limit exists:

$$\lim_{n\to\infty} \frac{f'(n)}{g'(n)} = \lim_{n\to\infty} \frac{2n + 3}{\ln(3) \cdot 3^n}. \tag{2}$$

To do this, we seek to apply L'Hopital's Rule to (2). Note that (2) has the indeterminate form $\frac{\infty}{\infty}$. Furthermore, note that $f'(n) = 2n + 3$ and $g'(n) = \ln(3) \cdot 3^n$ are differentiable functions. Now:

$$\lim_{n \to \infty} \frac{f''(n)}{g''(n)} = \lim_{n \to \infty} \frac{2}{(\ln(3))^2 \cdot 3^n} = 0.$$

So by L'Hopital's Rule, we have that:

$$\lim_{n \to \infty} \frac{2n+3}{\ln(3) \cdot 3^n} = \lim_{n \to \infty} \frac{2}{(\ln(3))^2 \cdot 3^n} = 0.$$

Thus, we can apply L'Hopital's Rule to the original limit $\lim_{n \to \infty} \frac{n^2+3n+5}{3^n}$, to deduce that:

$$\begin{aligned}
&\lim_{n \to \infty} \frac{n^2+3n+5}{3^n} \\
&= \lim_{n \to \infty} \frac{2n+3}{\ln(3) \cdot 3^n} \\
&= \lim_{n \to \infty} \frac{2}{(\ln(3))^2 \cdot 3^n} = 0.
\end{aligned}$$

So by the Limit Comparison Test, $n^2 + 3n + 5 \in \mathcal{O}(3^n)$.

3 Convergence Tests

L'Hopital's Rule is useful in situations where both $f(n)$ and $g(n)$ are differentiable, and when differentiating $f(n)$ and $g(n)$ simplifies the limit. This is often not the case. For instance, suppose that one of our functions is $n!$, which is not differentiable or even continuous. Thus, L'Hopital's Rule cannot be applied to compute (1).

Example 7. We consider another example of when L'Hopital's rule is unhelpful in applying the Limit Comparison Test. Let $f(n) = 2^n$ and $g(n) = n^n$. We consider:

$$\lim_{n \to \infty} \frac{2^n}{n^n}, . \tag{3}$$

which has the indeterminate form $\frac{\infty}{\infty}$. Now $f(n) = 2^n$ and $g(n) = n^n$ are both differentiable, with derivatives $f'(n) = \ln(2) \cdot 2^n$ and $g'(n) = (\ln(n) + 1) \cdot n^n$.[1] Now in order to apply L'Hopital's Rule, it remains to show that the following limit exists.

$$\lim_{n \to \infty} \frac{f'(n)}{g'(n)} \tag{4}$$

$$= \lim_{n \to \infty} \frac{\ln(2) \cdot 2^n}{(\ln(n)+1) \cdot n^n} \tag{5}$$

$$= \ln(2) \cdot \lim_{n \to \infty} \frac{2^n}{(\ln(n)+1) \cdot n^n}. \tag{6}$$

Notice that the limit at (6) is not simpler to evaluate than the original limit at (3). For this reason, L'Hopital's Rule is ineffectual in helping us to apply the Limit Comparison Test.

In order to apply the Limit Comparison Test in cases such as in Example 7 or when one of our functions is not continuous, we use series convergence tests from calculus. The two most commonly used convergence tests in an algorithms course are the Ratio Test and the Root Test, which test whether a series of the form $\sum_{n=0}^{\infty} a_n$ converges. Note that if $\sum_{n=0}^{\infty} a_n$ converges, then $\lim_{n \to \infty} a_n = 0$. This is applied in asymptotic analysis in the following manner. Let $f(n)$ and $g(n)$ be functions. Let $k \in \mathbb{N}$. If the following series

$$\sum_{n=k}^{\infty} \frac{f(n)}{g(n)}$$

converges, then

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0.$$

By the Limit Comparison Test, we have that $f(n) \in \mathcal{O}(g(n))$.

[1] Recall that we use *logarithmic differentiation* to compute the derivative of n^n, and **not** the power or exponential rules.

3.1 Ratio Test

Theorem 3.1 (Ratio Test)**.** *Let $k \in \mathbb{N}$. Suppose that we have the series $\sum_{n=k}^{\infty} a_n$. Suppose that the following limit exists.*

$$L := \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

(a) *If $L < 1$, then the series is convergent. (In particular, the series converges absolutely).*

(b) *If $L > 1$, then the series diverges. In particular, if $L > 1$, then the sequence $(a_n)_{n \in \mathbb{N}}$ diverges to either ∞ or $-\infty$ as well. That is, $\lim_{n \to \infty} |a_n| = \infty$.*

(c) *If $L = 1$, the Ratio Test is inconclusive and another convergence test is needed.*

The Ratio Test works particularly well when functions have terms dealing with factorials or exponentials. In contrast, the Ratio Test is less helpful when dealing with polynomials. We provide some examples illustrating the Ratio Test.

Example 8. Let $f(n) = 2^n$ and $g(n) = n!$. Recall that $g(n) = n!$ is not even continuous, let alone differentiable. So we cannot use L'Hopital's Rule when applying the Limit Comparison Test. Rather, we appeal to the Ratio Test. Consider the following series:

$$\sum_{n=0}^{\infty} \frac{2^n}{n!}.$$

Here, $a_n = \frac{2^n}{n!}$. Now consider:

$$\begin{aligned} L &:= \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \\ &= \lim_{n \to \infty} \frac{2^{n+1} n!}{2^n (n+1)!} \\ &= \lim_{n \to \infty} \frac{2}{n+1} = 0. \end{aligned}$$

As $L = 0$, the Ratio Test tells us that our series converges. Thus,

$$\lim_{n \to \infty} \frac{2^n}{n!} = 0.$$

So $2^n \in \mathcal{O}(n!)$, as desired.

Example 9. Let $f(n) = n!$ and $g(n) = 5^n$. We apply the Ratio Test. Consider the following series:

$$\sum_{n=0}^{\infty} \frac{n!}{5^n}.$$

Here, $a_n = \frac{n!}{5^n}$. Now consider:

$$\begin{aligned} L &:= \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \\ &= \lim_{n \to \infty} \frac{(n+1)! \cdot 5^n}{n! \cdot 5^{n+1}} \\ &= \lim_{n \to \infty} \frac{n+1}{5} = \infty. \end{aligned}$$

So as $L = \infty$, the Ratio Test tells us that not only does our series diverge, but that:

$$\lim_{n \to \infty} \frac{n!}{5^n} = \infty,$$

as well. So by the Limit Comparison Test, we have that $n! \in \Omega(5^n)$.

3.2 Root Test

We conclude with a discussion of the Root Test.

Theorem 3.2 (Root Test)**.** *Let $k \in \mathbb{N}$. Suppose that we have the series $\sum_{n=k}^{\infty} a_n$. Suppose that the following limit exists.*

$$L := \lim_{n \to \infty} |a_n|^{1/n}.$$

(a) *If $L < 1$, then the series is convergent. (In particular, the series converges absolutely).*

(b) *If $L > 1$, then the series diverges. In particular, if $L > 1$, then the sequence $(a_n)_{n \in \mathbb{N}}$ diverges to either ∞ or $-\infty$ as well. That is, $\lim_{n \to \infty} |a_n| = \infty$.*

(c) *If $L = 1$, the Ratio Test is inconclusive and another convergence test is needed.*

Example 10. Take $f(n) = 2^n$ and $g(n) = n^n$. Recall from Example 7 that L'Hopital's Rule is insufficient to help us in applying the Limit Comparison Test. Rather, we appeal to the Root Test. Consider the following series:

$$\sum_{n=1}^{\infty} \frac{2^n}{n^n}.$$

So $a_n = \frac{2^n}{n^n}$. Now consider the following limit.

$$\begin{aligned} L &:= \lim_{n \to \infty} |a_n|^{1/n} \\ &= \lim_{n \to \infty} \left(\frac{2^n}{n^n} \right)^{1/n} \\ &= \lim_{n \to \infty} \frac{2}{n} = 0. \end{aligned}$$

As $L = 0$, our series converges. Thus, $\lim_{n \to \infty} \frac{2^n}{n^n} = 0$. So by the Limit Comparison Test, $2^n \in \mathcal{O}(n^n)$.