Derivation of the Time Dependent Gross-Pitaevskii Equation in Two Dimensions

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Abstract

We present a microscopic derivation of the defocusing two-dimensional cubic nonlinear Schrödinger equation as a mean field equation starting from an interacting N-particle system of Bosons. We consider the interaction potential to be given either by $W_\beta(x) = N^{-1+2\beta}W(N^\beta x)$, for any $\beta > 0$, or to be given by $V_N(x) = e^{2N}V(e^N x)$, for some spherical symmetric, nonnegative and compactly supported $W, V \in L^\infty(\mathbb{R}^2, \mathbb{R})$. In both cases we prove the convergence of the reduced density matrix corresponding to the exact time evolution to the projector onto the solution of the corresponding nonlinear Schrödinger equation in trace norm. For the latter potential V_N we show that it is crucial to take the microscopic structure of the condensate into account in order to obtain the correct dynamics.

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1 Introduction

This paper deals with the effective dynamics of a two dimensional condensate of N interacting bosons. Fundamentally, the evolution of the system is described by a time-dependent wave-function $\Psi_t \in L^2_s(\mathbb{R}^{2N},\mathbb{C}), \|\Psi_t\| = 1$ (Here and below norms without index $\|\cdot\|$ always denote the L^2-norm on the appropriate Hilbert space.). $L^2_s(\mathbb{R}^{2N},\mathbb{C})$ denotes the set of all $\Psi \in L^2(\mathbb{R}^{2N},\mathbb{C})$ which are symmetric under pairwise permutations of the variables $x_1,\dots,x_N \in \mathbb{R}^2$. Assuming that $\Psi_t \in H^2(\mathbb{R}^{2N},\mathbb{C})$ holds, Ψ_t then solves the N-particle Schrödinger equation

$$\mathrm{i}\partial_t\Psi_t = H_U\Psi_t \tag{1}$$

where the (non-relativistic) Hamiltonian $H_U : H^2(\mathbb{R}^{2N},\mathbb{C}) \to L^2(\mathbb{R}^{2N},\mathbb{C})$ is given by

$$H_U = -\sum_{j=1}^{N}\Delta_j + \sum_{1\leq j<k\leq N} U(x_j - x_k) + \sum_{j=1}^{N} A_t(x_j)\ . \tag{2}$$

In general, even for small particle numbers N, (1) cannot be solved neither exactly nor numerically for Ψ_t. Nevertheless, for a certain class of scaled potentials U and certain initial conditions Ψ_0 it is possible to derive an approximate solution of (1) in the trace class topology of reduced density matrices. The picture we have in mind is the description of a Bose-Einstein condensate. Initially one starts with the ground state of a trapped, dilute gas and then removes or changes the trap subsequently. In this paper, we will consider two choices for the interaction potential U.

- Let $U(x) = V_N(x) = e^{2N}V(e^N x)$ for a compactly supported, spherically symmetric and nonnegative potential $V \in L^\infty_c(\mathbb{R}^2,\mathbb{R})$. Below, the exponential scaling of V_N will be explained in detail. Note that, in contrast to existing dynamical mean-field results, $\|V_N\|_1 = \mathcal{O}(1)$ does not decay like $1/N$.
- Let, for any fixed $\beta > 0$, $U(x) = W_\beta(x) = N^{-1+2\beta}W(N^\beta x)$ for a compactly supported, spherically symmetric and nonnegative potential $W \in L^\infty_c(\mathbb{R}^2,\mathbb{R})$. This scaling can be motivated by formally imposing that the total potential energy is of the same order as the total kinetic energy, namely of order N, if Ψ_0 is close to the ground state.

Define the one particle reduced density matrix $\gamma^{(1)}_{\Psi_0}$ of Ψ_0 with integral kernel

$$\gamma^{(1)}_{\Psi_0}(x,x') = \int_{\mathbb{R}^{2N-2}} \Psi_0^*(x,x_2,\dots,x_N)\Psi_0(x',x_2,\dots,x_N)d^2x_2\dots d^2x_N\ .$$

To account for the physical situation of a Bose-Einstein condensate, we assume complete condensation in the limit of large particle number N. This amounts to assume that, for $N\to\infty$, $\gamma^{(1)}_{\Psi_0} \to |\varphi_0\rangle\langle\varphi_0|$ in trace norm for some $\varphi_0 \in L^2(\mathbb{R}^2,\mathbb{C}), \|\varphi_0\| = 1$. Our main goal is to show the persistence of condensation over time. This is of particular interest in experiments if one switches off the trapping potential A_t and monitors the expansion of the condensate. We prove that the time evolved reduced density matrix $\gamma^{(1)}_{\Psi_t}$ converges to $|\varphi_t\rangle\langle\varphi_t|$ in trace norm as $N\to\infty$ with convergence rate of order $N^{-\eta}$ for some $\eta > 0$. φ_t then solves the nonlinear Schrödinger equation

$$\mathrm{i}\partial_t\varphi_t = (-\Delta + A_t)\,\varphi_t + b_U|\varphi_t|^2\varphi_t =: h^{GP}_{b_U}\varphi_t \tag{3}$$

with initial datum φ_0. Depending on the interaction potential U, we obtain different coupling constants b_U. For $U = W_\beta$, we obtain $b_{W_\beta} = N\|W_\beta\|_1 = \|W\|_1$. This result is already expected from a heuristic law of large numbers argument, see below. In the case $U = V_N$, we have $b_{V_N} = 4\pi$. We like to remark that it is well known that convergence of $\gamma^{(1)}_{\Psi_t}$ to $|\varphi_t\rangle\langle\varphi_t|$ in trace norm is equivalent to the respective convergence in operator norm since $|\varphi_t\rangle\langle\varphi_t|$ is a rank-1-projection, see Remark 1.4. in [33]. Furthermore, the convergence of the one-particle reduced density matrix $\gamma^{(1)}_{\Psi_t} \to |\varphi_t\rangle\langle\varphi_t|$ in trace norm implies convergence of any k-particle reduced density matrix $\gamma^{(k)}_{\Psi_t}$ against $|\varphi_t^{\otimes k}\rangle\langle\varphi_t^{\otimes k}|$ in trace norm as $N \to \infty$ and k fixed, see for example [17].

In the case that the time evolution of Ψ_t is generated by H_{V_N} it is interesting to note that the effective evolution equation of φ_t does not depend on the scattering length a. This contrasts the three dimensional case, where the correct mean field coupling is given by $8\pi a_{3D}$, a_{3D} denoting the scattering length of the potential in three dimensions. The universal coupling 4π in the case of a positive scattering length is known within the physical literature, see e.g. (30) and (A3) in [7] (note that $\hbar = 1, m = \frac{1}{2}$ in our choice of coordinates). Actually, our dynamical result complements a more general theory describing the ground state properties of dilute Bose gases. It was shown in [23] that for such a gas with repulsive interaction $V \geq 0$, the ground state energy per particle is to leading order given by either the Gross-Pitaevskii energy functional with coupling parameter $8\pi/|\ln(\overline{\rho}a^2)|$ or a Thomas-Fermi type functional. Here, $\overline{\rho}$ denotes the mean density of the gas, see Equation (1.6) in [23] for a precise definition. The authors prove further that only if $N/|\ln(\overline{\rho}a^2)| = \mathcal{O}(1)$ holds, one obtains the Gross-Pitaevskii regime. This directly implies that scattering length of the interaction potential needs to have an exponential decrease in N. In our case, the scattering length of the potential V_N is given by ae^{-N}, a denoting the scattering length of V. The mean density of the system we consider is of order one, i.e. $\overline{\rho} = \mathcal{O}(1)$. This yields $8\pi N/|\ln(\overline{\rho}(e^{-N}a)^2)| \approx 4\pi$ which is in agreement with our findings. It should be pointed out that there has been some debate about the question whether two dimensional Bose-Einstein condensation can be observed experimentally. This amounts to the question whether condensation takes place for temperatures $T > 0$. For an ideal, noninteracting gas in box, the standard grand canonical computation for the critical temperature T_c of a Bose-Einstein condensate shows that there is no condensation for $T > 0$. For trapped, noninteracting Bosons in a confining power-law potential, the findings in [1] however show that in that case $T_c > 0$ holds. Finally, it was proven in [20] that $\gamma^{(1)}_{\Psi}$ converges to $|\varphi\rangle\langle\varphi|$ in trace norm if Ψ the ground state of H_{V_N} and φ is the ground state of the Gross-Pitaevskii energy functional, see (5). The assumptions made in the paper are that and the external potential A tends to $+\infty$ as $|x| \to \infty$ and the interaction potential V is nonnegative. It is also remarked that one does not observe 100 % condensation in the ground state of a interacting homogenous system. The emergence of 100 % Bose-Einstein condensation as a ground state phenomena thus highly depends on the particular physical system one considers. Our approach is the following: Initially, we assume the convergence of $\gamma^{(1)}_{\Psi_0}$ to $|\varphi_0\rangle\langle\varphi_0|$. We then show the persistence this condensation for time scales of order one. Our assumption is thus in agreement with the findings in [20]. We like to remark that the two dimensional Thomas-Fermi regime could be observed experimentally [12].

Next, we want to explain how the different coupling constants b_U are obtained in the dynamical setting. For this, we first recall known results from the three dimensional Bose gas. There, one considers the interaction potential to be given by $V_\beta(x) = N^{-1+3\beta}V(N^\beta x)$ for $0 \leq \beta \leq 1$.

For $0 < \beta < 1$, one obtains the cubic nonlinear Schrödinger equation with coupling constant $\|V\|_1$. This can be seen as a singular mean-field limit, where the full interaction is replaced by its corresponding mean value $\int_{\mathbb{R}^3} d^3y N^{3\beta} V(N^\beta(x-y))|\varphi_t(y)|^2 \to \|V\|_1 |\varphi_t(x)|^2$. For $\beta = 1$, however, the system develops correlations between the particles which cannot be neglected. As already mentioned, the correct mean field coupling is then given by $8\pi a_{3D}$. This is different for a two dimensional condensate. Let us first explain, why the short scale correlation structure is negligible if the potential is given by $W_\beta(x) = N^{-1+2\beta} W(N^\beta x)$ for any $\beta > 0$. Assuming that the energy of Ψ_t is comparable to the ground state energy, the wave function will develop short scale correlations between the particles. One may heuristically think of Ψ_t of Jastrow-type, i.e. $\Psi_t(x_1, \dots, x_N) \approx \prod_{i<j} F(x_i - x_j) \prod_{k=1}^N \varphi_t(x_k)$ [1]. The function F accounts for the pair correlations between the particles at short scales of order $N^{-\beta}$. It is well known that the correlation function F should be described by the zero energy scattering state $j_{N,R}$ of the potential W_β, where $j_{N,R}$ satisfies

$$\begin{cases} \left(-\Delta_x + \frac{1}{2} W_\beta(x)\right) j_{N,R}(x) = 0, \\ j_{N,R}(x) = 1 \text{ for } |x| = R. \end{cases}$$

Here, the boundary radius R is chosen of order $N^{-\beta}$. That is, $F(x_i - x_j) \approx j_{N,R}(x_i - x_j)$ for $|x_i - x_j| = \mathcal{O}(N^{-\beta})$ and $F(x_i - x_j) = 1$ for $|x_i - x_j| \gg \mathcal{O}(N^{-\beta})$. Rescaling to coordinates $y = N^\beta x$, the zero energy scattering state satisfies

$$\left(-\Delta_y + \frac{1}{2} N^{-1} W(y)\right) j_{N,N^\beta R}(y) = 0 \, . \tag{4}$$

Due to the factor N^{-1} in front of W, the zero energy scattering equation is almost constant, that is $j_{N,R}(x) \approx 1$, for all $|x| \leq R$. As a consequence, the microscopic structure F, induced by the zero energy scattering state, vanishes for any $\beta > 0$ and does not effect the dynamics of the reduced density matrix $\gamma^{(1)}_{\Psi_t}$. Assuming $\gamma^{(1)}_{\Psi_0} \approx |\varphi_0\rangle\langle\varphi_0|$, one may thus apply a law of large numbers argument and conclude that the interaction on each particle is then approximately given by its mean value

$$\int_{\mathbb{R}^2} d^2y N W_\beta(x-y) |\varphi_t|^2(y) \to \|W\|_1 |\varphi_t|^2(x) \, .$$

This yields to the correct coupling in the effective equation (3) in the case $U(x) = W_\beta(x)$.
Let us now consider the case for which the dynamics of Ψ_t is generated by the Hamiltonian H_{V_N}. If one would guess the effective coupling of φ_t to be also given by its mean value w.r.t. the distribution $|\varphi_t|^2$, one would end up with the N-dependent equation $\mathrm{i}\partial_t \varphi_t = (-\Delta + A_t)\varphi_t + N \int_{\mathbb{R}^2} d^2x V(x) |\varphi_t|^2 \varphi_t$. Note that the coupling constant of the self interaction differs from its correct value by a factor of $\mathcal{O}(N)$. As in the three dimensional Gross-Pitaevskii regime $\beta = 1$, it is now important to take the correlations explicitly into account. The scaling of the potential yields to $j_{N,R}(x) = j_{0,e^N R}(e^N x)$, which implies that the correlation function will influence the dynamics whenever two particles collide. The coupling parameter can then be inferred from

[1] One should however note that Ψ_t will not be close to a full product $\prod_{k=1}^N \varphi_t(x_k)$ in norm. For certain types of interactions, it has been shown rigorously that Ψ_t can be approximated by a quasifree state satisfying a Bogoloubov-type dynamics, see [3], [26], [27] and [28] for precise statements.

the relation

$$\int_{\mathbb{R}^2} d^2x V_N(x) j_{N,R}(x) = \frac{4\pi}{\ln\left(\frac{R}{ae^{-N}}\right)},$$

where a denotes the scattering length of the potential V. As mentioned, the logarithmic dependence of the integral above on a is special in two dimensions. Since $\frac{4\pi}{\ln\left(\frac{R}{ae^{-N}}\right)} \approx \frac{4\pi}{N}$ holds for $a > 0$, the effective equation for φ_t will not depend on a anymore. Consequently, one obtains as an effective coupling

$$\int_{\mathbb{R}^2} d^2y N V_N(x-y) j_{N,R}(x-y) |\varphi_t|^2(y) \to 4\pi |\varphi_t|^2(x).$$

We like to remark that it is easy to verify that, for any $s > 0$, the potential $V_{sN}(x) = e^{2Ns}V(e^{Ns}x)$ yields to an effective coupling $4\pi/s$. For the sake of simplicity, we will not consider this slight generalization, although our proof is also valid in this case.

The rigorous derivation of effective evolution equations is well known in the literature, see e.g. [3, 4, 8, 9, 10, 11, 17, 26, 27, 30, 31, 32, 33] and references therein. For the two-dimensional case we consider, it has been proven, for $0 < \beta < 3/4$ and W nonnegative, that $\gamma^{(1)}_{\Psi_t}$ converges to $|\varphi_t\rangle\langle\varphi_t|$ as $N \to \infty$ [13]. For $0 < \beta < 1/6$, it has been established in [6] that the reduced density matrices converge, assuming that the potential W is attractive, i.e. $W \leq 0$. This result was later extended to $0 < \beta < 3/4$, using stability properties of the ground state energy [18].

Another approach which relates more closely to the experimental setup is to consider a three-dimensional gas of Bosons which is strongly confined in one spatial dimension. Then, one obtains an effective two dimensional system in the unconfined directions. We remark that in this dimensional reduction two limits appear, the length scale in the confined direction and the scaling of the interaction in the unconfined directions. Results in this direction can be found in [2] and [14], see also [15]. It is still an open problem to derive our dynamical result starting from a strongly confined three dimensional system. For known results regarding the ground state properties of dilute Bose gases, we refer to the monograph [22], which also summarizes the papers [20], [23] and[24].

Our proof is based on [31], where the emergence of the Gross-Pitaevskii equation was proven by one of us (P.P.) in three dimensions for $\beta = 1$. In particular, we adapt some crucial ideas which allow us to control the microscopic structure of Ψ_t.

We shall shortly discuss the physical relevance of the different scalings. On the first view, the interactions discussed above do look rather unphysical. It is questionable to assume that the coupling constant and/or the range of the interaction change as the particle number increases. Nevertheless, one can think of situations, where for example the support of the interaction is small and the particle number of the system is adjusted accordingly.

The exponential scaling $V_N(x) = e^{2N}V(e^N x)$ is special. In this case it is possible to rescale space- and time-coordinates in such a way that in the new coordinates the interaction is *not* N dependent. Choosing $y = e^N x$ and $\tau = e^{2N}t$ the Schrödinger equation reads

$$\mathrm{i}\frac{d}{d\tau}\Psi_{e^{-2N}\tau} = \left(-\sum_{j=1}^N \Delta_{y_j} + \sum_{1\leq j<k\leq N} V(y_j - y_k) + \sum_{j=1}^N A_{e^{-2N}\tau}(e^{-N}y_j)\right)\Psi_{e^{-2N}\tau}\ .$$

The latter equation thus corresponds to an extremely dilute gas of bosons with density $\sim e^{-2N}$. In order to observe a nontrivial dynamics, this condensate is then monitored over time scales of order $\tau \sim e^{2N}$. Since the trapping potential is adjusted according to the density of the gas in the experiment, the N dependence of $A_{e^{-2N}\tau}(e^{-N}\cdot)$ is reasonable.

2 Main result

We will bound expressions which are uniformly bounded in N and t by some constant C. Constants appearing in a sequence of estimates will not be distinguished, i.e. in $X \leq CY \leq CZ$ the constants may differ.
For $U \in \{W_\beta, V_N\}$, define the energy functional $\mathcal{E}_U : H^2(\mathbb{R}^{2N}, \mathbb{C}) \to \mathbb{R}$

$$\mathcal{E}_U(\Psi) = N^{-1}\langle\!\langle \Psi, H_U \Psi \rangle\!\rangle ,$$

where $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ denotes the scalar product on $L^2(\mathbb{R}^{2N}, \mathbb{C})$. Furthermore, define the Gross-Pitaevskii energy functional $\mathcal{E}^{GP}_{b_U} : H^2(\mathbb{R}^2, \mathbb{C}) \to \mathbb{R}$

$$\mathcal{E}^{GP}_{b_U}(\varphi) = \langle \nabla\varphi, \nabla\varphi \rangle + \langle \varphi, (A_t + \frac{1}{2}b_U|\varphi|^2)\varphi \rangle = \langle \varphi, (h^{GP}_{b_U} - \frac{1}{2}b_U|\varphi|^2)\varphi \rangle \tag{5}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product on $L^2(\mathbb{R}^2, \mathbb{C})$. Note that both $\mathcal{E}_U(\Psi)$ and $\mathcal{E}^{GP}_{b_U}(\varphi)$ depend on t, due to the time varying external potential A_t. For the sake of readability, we will not indicate this time dependence explicitly. We now state our main Theorem:

Theorem 2.1 *Let $\Psi_0 \in L^2_s(\mathbb{R}^{2N}, \mathbb{C}) \cap H^2(\mathbb{R}^{2N}, \mathbb{C})$ with $\|\Psi_0\| = 1$. Let $\varphi_0 \in L^2(\mathbb{R}^2, \mathbb{C})$ with $\|\varphi_0\| = 1$ and assume $\lim_{N\to\infty} \gamma^{(1)}_{\Psi_0} = |\varphi_0\rangle\langle\varphi_0|$ in trace norm. Let the external potential A_t, which is defined in* (2)*, statisfy $A_t \in L^\infty(\mathbb{R}^2, \mathbb{R})$, $\dot{A}_t \in L^\infty(\mathbb{R}^2, \mathbb{R})$, for all $t \in \mathbb{R}$.*

(a) For any $\beta > 0$, let W_β be given by $W_\beta(x) = N^{-1+2\beta}W(N^\beta x)$, for $W \in L^\infty_c(\mathbb{R}^2, \mathbb{R})$, $W \geq 0$ and W spherically symmetric. Let Ψ_t the unique solution to $i\partial_t \Psi_t = H_{W_\beta}\Psi_t$ with initial datum Ψ_0. Let φ_t the unique solution to $i\partial_t\varphi_t = h^{GP}_{\|W\|_1}\varphi_t$ with initial datum φ_0 and assume that $\varphi_t \in H^3(\mathbb{R}^2, \mathbb{C})$. Let $\lim_{N\to\infty}\left(\mathcal{E}_{W_\beta}(\Psi_0) - \mathcal{E}^{GP}_{\|W\|_1}(\varphi_0)\right) = 0$. Then, for any $\beta > 0$ and for any $t > 0$

$$\lim_{N\to\infty} \gamma^{(1)}_{\Psi_t} = |\varphi_t\rangle\langle\varphi_t| \tag{6}$$

in trace norm.

(b) Let V_N be given by $V_N(x) = e^{2N}V(e^N x)$, for $V \neq 0; V \in L^\infty_c(\mathbb{R}^2, \mathbb{R})$, $V \geq 0$ and V spherically symmetric. Let Ψ_t the unique solution to $i\partial_t\Psi_t = H_{V_N}\Psi_t$ with initial datum Ψ_0. Let φ_t the unique solution to $i\partial_t\varphi_t = h^{GP}_{4\pi}\varphi_t$ with initial datum φ_0 and assume that $\varphi_t \in H^3(\mathbb{R}^2, \mathbb{C})$. Let $\lim_{N\to\infty}\left(\mathcal{E}_{V_N}(\Psi_0) - \mathcal{E}^{GP}_{4\pi}(\varphi_0)\right) = 0$.
Then, for any $t > 0$

$$\lim_{N\to\infty} \gamma^{(1)}_{\Psi_t} = |\varphi_t\rangle\langle\varphi_t| \tag{7}$$

in trace norm.

Remark:

(a) We expect that for regular enough external potentials A_t, the regularity assumption $\varphi_t \in H^3(\mathbb{R}^2,\mathbb{C})$ to follow from regularity assumptions on the initial datum φ_0. In particular, if $\varphi_0 \in \Sigma^3(\mathbb{R}^2,\mathbb{C}) = \{f \in L^2(\mathbb{R}^2,\mathbb{C}) | \sum_{\alpha+\beta\leq 3} \|x^\alpha \partial_x^\beta f\| < \infty\}$ holds, the bound $\|\varphi_t\|_{H^3} < \infty$ has been proven for external potentials which are at most quadratic in space, see [5] and Lemma 4.7. In particular, for $\varphi_0 \in \Sigma^3(\mathbb{R}^2,\mathbb{C})$, the bound $\|\varphi_t\|_{H^3} \leq C$ holds if the external potential is not present, i.e. $A_t = 0$.

(b) As already mentioned, the convergence of $\gamma^{(1)}_{\Psi_t}$ to $|\varphi_t\rangle\langle\varphi_t|$ in trace norm is equivalent to convergence in operator norm, since $|\varphi_t\rangle\langle\varphi_t|$ is a rank one projection [33]. Other equivalent definitions of asymptotic 100% condensation can be found in [25].

(c) In our proof we will give explicit error estimates in terms of the particle number N. We shall show that the rate of convergence is of order $N^{-\delta}$ for some $\delta > 0$, assuming that also initially $\gamma^{(1)}_{\Psi_0} \to |\varphi_0\rangle\langle\varphi_0|$ converges in trace norm with rate of at least $N^{-\delta}$.

(d) One can relax the conditions on the initial condition and only require $\Psi_0 \in L^2_s(\mathbb{R}^{2N},\mathbb{C})$ using a standard density argument.

(e) It has been shown that in the limit $N \to \infty$ the energy-difference $\mathcal{E}_{V_N}(\Psi^{gs}) - \mathcal{E}^{GP}_{4\pi}(\varphi^{gs}) \to 0$, where Ψ^{gs} is the ground state of a trapped Bose gas and φ^{gs} the ground state of the respective Gross-Pitaevskii energy functional, see [23], [24].

3 Organization of the proof

The method we use in this paper is introduced in detail in [32] and was generalized to derive various mean-field equations. As we have mentioned, our proof is based on [31], which covers the three-dimensional counterpart of our system. Heuristically speaking, the method we are going to employ is based on the idea of counting for each time t the relative number of those particles which are not in the state φ_t. It is then possible to show that the rate of particles which leave the condensate is small, if initially almost all particles are in the state φ_0. In order to compare the exact dynamic, generated by H_U, with the effective dynamic, generated by $h^{GP}_{b_U}$, we define the projectors p^φ_j and q^φ_j.

Definition 3.1 *Let $\varphi \in L^2(\mathbb{R}^2,\mathbb{C})$ with $\|\varphi\| = 1$.*

(a) For any $1 \leq j \leq N$ the projectors $p^\varphi_j : L^2(\mathbb{R}^{2N},\mathbb{C}) \to L^2(\mathbb{R}^{2N},\mathbb{C})$ and $q^\varphi_j : L^2(\mathbb{R}^{2N},\mathbb{C}) \to L^2(\mathbb{R}^{2N},\mathbb{C})$ are defined as

$$p^\varphi_j \Psi = \varphi(x_j) \int \varphi^*(\tilde{x}_j)\Psi(x_1,\dots,\tilde{x}_j,\dots,x_N)d^2\tilde{x}_j \quad \forall\, \Psi \in L^2(\mathbb{R}^{2N},\mathbb{C})$$

and $q^\varphi_j = 1 - p^\varphi_j$.
We shall also use, with a slight abuse of notation, the bra-ket notation $p^\varphi_j = |\varphi(x_j)\rangle\langle\varphi(x_j)|$.

(b) For any $0 \leq k \leq N$ we define the set

$$\mathcal{S}_k = \left\{ (s_1, s_2, \dots, s_N) \in \{0,1\}^N \; ; \; \sum_{j=1}^N s_j = k \right\}$$

and the orthogonal projector $P_k^\varphi : L^2(\mathbb{R}^{2N}, \mathbb{C}) \to L^2(\mathbb{R}^{2N}, \mathbb{C})$ *as*

$$P_k^\varphi = \sum_{\vec{a} \in \mathcal{S}_k} \prod_{j=1}^{N} \left(p_j^\varphi\right)^{1-s_j} \left(q_j^\varphi\right)^{s_j} .$$

For negative k *and* $k > N$ *we set* $P_k^\varphi = 0$*.*

(c) *For any function* $m : \mathbb{N}_0 \to \mathbb{R}_0^+$ *we define the operator* $\widehat{m}^\varphi : L^2(\mathbb{R}^{2N}, \mathbb{C}) \to L^2(\mathbb{R}^{2N}, \mathbb{C})$ *as*

$$\widehat{m}^\varphi = \sum_{j=0}^{N} m(j) P_j^\varphi . \tag{8}$$

We also need the shifted operators $\widehat{m}_d^\varphi : L^2(\mathbb{R}^{2N}, \mathbb{C}) \to L^2(\mathbb{R}^{2N}, \mathbb{C})$ *given by*

$$\widehat{m}_d^\varphi = \sum_{j=-d}^{N-d} m(j+d) P_j^\varphi .$$

Following a general strategy, which is described in detail in [32], we define a functional $\alpha : L^2(\mathbb{R}^{2N}, \mathbb{C}) \times L^2(\mathbb{R}^2, \mathbb{C}) \to \mathbb{R}_0^+$ such that

(a) $\frac{\mathrm{d}}{\mathrm{d}t}\alpha(\Psi_t, \varphi_t)$ can be estimated by $\alpha(\Psi_t, \varphi_t) + \mathcal{O}(1)$. Using a Grönwall type estimate, it then follows that $\alpha(\Psi_t, \varphi_t) \leq C_1 e^{C_2 t}(\alpha(\Psi_0, \varphi_0) + \mathcal{O}(1))$, for some constants $C_1, C_2 > 0$.

(b) $\alpha(\Psi, \varphi) \to 0$ implies convergence of the reduced one particle density matrix of Ψ to $|\varphi\rangle\langle\varphi|$ in trace norm.

In the case $\beta = 0$ it was shown that the choice

$$\alpha(\Psi, \varphi) = \left\langle\!\!\left\langle \Psi, (\widehat{n}^\varphi)^j \, \Psi \right\rangle\!\!\right\rangle ,$$

where $n(k) = \sqrt{k/N}$ and $\langle\!\langle \cdot \rangle\!\rangle$ is scalar product on $L^2(\mathbb{R}^{2N}, \mathbb{C})$ fulfills these requirements, for arbitrary $j > 0$, see for example [32] and [17]. For the more involved scaling we consider, it is however necessary to adjust this definition in order to obtain a Grönwall type estimate.
Our proof is organized as follows:

(a) In Section 4 we recall some important properties of the operator $\widehat{m}$.

(b) For the most difficult scaling given by the potential V_N, it is crucial to take the interaction-induced correlations between the particles into account. In Section 5 we provide some estimates on the zero-energy scattering state. Furthermore, we explain how the effective coupling parameter b_{V_N} can be inferred from the microscopic structure.

(c) In Section 6 we prove our main Theorem stated above. We first consider the potential W_β and define a counting measure which allows us to establish a Grönwall estimate for all $\beta > 0$. We will explain in detail how one arrives at this Grönwall estimate.

    Afterwards, the counting measure is adjusted to the case V_N, taking the microscopic structure $j_{N,R}$ of the wave function into account. We then establish a Grönwall estimate and finally prove the Theorem for V_N.

    The needed estimates in Section 6 are then proven in Section 7.

4 Preliminaries

We will first fix the notation we are going to employ during the rest of the paper.

Notation 4.1 *(a) Throughout the paper hats* $\widehat{\cdot}$ *will always be used in the sense of Definition 3.1 (c). The label* n *will always be used for the function* $n(k) = \sqrt{k/N}$.

(b) For better readability, we will often omit the upper index φ *on* p_j, q_j, P_j, $P_{j,k}$ *and* $\widehat{\cdot}$. *It will be placed exclusively formulas where the* φ*-dependence is crucial.*

(c) The operator norm, defined for any linear operator $f : L^2(\mathbb{R}^{2N}, \mathbb{C}) \to L^2(\mathbb{R}^{2N}, \mathbb{C})$, *will be denoted by*

$$\|f\|_{op} = \sup_{\psi \in L^2(\mathbb{R}^{2N},\mathbb{C}), \|\Psi\|=1} \|f\Psi\| \ .$$

(d) We will denote by $\mathcal{K}(\varphi_t, A_t)$ *a generic polynomial with finite degree in* $\|\varphi_t\|_\infty, \|\nabla\varphi_t\|_\infty, \|\nabla\varphi_t\|, \|\Delta\varphi_t\|, \|A_t\|_\infty, \int_0^t ds \|\dot{A}_s\|_\infty$ *and* $\|\dot{A}_t\|_\infty$. *Note, in particular, that for a generic constant* C *the inequality* $C \leq \mathcal{K}(\varphi_t, A_t)$ *holds. The exact form of* $\mathcal{K}(\varphi_t, A_t)$ *which appears in the final bounds can be reconstructed, collecting all contributions from the different estimates.*

(e) We will denote for any multiplication operator $F : L^2(\mathbb{R}^2, \mathbb{C}) \to L^2(\mathbb{R}^2, \mathbb{C})$ *the corresponding operator*

$$\mathbb{1}^{\otimes(k-1)} \otimes F \otimes \mathbb{1}^{\otimes(N-k)} : L^2(\mathbb{R}^{2N}, \mathbb{C}) \to L^2(\mathbb{R}^{2N}, \mathbb{C})$$

acting on the N*-particle Hilbert space by* $F(x_k)$. *In particular, we will use, for any* $\Psi, \Omega \in L^2(\mathbb{R}^{2N}, \mathbb{C})$ *the notation*

$$\langle\!\langle \Omega, \mathbb{1}^{\otimes(k-1)} \otimes F \otimes \mathbb{1}^{\otimes(N-k)}\Psi \rangle\!\rangle = \langle\!\langle \Omega, F(x_k)\Psi \rangle\!\rangle \ .$$

In analogy, for any two-particle multiplication operator $K : L^2(\mathbb{R}^2, \mathbb{C})^{\otimes 2} \to L^2(\mathbb{R}^2, \mathbb{C})^{\otimes 2}$, *we denote the operator acting on any* $\Psi \in L^2(\mathbb{R}^{2N}, \mathbb{C})$ *by multiplication in the variable* x_i *and* x_j *by* $K(x_i, x_j)$. *In particular, we denote*

$$\langle\!\langle \Omega, K(x_i, x_j)\Psi \rangle\!\rangle = \int_{\mathbb{R}^{2N}} K(x_i, x_j)\Omega^*(x_1, \ldots, x_N)\Psi(x_1, \ldots, x_N) d^2x_1 \ldots d^2x_N \ .$$

First we prove some properties of the projectors p_j, q_j, which were defined in Definition 3.1.

Lemma 4.2 *(a) For any weights* $m, r : \mathbb{N}_0 \to \mathbb{R}_0^+$ *the commutation relations*

$$\widehat{m}\widehat{r} = \widehat{mr} = \widehat{r}\,\widehat{m} \qquad \widehat{m}p_j = p_j\widehat{m} \qquad \widehat{m}q_j = q_j\widehat{m} \qquad \widehat{m}P_k = P_k\widehat{m}$$

hold.

(b) Let $n : \mathbb{N}_0 \to \mathbb{R}_0^+$ *be given by* $n(k) = \sqrt{k/N}$. *Then, the square of* $\widehat{n}$ *equals the relative particle number operator of particles not in the state* φ, *i.e.*

$$(\widehat{n})^2 = N^{-1} \sum_{j=1}^{N} q_j \ . \tag{9}$$

(c) *For any weight* $m : \mathbb{N}_0 \to \mathbb{R}_0^+$ *and any function* $f \in L^\infty\left(\mathbb{R}^4, \mathbb{R}\right)$ *and any* $j,k = 0,1,2$

$$\widehat{m} Q_j f(x_1,x_2) Q_k = Q_j f(x_1,x_2) \widehat{m}_{j-k} Q_k \ ,$$

where $Q_0 = p_1p_2$, $Q_1 \in \{p_1q_2, q_1p_2\}$ *and* $Q_2 = q_1q_2$. *Furthermore, for* $j,k \in \{0,1\}$

$$\widehat{m} \widetilde{Q}_j \nabla_1 \widetilde{Q}_k = \widetilde{Q}_j \nabla_1 \widehat{m}_{j-k} \widetilde{Q}_k \ ,$$

where $\widetilde{Q}_0 = p_1$ *and* $\widetilde{Q}_1 = q_1$.

(d) *For any weight* $m : \mathbb{N}_0 \to \mathbb{R}_0^+$ *and any function* $f \in L^\infty\left(\mathbb{R}^4, \mathbb{C}\right)$

$$[f(x_1,x_2), \widehat{m}] = [f(x_1,x_2), p_1p_2(\widehat{m} - \widehat{m}_2) + (p_1q_2 + q_1p_2)(\widehat{m} - \widehat{m}_1)] \ .$$

(e) *Let* $f \in L^1\left(\mathbb{R}^2, \mathbb{C}\right)$, $g \in L^2\left(\mathbb{R}^2, \mathbb{C}\right)$. *Then,*

$$\|p_j f(x_j - x_k) p_j\|_{op} \leq \|f\|_1 \|\varphi\|_\infty^2 \ , \tag{10}$$

$$\|p_j g^*(x_j - x_k)\|_{op} = \|g(x_j - x_k) p_j\|_{op} \leq \|g\| \ \|\varphi\|_\infty \tag{11}$$

$$\||\varphi(x_j)\rangle\langle\nabla_j\varphi(x_j)| h^*(x_j - x_k)\|_{op} = \|h(x_j - x_k)\nabla_j p_j\|_{op} \leq \|h\| \|\nabla\varphi\|_\infty \ . \tag{12}$$

Proof:

(a) follows immediately from Definition 3.1, using that p_j and q_j are orthogonal projectors.

(b) Note that $\cup_{k=0}^N \mathcal{S}_k = \{0,1\}^N$, so $1 = \sum_{k=0}^N P_k$. Using also $(q_j)^2 = q_j$ and $q_jp_j = 0$ we get

$$\sum_{j=1}^N q_j = \sum_{j=1}^N q_j \sum_{k=0}^N P_k = \sum_{k=0}^N \sum_{j=1}^N q_j P_k = \sum_{k=0}^N k P_k = N\widehat{n^2} = N\widehat{n}^2 \ .$$

(c) Using the definitions above we have

$$\widehat{m} Q_j f(x_1,x_2) Q_k = \sum_{l=0}^N m(l) P_l Q_j f(x_1,x_2) Q_k \ .$$

The number of projectors q_j in P_lQ_j in the coordinates $j = 3, \ldots, N$ is equal to $l - j$. The p_j and q_j with $j = 3, \ldots, N$ commute with $Q_j f(x_1,x_2) Q_k$. Thus $P_l Q_j f(x_1,x_2) Q_k = Q_j f(x_1,x_2) Q_k P_{l-j+k}$ and

$$\begin{aligned}\widehat{m} Q_j f(x_1,x_2) Q_k &= \sum_{l=0}^N m(l) Q_j f(x_1,x_2) Q_k P_{l-j+k} \\ &= \sum_{\widetilde{l}=k-j}^{N+k-j} Q_j f(x_1,x_2) m(\widetilde{l} + j - k) P_{\widetilde{l}} Q_k = Q_j f(x_1,x_2) \widehat{m}_{j-k} Q_k \ .\end{aligned}$$

Similarly one gets the second formula.

(d) First note that

$$
\begin{aligned}
&[f(x_1,x_2),\widehat{m}]-[f(x_1,x_2),p_1p_2(\widehat{m}-\widehat{m}_2)+p_1q_2(\widehat{m}-\widehat{m}_1)+q_1p_2(\widehat{m}-\widehat{m}_1)]\\
&\quad=[f(x_1,x_2),q_1q_2\widehat{m}]+[f(x_1,x_2),p_1p_2\widehat{m}_2+p_1q_2\widehat{m}_1+q_1p_2\widehat{m}_1]\ .
\end{aligned}
\tag{13}
$$

We will show that the right hand side is zero. Multiplying the right hand side with p_1p_2 from the left and using (c) one gets

$$
\begin{aligned}
&p_1p_2f(x_1,x_2)q_1q_2\widehat{m}+p_1p_2f(x_1,x_2)p_1p_2\widehat{m}_2-p_1p_2\widehat{m}_2f(x_1,x_2)\\
&\quad+p_1p_2f(x_1,x_2)p_1q_2\widehat{m}_1+p_1p_2f(x_1,x_2)q_1p_2\widehat{m}_1\\
=&p_1p_2\widehat{m}_2f(x_1,x_2)q_1q_2+p_1p_2\widehat{m}_2f(x_1,x_2)p_1p_2-p_1p_2\widehat{m}_2f(x_1,x_2)\\
&\quad+p_1p_2\widehat{m}_2f(x_1,x_2)p_1q_2+p_1p_2\widehat{m}_2f(x_1,x_2)q_1p_2\\
&\quad=0\ .
\end{aligned}
$$

Multiplying (13) with p_1q_2 from the left one gets

$$
\begin{aligned}
&p_1q_2f(x_1,x_2)q_1q_2\widehat{m}+p_1q_2f(x_1,x_2)p_1p_2\widehat{m}_2+p_1q_2f(x_1,x_2)p_1q_2\widehat{m}_1\\
&+p_1q_2f(x_1,x_2)q_1p_2\widehat{m}_1-p_1q_2\widehat{m}_1f(x_1,x_2)\ .
\end{aligned}
$$

Using (c) the latter is zero. Also multiplying with q_1p_2 yields zero due to symmetry in interchanging x_1 with x_2. Multiplying (13) with q_1q_2 from the left one gets

$$
\begin{aligned}
&q_1q_2f(x_1,x_2)\widehat{m}q_1q_2-q_1q_2\widehat{m}f(x_1,x_2)+q_1q_2f(x_1,x_2)p_1p_2\widehat{m}_2+\\
&q_1q_2f(x_1,x_2)p_1q_2\widehat{m}_1+q_1q_2f(x_1,x_2)q_1p_2\widehat{m}_1
\end{aligned}
$$

which is again zero and so is (13).

(e) First note that, for bounded operators A,B, $\|AB\|_{\mathrm{op}}=\|B^*A^*\|_{\mathrm{op}}$ holds, where A^* is the adjoint operator of A. To show (10), note that

$$
p_jf(x_j-x_k)p_j=p_j(f\star|\varphi|^2)(x_k)\ . \tag{14}
$$

It follows that

$$
\|p_jf(x_j-x_k)p_j\|_{\mathrm{op}}\le\|f\|_1\|\varphi\|_\infty^2\ .
$$

For (11) we write

$$
\begin{aligned}
\|g(x_j-x_k)p_j\|_{\mathrm{op}}^2=&\sup_{\|\Psi\|=1}\|g(x_j-x_k)p_j\Psi\|^2=\\
=&\sup_{\|\Psi\|=1}\langle\!\langle\Psi,p_j|g(x_j-x_k)|^2p_j\Psi\rangle\!\rangle\\
\le&\|p_j|g(x_j-x_k)|^2p_j\|_{\mathrm{op}}\ .
\end{aligned}
$$

With (10) we get (11). For (12) we use

$$
\begin{aligned}
\|g(x_j-x_k)\nabla_jp_j\|_{\mathrm{op}}^2=&\sup_{\|\Psi\|=1}\langle\!\langle\Psi,p_j(|g|^2*|\nabla\varphi|^2)(x_k)\Psi\rangle\!\rangle\le\||g|^2*|\nabla\varphi|^2\|_\infty\\
\le&\|g\|^2\|\nabla\varphi\|_\infty^2
\end{aligned}
$$

□

Within our estimates we will encounter wave functions where some of the symmetry is broken (at this point the reader should exemplarily think of the wave function $V_\beta(x_1 - x_2)\Psi$ which is not symmetric under exchange of the variables x_1 and x_3, for example). This leads to the following definition

Definition 4.3 *For any finite set $\mathcal{M} \subset \{1, 2, \dots, N\}$, define the space $\mathcal{H}_\mathcal{M} \subset L^2(\mathbb{R}^{2N}, \mathbb{C})$ as the set of functions which are symmetric in all variables in $\mathcal{M}$*

$$\begin{aligned}\Psi \in \mathcal{H}_\mathcal{M} \Leftrightarrow &\Psi(x_1, \dots, x_j, \dots, x_k, \dots, x_N) = \Psi(x_1, \dots, x_k, \dots, x_j, \dots, x_N)\\ &\textit{for all } j, k \in \mathcal{M} \; .\end{aligned}$$

Based on the combinatorics of the p_j and q_j, we obtain the following

Lemma 4.4 *For any $f : \mathbb{N}_0 \to \mathbb{R}_0^+$ and any finite set $\mathcal{M}_a \subset \{1, 2, \dots, N\}$ with $1 \in \mathcal{M}_a$ and any finite set $\mathcal{M}_b \subset \{1, 2, \dots, N\}$ with $1, 2 \in \mathcal{M}_b$*

$$\left\|\widehat{f} q_1 \Psi\right\|^2 \leq \frac{N}{|\mathcal{M}_a|} \|\widehat{f}\widehat{n}\Psi\|^2 \qquad \textit{for any } \Psi \in \mathcal{H}_{\mathcal{M}_a}, \tag{15}$$

$$\left\|\widehat{f} q_1 q_2 \Psi\right\|^2 \leq \frac{N^2}{|\mathcal{M}_b|(|\mathcal{M}_b| - 1)} \|\widehat{f}(\widehat{n})^2\Psi\|^2 \qquad \textit{for any } \Psi \in \mathcal{H}_{\mathcal{M}_b} \; . \tag{16}$$

Proof: Let $\Psi \in \mathcal{H}_{\mathcal{M}_a}$ for some finite set $1 \in \mathcal{M}_a \subset \{1, 2, \dots, N\}$. By Lemma 4.2 (b), (15) can be estimated as

$$\begin{aligned}\|\widehat{f}\widehat{n}\Psi\|^2 =& \langle\!\langle \Psi, (\widehat{f})^2 (\widehat{n})^2 \Psi \rangle\!\rangle = N^{-1} \sum_{k=1}^{N} \langle\!\langle \Psi, (\widehat{f})^2 q_k \Psi \rangle\!\rangle\\ \geq& N^{-1} \sum_{k \in \mathcal{M}_a} \langle\!\langle \Psi, (\widehat{f})^2 q_k \Psi \rangle\!\rangle = \frac{|\mathcal{M}_a|}{N} \langle\!\langle \Psi, (\widehat{f})^2 q_1 \Psi \rangle\!\rangle\\ =& \frac{|\mathcal{M}_a|}{N} \|\widehat{f} q_1 \Psi\|^2 \; .\end{aligned}$$

Similarly, we obtain for $\Psi \in \mathcal{H}_{\mathcal{M}_b}$

$$\begin{aligned}\|\widehat{f}(\widehat{n})^2\Psi\|^2 =& \langle\!\langle \Psi, (\widehat{f})^2 (\widehat{n})^4 \Psi \rangle\!\rangle \geq N^{-2} \sum_{j,k \in \mathcal{M}_b} \langle\!\langle \Psi, (\widehat{f})^2 q_j q_k \Psi \rangle\!\rangle\\ =& \frac{|\mathcal{M}_b|(|\mathcal{M}_b| - 1)}{N^2} \langle\!\langle \Psi, (\widehat{f})^2 q_1 q_2 \Psi \rangle\!\rangle + \frac{|\mathcal{M}_b|}{N^2} \langle\!\langle \Psi, (\widehat{f})^2 q_1 \Psi \rangle\!\rangle\\ \geq& \frac{|\mathcal{M}_b|(|\mathcal{M}_b| - 1)}{N^2} \|\widehat{f} q_1 q_2 \Psi\|^2\end{aligned}$$

which concludes the Lemma.

□

Corollary 4.5 *For any weight $m : \mathbb{N}_0 \to \mathbb{R}_0^+$*

$$\|\nabla_2 \widehat{m} q_2 \Psi\| \leq 2 \|\widehat{m}\|_{op} \|\nabla_2 q_2 \Psi\|, \tag{17}$$

$$\|\nabla_2 \widehat{m} q_1 q_2 \Psi\| \leq C \|\widehat{m}\widehat{n}\|_{op} \|\nabla_2 q_2 \Psi\| \; . \tag{18}$$

Proof: Using $p_2 + q_2 = 1$ and triangle inequality,

$$\|\nabla_2 \widehat{m} q_2 \Psi\| \le \|p_2 \nabla_2 \widehat{m} q_2 \Psi\| + \|q_2 \nabla_2 \widehat{m} q_2 \Psi\|, \tag{19}$$
$$\|\nabla_2 \widehat{m} q_1 q_2 \Psi\| \le \|p_2 \nabla_2 \widehat{m} q_1 q_2 \Psi\| + \|q_2 \nabla_2 \widehat{m} q_1 q_2 \Psi\| \,. \tag{20}$$

With Lemma 4.2 (c) we get

$$(19) = \|\widehat{m}_1 p_2 \nabla_2 q_2 \Psi\| + \|\widehat{m} q_2 \nabla_2 q_2 \Psi\| \le (\|\widehat{m}_1\|_{\mathrm{op}} + \|\widehat{m}\|_{\mathrm{op}}) \|\nabla_2 q_2 \Psi\| \,.$$

Note that the wave function $p_2 \nabla_2 q_2 \Psi$ is symmetric under the exchange of any two variables but x_2. Thus we can use Lemma 4.4 to get

$$\begin{aligned}(20) &= \|q_1 \widehat{m}_1 p_2 \nabla_2 q_2 \Psi\| + \|q_1 \widehat{m} q_2 \nabla_2 q_2 \Psi\| \\ &\le \frac{N}{N-1} (\|\widehat{m}_1 \widehat{n}\|_{\mathrm{op}} + \|\widehat{m} \widehat{n}\|_{\mathrm{op}}) \|\nabla_2 q_2 \Psi\| \,.\end{aligned}$$

Since $\sqrt{k} \le \sqrt{k+1}$ for $k \ge 0$ it follows that the latter is bounded by

$$C(\|\widehat{m}_1 \widehat{n}_1\|_{\mathrm{op}} + \|\widehat{m} \widehat{n}\|_{\mathrm{op}}) \|\nabla_2 q_2 \Psi\| \,.$$

Using that $\|\widehat{r}\|_{\mathrm{op}} = \sup_{0 \le k \le N} \{r(k)\} = \|\widehat{r}_d\|_{\mathrm{op}}$ for any $d \in \mathbb{N}$ and any weight r, the Corollary follows.

□

Lemma 4.6 *Let $\Omega, \chi \in \mathcal{H}_{\mathcal{M}}$ for some $\mathcal{M}$, let $1 \notin \mathcal{M}$ and $2, 3 \in \mathcal{M}$. Let $O_{j,k}$ be an operator acting on the j^{th} and k^{th} coordinate. Then*

$$|\langle\!\langle \Omega, O_{1,2} \chi \rangle\!\rangle| \le \|\Omega\|^2 + |\langle\!\langle O_{1,2} \chi, O_{1,3} \chi \rangle\!\rangle| + (|\mathcal{M}|)^{-1} \|O_{1,2} \chi\|^2 \,.$$

Proof: Using symmetry and Cauchy Schwarz

$$|\langle\!\langle \Omega, O_{1,2} \chi \rangle\!\rangle| = |\mathcal{M}|^{-1} |\langle\!\langle \Omega, \sum_{j \in \mathcal{M}} O_{1,j} \chi \rangle\!\rangle| \le |\mathcal{M}|^{-1} \|\Omega\| \, \|\sum_{j \in \mathcal{M}} O_{1,j} \chi\|$$

For the second factor we can write

$$\begin{aligned}\|\sum_{j \in \mathcal{M}} O_{1,j} \chi\|^2 &= \langle\!\langle \sum_{j \in \mathcal{M}} O_{1,j} \chi, \sum_{k \in \mathcal{M}} O_{1,k} \chi \rangle\!\rangle \\ &\le \sum_{j \in \mathcal{M}} |\langle\!\langle O_{1,j} \chi, O_{1,j} \chi \rangle\!\rangle| + |\sum_{j \ne k \in \mathcal{M}} \langle\!\langle O_{1,j} \chi, O_{1,k} \chi \rangle\!\rangle| \\ &\le |\mathcal{M}| |\langle\!\langle O_{1,2} \chi, O_{1,2} \chi \rangle\!\rangle| + |\mathcal{M}| (|\mathcal{M}| - 1) |\langle\!\langle O_{1,2} \chi, O_{1,3} \chi \rangle\!\rangle|\end{aligned}$$

Since $ab \le 1/2a^2 + 1/2b^2$ and $(a+b)^2 \le 2a^2 + 2b^2$ holds for any real numbers a, b, the Lemma follows.

□

In our estimates, we need the regularity conditions

$$\|\nabla\varphi_t\|_\infty < \infty, \qquad \|\varphi_t\|_\infty < \infty, \qquad \|\nabla\varphi_t\| < \infty, \qquad \|\Delta\varphi_t\| < \infty .$$

That is, we need $\varphi_t \in H^2(\mathbb{R}^2,\mathbb{C}) \cap W^{1,\infty}(\mathbb{R}^2,\mathbb{C})$. Then, $\|\Delta|\varphi_t|^2\|, \|\Delta|\varphi_t|^2\|_1$ and $\|\varphi_t^2\|$, which also appear in our estimates, can be bounded by

$$\begin{aligned}
\Delta|\varphi_t|^2 =& \varphi_t^*\Delta\varphi_t + \varphi_t\Delta\varphi_t^* + 2(\nabla\varphi_t^*)\cdot(\nabla\varphi_t) \\
\|\Delta|\varphi_t|^2\| \leq& 2\|\Delta\varphi_t\|\|\varphi_t\|_\infty + 2\|\nabla\varphi_t\|\|\nabla\varphi_t\|_\infty \\
\|\Delta|\varphi_t|^2\|_1 \leq& 4\|\Delta\varphi_t\| \\
\|\varphi_t^2\| \leq& \|\varphi_t\|_\infty\|\varphi_t\| .
\end{aligned}$$

Recall the Sobolev embedding Theorem, which implies in particular $H^k(\mathbb{R}^2,\mathbb{C}) = W^{k,2}(\mathbb{R}^2,\mathbb{C}) \subset C^{k-2}(\mathbb{R}^2,\mathbb{C})$. If $\varphi \in C^1(\mathbb{R}^2,\mathbb{C}) \cap H^1(\mathbb{R}^2,\mathbb{C})$, then $\varphi \in W^{1,\infty}(\mathbb{R}^2,\mathbb{C})$ follows since both φ and $\nabla\varphi$ have to decay at infinity. Thus, $\varphi_t \in H^3(\mathbb{R}^2,\mathbb{C})$ implies $\varphi_t \in H^2(\mathbb{R}^2,\mathbb{C}) \cap W^{1,\infty}(\mathbb{R}^2,\mathbb{C})$, which suffices for our estimates. Since φ_t obeys a defocusing nonlinear Schrödinger equation, we expect the regularity of the solution φ_t to follow from the regularity of the initial datum φ_0. For a certain class of external potentials A_t this has been proven in [5]:

Lemma 4.7 *Let $\varphi_0 \in \Sigma^k(\mathbb{R}^2,\mathbb{C}) = \{f \in L^2(\mathbb{R}^2,\mathbb{C}) | \sum_{\alpha+\beta\leq k} \|x^\alpha\partial_x^\beta f\| < \infty\}$, for $k \geq 2$. Let, for $b > 0$, φ_t the unique solution to*

$$i\partial_t\varphi_t = (-\Delta + A_t + b|\varphi_t|^2)\varphi_t .$$

Let $A_\cdot \in L^\infty_{loc}(\mathbb{R}_t \times \mathbb{R}^2_x, \mathbb{C})$ real valued and smooth with respect to the space variable: for (almost) all $t \in \mathbb{R}$, the map $x \mapsto A_t(x)$ is C^∞. Moreover, A_t is at most quadratic in space, uniformly w.r.t. time t:

$$\forall\alpha \in \mathbb{N}^2, |\alpha| \geq 2, \qquad \partial_x^\alpha A_\cdot \in L^\infty(\mathbb{R}_t \times \mathbb{R}^d_x, \mathbb{C}).$$

In addition, $t \mapsto \sup_{|x|\leq 1} |A_t(x)|$ belongs to $L^\infty(\mathbb{R},\mathbb{C})$. Then

(a) $\varphi_t \in \Sigma^k(\mathbb{R}^2,\mathbb{C})$, which implies $\varphi_t \in H^k(\mathbb{R}^2,\mathbb{C})$.

(b) $\|\varphi_t\| = \|\varphi_0\|$.

(c) Let $\varphi_0 \in \Sigma^3(\mathbb{R}^2,\mathbb{C})$. Assume in addition that $\|A_t\|_\infty < \infty$ and $\|\dot{A}_t\|_\infty < \infty$. Then, for any fixed $t \geq 0$, $\mathcal{K}(\varphi_t, A_t) < \infty$ follows.

Proof: Part (a) is Corollary 1.4. in [5]. We like to remark that $\|\varphi_t\|_{H^k} \leq C$ holds, if $A_t = 0$, see Section 1.2. in [5]. The conditions on A_t are for example satisfied if $A_t \in C_c^\infty(\mathbb{R}^2,\mathbb{R})$ for all $t \in \mathbb{R}$, $A_t(x) = 0$, for all $|t| \geq T$. Part (b) can be verified directly, using the existence of global in time solutions. Part (c) follows from (a) and the embedding $H^3(\mathbb{R}^2,\mathbb{C}) \subset H^2(\mathbb{R}^2,\mathbb{C}) \cap W^{1,\infty}(\mathbb{R}^2,\mathbb{C})$.

□

5 Microscopic structure in 2 dimensions

5.1 The scattering state

In this section we analyze the microscopic structure which is induced by V_N. In particular, we explain why the dynamical properties of the system are determined by the low energy scattering regime.

Definition 5.1 *Let $V \in L_c^\infty(\mathbb{R}^2, \mathbb{R})$, $V(x) \geq 0$, V spherically symmetric and let V_N be given by $V_N(x) = e^{2N} V(e^N x)$. For any $R \geq diam(supp(V_N))$, we define the zero energy scattering state $j_{N,R}$ by*

$$\begin{cases} \left(-\Delta_x + \frac{1}{2} e^{2N} V(e^N x)\right) j_{N,R}(x) = 0, \\ j_{N,R}(x) = 1 \textit{ for } |x| = R \,. \end{cases} \tag{21}$$

Next, we want to recall some important properties of the scattering state $j_{N,R}$, see also Appendix C of [22].

Lemma 5.2 *Let $V \in L_c^\infty(\mathbb{R}^2, \mathbb{R})$, $V(x) \geq 0$ and spherically symmetric. Define $I_R = \int_{\mathbb{R}^2} d^2x V_N(x) j_{N,R}(x)$. For the scattering state defined previously the following relations hold:*

(a) There exists a nonnegative number a, called scattering length of the potential V, such that

$$I_R = \frac{4\pi}{\ln\left(\frac{e^N R}{a}\right)}$$

(in the case $a = 0$ we have $I_R = 0$). The scattering length a does not depend on R and fulfills $a \leq diam(supp(V))$. Furthermore, $I_R \geq 0$ holds.

(b) $j_{N,R}$ is a nonnegative function which is spherically symmetric in $|x|$. For $|x| \geq diam(supp(V_N))$, $j_{N,R}$ is given by

$$j_{N,R}(x) = 1 + \frac{1}{\ln\left(\frac{e^N R}{a}\right)} \ln\left(\frac{|x|}{R}\right) .$$

Proof:

(a)+(b) Rescaling $x \to e^N x = y$, we obtain, setting $\tilde{R} = e^N R$ and $s_{\tilde{R}}(y) = j_{0,e^N R}(y)$, the unscaled scattering equation

$$\begin{cases} \left(-\Delta_y + \frac{1}{2} V(y)\right) s_{\tilde{R}}(y) = 0, \\ s_{\tilde{R}}(y) = 1 \text{ for } |y| = \tilde{R} \,. \end{cases} \tag{22}$$

Since we assume V to be nonnegative, one can define the scattering state $s_{\tilde{R}}$ by a variational principle. Theorem C.1 in [22] then implies that $s_{\tilde{R}}$ is a nonnegative, spherically symmetric function in $|y|$. It is then easy to verify that for diam $(\text{supp}(V)) \leq |y|$ there exists a number $A \in \mathbb{R}$ such that

$$s_{\tilde{R}}(y) = 1 + \frac{A}{4\pi} \ln\left(\frac{|y|}{\tilde{R}}\right) \,. \tag{23}$$

Next, we show that $A = \int_{\mathbb{R}^2} d^2y V(y) s_{\tilde{R}}(y)$. This can be seen by noting that, for $r >$ diam (supp(V)),

$$\begin{aligned}\int_{\mathbb{R}^2} d^2y V(y) s_{\tilde{R}}(y) =& 2\int_{B_r(0)} d^2y \Delta s_{\tilde{R}}(y) = 2\int_{\partial B_r(0)} \nabla s_{\tilde{R}}(y) \cdot ds \\ =& \frac{A}{2\pi}\int_{\partial B_r(0)} \nabla \ln(|y|) \cdot ds = \frac{A}{2\pi}\int_0^{2\pi} \frac{1}{r} r d\varphi \\ =& A\ .\end{aligned}$$

By Theorem C.1 in [22], there exists a number $a \geq 0$, not depending on $\tilde{R}$, such that for all $|y| \geq$ diam (supp(V))

$$s_{\tilde{R}}(y) = \frac{\ln(|y|/a)}{\ln(\tilde{R}/a)}\ .$$

Comparing this with (23), we obtain

$$\int_{\mathbb{R}^2} V(y) s_{\tilde{R}}(y) dy^2 = \frac{4\pi}{\ln\left(\frac{\tilde{R}}{a}\right)}\ .$$

Since $s_{\tilde{R}}$ is nonnegative, it furthermore follows that $a \leq$ diam (supp(V)). This directly implies $A \geq 0$. By scaling, we obtain

$$I_R = \int_{\mathbb{R}^2} V_N(y) j_{N,R}(y) dy^2 = \int_{\mathbb{R}^2} V(y) s_{\tilde{R}}(y) dy^2 = \frac{4\pi}{\ln\left(\frac{e^N R}{a}\right)}\ .$$

□

Assuming that the energy per particle $\mathcal{E}_{V_N}(\Psi)$ is of order one, the wave function Ψ will have a microscopic structure near the interactions V_N, given by $j_{N,R}$. The interaction among two particles is then determined by $\frac{4\pi}{N+\ln\left(\frac{R}{a}\right)} \approx \frac{4\pi}{N}$. Keeping in mind that each particle interacts with all other $N-1$ particles, we obtain the effective Gross-Pitaevskii equation, for $\varphi_t \in H^2(\mathbb{R}^2, \mathbb{C})$

$$i\partial_t \varphi_t(x) = (-\Delta + A_t + 4\pi|\varphi_t(x)|^2)\varphi_t(x).$$

Thus, choosing $V_N(x) = e^{2N} V(e^N x)$ leads in our setting to an effective one-particle equation which is determined by the low energy scattering behavior of the particles. We remark that, for any $s > 0$, the potential $e^{2Ns} V(e^{Ns} x)$ yields to the coupling $4\pi/s$.

5.2 Properties of the scattering state

Note that the potential V_N is strongly peaked within an exponentially small region. In order to control the short scale structure of Ψ_t, we define, with a slight abuse of notation, a potential M_β with softer scaling behavior in such a way that the potential $V_N - M_\beta$ has scattering length zero. This allows us to "replace" V_N by M_β, which has better scaling behavior and is easier to control. In particular, $\|M_\beta\| \leq CN^{-1+\beta}$ can be controlled for β sufficiently small, while $\|V_N\| = \mathcal{O}(e^N)$ cannot be bounded by any finite polynomial in N. The potential M_β is *not* of the exact scaling $N^{-1+2\beta} M(N^\beta x)$. However, it is in the set $\mathcal{V}_\beta$, which we will define now.

Definition 5.3 *For any $\beta > 0$, we define the set of potentials $\mathcal{V}_\beta$ as*

$$\mathcal{V}_\beta = \Big\{ U \in L^2(\mathbb{R}^2, \mathbb{R}) | U(x) \geq 0 \; \forall x \in \mathbb{R}^2, \|U\|_1 \leq CN^{-1}, \|U\| \leq CN^{-1+\beta},$$
$$\|U\|_\infty \leq CN^{-1+2\beta}, U(x) = 0 \; \forall |x| \geq CN^{-\beta}, \; U \text{ is spherically symmetric} \Big\}.$$

Note that $N^{-1+2\beta} W(N^\beta x) \in \mathcal{V}_\beta$ holds, if W is positive, spherically symmetric and compactly supported.

All relevant estimates in this paper are formulated for $W_\beta \in \mathcal{V}_\beta$.

Definition 5.4 *Let $V \in L^\infty_c(\mathbb{R}^2, \mathbb{R})$, $V(x) \geq 0$ and spherically symmetric. For any $\beta > 0$ and any $R_\beta \geq N^{-\beta}$ we define the potential M_β via*

$$M_\beta(x) = \begin{cases} 4\pi N^{-1+2\beta} & \text{if } N^{-\beta} < |x| \leq R_\beta \\ 0 & \text{else} \end{cases} . \tag{24}$$

Furthermore, we define the zero energy scattering state f_β of the potential $\frac{1}{2}(V_N - M_\beta)$, that is

$$\begin{cases} \left(-\Delta_x + \frac{1}{2}\left(V_N(x) - M_\beta(x)\right)\right) f_\beta(x) = 0 \\ f_\beta(x) = 1 \text{ for } |x| = R_\beta \end{cases} . \tag{25}$$

Note that M_β and f_β depend on R_β. We choose R_β such that the scattering length of the potential $(V_N - M_\beta)$ is zero. This is equivalent to the condition $\int_{\mathbb{R}} d^2x (V_N(x) - M_\beta(x)) f_\beta(x) = 0$.

Lemma 5.5 *For the scattering state f_β, defined by (25), the following relations hold:*

(a) There exists a minimal value $R_\beta < \infty$ such that $\int_{\mathbb{R}^2} d^2x (V_N(x) - M_\beta(x)) f_\beta(x) = 0$.

For the rest of the paper we assume that R_β is chosen such that (a) holds.

(b) There exists $K_\beta \in \mathbb{R}$, $K_\beta > 0$ such that $K_\beta f_\beta(x) = j_{N,R_\beta}(x) \; \forall |x| \leq N^{-\beta}$.

(c) For N sufficiently large the supports of V_N and M_β do not overlap.

(d) f_β is a nonnegative, monotone nondecreasing function in $|x|$.

(e)

$$f_\beta(x) = 1 \text{ for } |x| \geq R_\beta \; . \tag{26}$$

(f)

$$1 \geq K_\beta \geq 1 + \frac{1}{N + \ln\left(\frac{R_\beta}{a}\right)} \ln\left(\frac{N^{-\beta}}{R_\beta}\right) \; . \tag{27}$$

(g) $R_\beta \leq CN^{-\beta}$.

For any fixed $0 < \beta$, N sufficiently large such that V_N and M_β do not overlap, we obtain

(h)

$$|N\|V_N f_\beta\|_1 - 4\pi| = |N\|M_\beta f_\beta\|_1 - 4\pi| \leq C\frac{\ln(N)}{N} .$$

(i) Define

$$g_\beta(x) = 1 - f_\beta(x) .$$

Then,

$$\|g_\beta\|_1 \leq CN^{-1-2\beta}\ln N , \qquad \|g_\beta\| \leq CN^{-1-\beta}\ln N , \qquad \|g_\beta\|_\infty \leq 1 .$$

(j)

$$|N\|M_\beta\|_1 - 4\pi| \leq C\frac{\ln(N)}{N} .$$

(k)

$$M_\beta \in \mathcal{V}_\beta , M_\beta f_\beta \in \mathcal{V}_\beta .$$

Proof:

(a) In the following, we will sometimes denote, with a slight abuse of notation, $f_\beta(x) = f_\beta(r)$ and $j_{N,R}(x) = j_{N,R}(r)$ for $r = |x|$ (for this, recall that f_β and $j_{N,R}$ are radially symmetric). We further denote by $f'_\beta(r)$ the derivative of f_β with respect to

We first show by contradiction that there exists a $x_0 \in \mathbb{R}^2$, $|x_0| \leq N^{-\beta}$, such that $f_\beta(x_0) \neq 0$. For this, assume that $f_\beta(x) = 0$ for all $|x| \leq N^{-\beta}$. Since f_β is continuous, there exists a maximal value $r_0 \geq N^{-\beta}$ such that the scattering equation (25) is equivalent to

$$\begin{cases} \left(-\Delta_x - \frac{1}{2}M_\beta(x)\right) f_\beta(x) = 0, \\ f_\beta(x) = 1 \text{ for } |x| = R_\beta, \\ f_\beta(x) = 0 \text{ for } |x| \leq r_0 . \end{cases} \tag{28}$$

Using (25) and Gauss'-theorem, we further obtain

$$f'_{\beta_1,1}(r) = \frac{1}{4\pi r}\int_{B_r(0)} d^2x(V_N(x) - M_\beta(x))f_\beta(x) . \tag{29}$$

(28) and (29) then imply for $r > r_0$

$$\left|f'_\beta(r)\right| = \frac{1}{4\pi r}\left|\int_{B_r(0)} d^2x M_\beta(x) f_\beta(x)\right| = \frac{2\pi N^{-1+2\beta}}{r}\left|\int_{r_0}^r dr' r' f_\beta(r')\right|$$
$$\leq \frac{2\pi N^{-1+2\beta}}{r}\left|\int_{r_0}^r dr' r'(r' - r_0) \sup_{r_0 \leq s \leq r} |f'_\beta(s)|\right| .$$

Taking the supreme over the interval $[r_0, r]$, the inequality above then implies that there exists a constant $C(r, r_0) \neq 0$, $\lim_{r \to r_0} C(r, r_0) = 0$ such that $\sup_{r_0 \leq s \leq r} |f'_\beta(s)| \leq C(r, r_0) N^{-1+3\beta_1} \sup_{r_0 \leq s \leq r} |f'_\beta(s)|$. Thus, for r close enough to r_0, the inequality above can only hold if $f'_\beta(s) = 0$ for $s \in [r_0, r]$, yielding a contradiction to the choice of r_0.

Consequently, there exists a $x_0 \in \mathbb{R}^2, |x_0| \leq N^{-\beta}$, such that $f_\beta(x_0) \neq 0$. We can thus define

$$h(x) = f_\beta(x) \frac{j_{N,R}(x_0)}{f_\beta(x_0)}$$

on the compact set $\overline{B_{x_0}(0)}$. One easily sees that $h(x) = j_{N,R}(x)$ on $\partial \overline{B_{x_0}(0)}$ and satisfies the zero energy scattering equation (21) for $x \in \overline{B_{N^{-\beta}}(0)}$. Note that the scattering equations (21) and (25) have a unique solution on any compact set. It then follows that $h(x) = j_{N,R}(x)\ \forall x \in \overline{B_{N^{-\beta}}(0)}$. Since $j_{N,R}(N^{-\beta}) \neq 0$, we then obtain $f_\beta(N^{-\beta_1}) \neq 0$. Applying Theorem C.1 in [22] once more, it then follows that either f_β or $-f_\beta$ is a nonnegative, monotone nondecrasing function in $|x|$ for all $|x| \leq N^{-\beta}$.

Recall that W_β and hence $f_\beta(x)$ depend on $R_\beta \in [N^{-\beta}, \infty[$. For conceptual clarity, we denote $W_\beta^{(R_\beta)}(x) = W_\beta(x)$ and $f_\beta^{(R_\beta)}(x) = f_\beta(x)$ for the rest of the proof of part (a). For β fixed, consider the function

$$s : [N^{-\beta}, \infty[\to \mathbb{R} \tag{30}$$

$$R_\beta \mapsto \int_{B_{R_\beta}(0)} d^2x (V_N(x) - W_\beta^{(R_\beta)}(x)) f_\beta^{(R_\beta)}(x). \tag{31}$$

We show by contradiction that the function s has at least one zero. Assume $s \neq 0$ were to hold. We can assume w.l.o.g. $s > 0$. It then follows from Gauss'-theorem that $f_\beta'^{(R_\beta)}(R_\beta) > 0$ for all $R_\beta \geq N^{-\beta}$. By uniqueness of the solution of the scattering equation (25), for $\tilde{R}_\beta < R_\beta$ there exists a constant $K_{\tilde{R}_\beta, R_\beta} \neq 0$, such that for all $|x| \leq \tilde{R}_\beta$ we have $f_\beta^{(\tilde{R}_\beta)}(x) = K_{\tilde{R}_\beta, R_\beta} f_\beta^{(R_\beta)}(x)$. Since $f_\beta^{(R_\beta)}$ and s are continuous, we can further conclude $K_{\tilde{R}_\beta, R_\beta} > 0$. From $s \neq 0$, it then follows that, for all $r \in [N^{-\beta}, \infty[$ and for all $R_\beta \in [N^{-\beta}, \infty[$, $f_\beta'^{(R_\beta)}(r) \neq 0$. Thus, for all $r \in [N^{-\beta}, \infty[$ and for all $R_\beta \in [N^{-\beta_1}, \infty[$, the function $f_\beta^{(R_\beta)}(r)$ doesn't change sign. From Lemma 5.2, the assumption $s(N^{-\beta}) > 0$ and $K_{\tilde{R}_\beta, R_\beta} > 0$, we obtain, for all $r \in [0, N^{-\beta}]$ and for all $R_\beta \in [N^{-\beta}, \infty[$, that $f_\beta^{(R_\beta)}(r) \geq 0$ holds. This, however, implies $\lim_{R_\beta \to \infty} s(R_\beta) = -\infty$ yielding to a contradiction. By continuity of s, there exists thus a minimal value $R_\beta \geq N^{-\beta}$ such that $s(R_\beta) = 0$.

Remark 5.6 *As mentioned, we will from now on fix $R_\beta \in [N^{-\beta}, \infty[$ as the minimal value such that $s(R_\beta) = 0$. Furthermore, we may assume $a > 0$ and $R_\beta > N^{-\beta}$ in the following. For $a = 0$, we can choose $R_\beta = N^{-\beta}$, such that $f_\beta(x) = j_{N,R}(x)$. It is then easy to verify that the Lemma stated is valid.*

(b) From (a), we can conclude that

$$K_\beta = \frac{j_{N,R_\beta}(N^{-\beta})}{f_\beta(N^{-\beta})}. \tag{32}$$

Next, we show that the constant K_β is positive. Since $j_{N,R_\beta}(N^{-\beta})$ is positive, it follows from Eq. (32) that K_β and $f_\beta(N^{-\beta})$ have equal sign. By (a), the sign of f_β is constant for $|x| \leq R_\beta$. Since j_{N,R_β} and V_N are nonnegative functions, we obtain by Gauss-theorem and the scattering equation (25)

$$\operatorname{sgn}\left(\frac{\partial f_\beta}{\partial r}|_{r=N^{-\beta}}\right) = \operatorname{sgn}(K_\beta). \tag{33}$$

Recall that R_β is the smallest value such that $\frac{\partial f_\beta}{\partial r}\big|_{r=R_\beta} = 0$. If it were now that K_β is negative, we could conclude from (32) and (33) that $\frac{\partial f_\beta}{\partial r}|_{r=N^{-\beta}} < 0$ and $f_\beta(N^{-\beta}) < 0$. Since R_β is by definition the smallest value where $\frac{\partial f_\beta}{\partial r} = 0$, we were able to conclude from the continuity of the derivative that $\frac{\partial f_\beta}{\partial r} < 0$ for all $r < R_\beta$ and hence $f(R_\beta) < 0$. However, this were in contradiction to the boundary condition of the zero energy scattering state (see (25)) and thus $K_\beta > 0$ follows.

(c) This directly follows from $e^{-N} < CN^{-\beta}$ for N sufficiently large.

(d) From the proof of property (b), we see that f_β and its derivative is positive at $N^{-\beta}$. From (29), we obtain $f_\beta'(r) = 0$ for all $r > R_\beta$. Due to continuity $f_\beta'(r) > 0$ for all $r < R_\beta$. Since f_β is continuous, positive at $N^{-\beta}$, and its derivative is a nonnegative function, it follows that f_β is a nonnegative, monotone nondecreasing function in $|x|$.

(e) By definition of R_β, it follows that $\tilde{I} = \int_{\mathbb{R}^2} d^2x(V_N(x) - W_\beta(x))f_\beta(x) = 0$. Therefore, for all $|x| \geq R_\beta$, f_β solves $-\Delta f_\beta(x) = 0$, which has the solution

$$f_\beta(x) = 1 + \frac{\tilde{I}}{4\pi}\ln\left(\frac{|x|}{R_\beta}\right) = 1$$

(f) Since f_β is a positive monotone nondecreasing function in $|x|$, we obtain

$$1 \geq f_\beta(N^{-\beta}) = j_{N,R_\beta}(N^{-\beta})/K_\beta = \left(1 + \frac{1}{N + \ln\left(\frac{R_\beta}{a}\right)}\ln\left(\frac{N^{-\beta}}{R_\beta}\right)\right)/K_\beta$$

We obtain the lower bound

$$K_\beta \geq 1 + \frac{1}{N + \ln\left(\frac{R_\beta}{a}\right)}\ln\left(\frac{N^{-\beta}}{R_\beta}\right).$$

For the upper bound we first prove that $f_\beta(x) \geq j_{N,R_\beta}(x)$ holds for all $|x| \leq R_\beta$. Using the scatting equations (23) and (25) we obtain

$$\Delta_x(f_\beta(x) - j_{N,R_\beta}(x)) = \frac{1}{2}V_N(x)(f_\beta(x) - j_{N,R_\beta}(x)) - W_\beta(x)f_\beta(x)$$

as well as $f_\beta(R_\beta) - j_{N,R_\beta}(R_\beta) = 0$. Since $W_\beta(x)f_\beta(x) \geq 0$, we obtain that $\Delta_x(f_\beta(x) - j_{N,R_\beta}(x)) \leq 0$ for $N^{-\beta} \leq |x| \leq R_\beta$. That is, $f_\beta(x) - j_{N,R_\beta}(x)$ is superharmonic for $N^{-\beta} < |x| < R_\beta$. Using the minimum principle, we obtain, using that $f_\beta - j_{N,R_\beta}$ is spherically symmetric

$$\min_{N^{-\beta} \leq |x| \leq R_\beta} (f_\beta - j_{N,R_\beta}) = \min_{|x| \in \{N^{-\beta}, R_\beta\}} (f_\beta - j_{N,R_\beta}) \tag{34}$$

If it were now that $\min_{|x|\in\{N^{-\beta},R_\beta\}}(f_\beta - j_{N,R_\beta}) = f_\beta(N^{-\beta}) - j_{N,R_\beta}(N^{-\beta}) \leq f_\beta(R_\beta) - j_{N,R_\beta}(R_\beta) = 0$, we could conclude that $f_\beta(x) - j_{N,R_\beta}(x) \leq 0$ for all $N^{-\beta} \leq |x| \leq R_\beta$. Since $f_\beta(x) - j_{N,R_\beta}(x)$ then obeys

$$\begin{cases} -\Delta(f_\beta(x) - j_{N,R_\beta}(x)) + \frac{1}{2}V_N(x)(f_\beta(x) - j_{N,R_\beta}(x)) = 0 & \text{for } |x| \leq N^{-\beta}, \\ f_\beta(x) - j_{N,R_\beta}(x) \leq 0 & \text{for } |x| = N^{-\beta}, \end{cases}$$

we could then conclude that $f_\beta(x) - j_{N,R_\beta}(x) \leq 0$ for all $|x| \leq R_\beta$. From this, we obtain that $\Delta(f_\beta(x) - j_{N,R_\beta}(x)) \leq 0$ for $|x| \leq R_\beta$. That is, $f_\beta(x) - j_{N,R_\beta}(x)$ is superharmonic for all $|x| \leq R_\beta$. Using the minimum principle once again, we then obtain

$$\min_{B_{R_\beta}(0)} (f_\beta - j_{N,R_\beta}) = f_\beta(R_\beta) - j_{N,R_\beta}(R_\beta) = 0$$

which contradicts $f_\beta(x) - j_{N,R_\beta}(x) \leq 0$ for $|x| \leq R_\beta$. Therefore, we can conclude in (34) that $\min_{N^{-\beta}\leq|x|\leq R_\beta}(f_\beta - j_{N,R_\beta}) = f_\beta(R_\beta) - j_{N,R_\beta}(R_\beta) = 0$ holds. Then, it follows that $f_\beta(x) - j_{N,R_\beta}(x) \geq 0$ for all $N^{-\beta} \leq |x| \leq R_\beta$. Using the zero energy scattering equation $-\Delta(f_\beta(x) - j_{N,R_\beta}(x)) + \frac{1}{2}V_N(x)(f_\beta(x) - j_{N,R_\beta}(x)) = 0$ for $|x| \leq N^{-\beta}$, we can, together with $f_\beta(N^{-\beta}) - j_{N,R_\beta}(N^{-\beta}) \geq 0$, conclude that $f_\beta(x) - j_{N,R_\beta}(x) \geq 0$ for all $|x| \leq R_\beta$.

As a consequence, we obtain the desired bound $K_\beta = \frac{j_{N,R_\beta}(N^{-\beta})}{f_\beta(N^{-\beta})} \leq 1$.

(g) Since f_β is a nonnegative, monotone nondecreasing function in $|x|$ with $f_\beta(x) = 1 \; \forall |x| \geq R_\beta$, it follows that

$$\begin{aligned} Cf_\beta(N^{-\beta}) =& f_\beta(N^{-\beta}) \int_{\mathbb{R}^2} d^2x V_N(x) \geq \int_{\mathbb{R}^2} d^2x V_N(x) f_\beta(x) \\ =& \int_{\mathbb{R}^2} d^2x M_\beta(x) f_\beta(x) \geq f_\beta(N^{-\beta}) \int_{\mathbb{R}^2} d^2x M_\beta(x) \; . \end{aligned}$$

Therfore, $\int_{\mathbb{R}^2} d^2x M_\beta(x) \leq C$ holds, which implies that $R_\beta \leq CN^{1/2-\beta}$.

From

$$\begin{aligned} \frac{1}{K_\beta} \frac{4\pi}{N + \ln\left(\frac{R_\beta}{a}\right)} =& \frac{1}{K_\beta} \int_{\mathbb{R}^2} d^2x V_N(x) j_{N,R_\beta}(x) = \int_{\mathbb{R}^2} d^2x V_N(x) f_\beta(x) \\ =& \int_{\mathbb{R}^2} d^2x M_\beta(x) f_\beta(x) = 8\pi^2 N^{-1+2\beta} \int_{N^{-\beta}}^{R_\beta} dr r f_\beta(r) \end{aligned}$$

we conclude that

$$\int_{N^{-\beta}}^{R_\beta} drr f_\beta(r) = \frac{N^{1-2\beta}}{2\pi K_\beta \left(N + \ln\left(\frac{R_\beta}{a}\right)\right)} \ .$$

Since f_β is a nonegative, monotone nondecreasing function in $|x|$,

$$\frac{1}{2}(R_\beta^2 - N^{-2\beta})\frac{j_{N,R_\beta}(N^{-\beta})}{K_\beta} = \frac{1}{2}(R_\beta^2 - N^{-2\beta}) f_\beta(N^{-\beta}) \leq \int_{N^{-\beta}}^{R_\beta} drr f_\beta(r)$$

which implies

$$R_\beta^2 N^{2\beta} \leq \frac{N}{\pi\left(N + \ln\left(\frac{R_\beta}{a}\right)\right) j_{N,R_\beta}(N^{-\beta})} + 1$$

Using $R_\beta \leq CN^{1/2-\beta}$, it then follows

$$j_{N,R_\beta}(N^{-\beta}) = 1 + \frac{1}{N + \ln\left(\frac{R_\beta}{a}\right)} \ln\left(\frac{N^{-\beta}}{R_\beta}\right) \geq 1 - \frac{C}{N} \ ,$$

which implies $R_\beta \leq CN^{-\beta}$.

(h) Using

$$\|M_\beta f_\beta\|_1 = \|V_N f_\beta\|_1 = K_\beta^{-1} \|V_N j_{N,R_\beta}\|_1 = K_\beta^{-1} \frac{4\pi}{N + \ln\left(\frac{R_\beta}{a}\right)} \ ,$$

we obtain

$$\begin{aligned} |N\|V_N f_\beta\|_1 - 4\pi| =& |N\|M_\beta f_\beta\|_1 - 4\pi| = 4\pi \left| K_\beta^{-1} \frac{N}{N + \ln\left(\frac{R_\beta}{a}\right)} - 1 \right| \\ =& \frac{4\pi}{K_\beta} \left| \frac{N - NK_\beta + K_\beta \ln\left(\frac{R_\beta}{a}\right)}{N + \ln\left(\frac{R_\beta}{a}\right)} \right| \leq C\frac{\ln(N)}{N} \ . \end{aligned}$$

(i) Using for $|x| \leq R_\beta$ the inequalities $j_{N,R_\beta}(x) \geq 1 + \frac{1}{N+\ln\left(\frac{R_\beta}{a}\right)} \ln\left(\frac{|x|}{R_\beta}\right)$ as well as $1 \geq f_\beta(x) \geq j_{N,R_\beta}(x)$, it follows for $|x| \leq R_\beta$

$$\begin{aligned} 0 \leq& g_\beta(x) = 1 - f_\beta(x) \leq 1 - j_{N,R_\beta}(x) \leq -\frac{1}{N + \ln\left(\frac{R_\beta}{a}\right)} \ln\left(\frac{|x|}{R_\beta}\right) \\ \leq& CN^{-1} |\ln(N|x|)| \ . \end{aligned}$$

Since $g_\beta(x) = 0$ for $|x| > R_\beta$, we conclude with $R_\beta \leq CN^{-\beta}$ that

$$\|g_\beta\|_1 \leq \frac{C}{N} \int_0^{R_\beta} drr |\ln(Nr)| \leq CN^{-1-2\beta} \ln N \ ,$$

as well as

$$\begin{aligned}\|g_\beta\|^2 \leq & \frac{C}{N^2}\int_0^{R_\beta} drr\,(\ln(Nr))^2 \\ =& CN^{-4}\Big[r^2(2(\ln(r))^2-2\ln(r)+1)\Big]_0^{NR_\beta} \\ \leq & CN^{-2-2\beta}\,(\ln(N))^2\,.\end{aligned}$$

$\|g_\beta\|_\infty = \|1-f_\beta\|_\infty \leq 1$, since f_β is a nonnegative, monotone nondecreasing function with $f_\beta(x)\leq 1$.

(j) Using (h) and (i), we obtain with $\|M_\beta\|_1 \leq CN^{-1}$

$$\begin{aligned}|N\|M_\beta\|_1-4\pi| &\leq |N\|M_\beta f_\beta\|_1-4\pi| + N\|M_\beta g_\beta\|_1 \\ &\leq C\left(\frac{\ln(N)}{N}+\|\mathbb{1}_{|\cdot|\geq N^{-\beta}}g_\beta\|_\infty\right)\,.\end{aligned}$$

Since $g_\beta(x)$ is a nonnegative, monotone nonincreasing function, it follows with $K_\beta \leq 1$

$$\begin{aligned}\|\mathbb{1}_{|\cdot|\geq N^{-\beta}}g_\beta\|_\infty = g_\beta(N^{-\beta}) = 1-f_\beta(N^{-\beta}) = 1-\frac{j_{N,R_\beta}(N^{-\beta})}{K_\beta} \\ \leq 1-\left(1+\frac{1}{N+\ln\left(\frac{R_\beta}{a}\right)}\ln\left(\frac{N^{-\beta}}{R_\beta}\right)\right)\,.\end{aligned}$$

and (j) follows.

(k) $M_\beta \in \mathcal{V}_\beta$ follows directly from $R_\beta \leq CN^{-\beta}$. Furthermore, $0 \leq M_\beta(x)f_\beta(x) \leq M_\beta(x)$ implies $M_\beta f_\beta \in \mathcal{V}_\beta$.

□

6 Proof of the Theorem

6.1 Proof for the potential W_β

6.1.1 Choosing the weight

As we have already mentioned, we define a functional $\alpha : L^2(\mathbb{R}^{2N},\mathbb{C})\times L^2(\mathbb{R}^2,\mathbb{C}) \to \mathbb{R}_0^+$ such that

(I) $\frac{\mathrm{d}}{\mathrm{d}t}\alpha(\Psi_t,\varphi_t)$ can be estimated by $\alpha(\Psi_t,\varphi_t)+\mathcal{O}(1)$, yielding to a bound of $\alpha(\Psi_t,\varphi_t)$ via a Grönwall estimate.

(II) $\alpha(\Psi,\varphi)\to 0$ implies convergence of the reduced one particle density matrix $\gamma_\psi^{(1)}$ to $|\varphi\rangle\langle\varphi|$ in trace norm.

For $\beta > 0$, the interaction gets peaked as $N \to \infty$ and one has to use smoothness properties of Ψ_t to be able to control the dynamics of the condensate. For small β and many different choices of the weight, one obtains

$$\alpha(\Psi_t, \varphi_t) \leq \alpha(\Psi_0, \varphi_0)$$
$$+ \int_0^t ds \left(\mathcal{K}(\varphi_s, A_s) \left(\alpha(\Psi_s, \varphi_s) + \mathcal{O}(1) + \langle\!\langle \Psi_s, \widehat{n}^{\varphi_s} \Psi_s \rangle\!\rangle + \left| \mathcal{E}_{W_\beta}(\Psi_s) - \mathcal{E}^{GP}_{N\|W_\beta\|_1}(\varphi_s) \right| \right) \right) .$$

This enables us to perform an integral type Grönwall estimate if we choose

$$\alpha(\Psi_t, \varphi_t) = \langle\!\langle \Psi_t, \widehat{n}^{\varphi_t} \Psi_t \rangle\!\rangle + \left| \mathcal{E}_{W_\beta}(\Psi_t) - \mathcal{E}^{GP}_{N\|W_\beta\|_1}(\varphi_t) \right| .$$

For large β, however, it is necessary to adjust the weight function for the following reason: Taking the time derivative of $\langle\!\langle \Psi_t, \widehat{n}^{\varphi_t} \Psi_t \rangle\!\rangle$, terms of the form $\widehat{n} - \widehat{n}_1$ and $\widehat{n} - \widehat{n}_2$ appear. The bound $N\|\widehat{n} - \widehat{n}_i\|_{\text{op}} = \mathcal{O}(N^{1/2})$, $i = 1, 2$ can then be easily verified. For $\beta > 1/2$ it is not possible to obtain a sufficient decay in N, see Lemma 7.7, part (b). For this reason, it is necessary to choose another weight function $\widehat{m}$ in such a way that $N\|\widehat{m} - \widehat{m}_i\|_{\text{op}}$ is better to control.

Definition 6.1 *For $0 < \xi < \frac{1}{2}$ define*

$$m(k) = \begin{cases} \sqrt{k/N}, & \textit{for } k \geq N^{1-2\xi}; \\ 1/2(N^{-1+\xi}k + N^{-\xi}), & \textit{else.} \end{cases}$$

and

$$\alpha^<(\Psi, \varphi) = \langle\!\langle \Psi, \widehat{m}^{\varphi} \Psi \rangle\!\rangle + \left| \mathcal{E}_{W_\beta}(\Psi) - \mathcal{E}^{GP}_{N\|W_\beta\|_1}(\varphi) \right| .$$

With this definition, we obtain $N\|\widehat{m} - \widehat{m}_1\|_{\text{op}} \leq CN^{\xi}$, see (57).

Lemma 6.2 *Let $\Psi \in L^2_s(\mathbb{R}^{2N}, \mathbb{C})$ and let $\varphi \in L^2(\mathbb{R}^2, \mathbb{C})$. Let $\alpha^<(\Psi, \varphi)$ be defined as above. Then,*

$$\lim_{N\to\infty} \alpha^<(\Psi, \varphi) = 0 \quad \Leftrightarrow \quad \lim_{N\to\infty} \gamma^{(1)}_\Psi = |\varphi\rangle\langle\varphi| \textit{ in trace norm}$$
$$\textit{and } \lim_{N\to\infty} (\mathcal{E}_{W_\beta}(\Psi) - \mathcal{E}^{GP}_{N\|W_\beta\|_1}(\varphi)) = 0 .$$

A proof of this Lemma can be found in [31]. Thus, $\alpha(\Psi_t, \varphi_t)$ satisfies condition (II). To obtain the desired Grönwall estimate, we will calculate $\frac{\mathrm{d}}{\mathrm{d}t}\langle\!\langle \Psi, \widehat{m}^{\varphi} \Psi \rangle\!\rangle$ and $\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{E}_{W_\beta}(\Psi_t) - \mathcal{E}^{GP}_{N\|W_\beta\|_1}(\varphi_t))$. For this, define

Definition 6.3 *Let $W_\beta \in \mathcal{V}_\beta$. Define*

$$Z^{\varphi}_\beta(x_j, x_k) = W_\beta(x_j - x_k) - \frac{N\|W_\beta\|_1}{N-1}|\varphi|^2(x_j) - \frac{N\|W_\beta\|_1}{N-1}|\varphi|^2(x_k) . \tag{35}$$

Note, for $W_\beta(x) = N^{-1+2\beta}W(N^\beta x)$, we have $N\|W_\beta\|_1 = \|W\|_1$. With

$$m^a(k) = m(k) - m(k+1), \qquad m^b(k) = m(k) - m(k+2)$$

and
$$\widehat{r} = \widehat{m}^b p_1 p_2 + \widehat{m}^a (p_1 q_2 + q_1 p_2) \ ,$$
we define the functionals $\gamma^<_{a,b} : L^2(\mathbb{R}^{2N}, \mathbb{C}) \times L^2(\mathbb{R}^2, \mathbb{C}) \to \mathbb{R}^+_0$ *by*

$$\begin{aligned}
\gamma^<_a(\Psi, \varphi) =& \langle\!\langle \Psi, \dot{A}_t(x_1)\Psi \rangle\!\rangle - \langle \varphi, \dot{A}_t \varphi \rangle && (36)\\
\gamma^<_b(\Psi, \varphi) =& N(N-1)\Im\left(\langle\!\langle \Psi, Z^\varphi_\beta(x_1, x_2)\widehat{r}\Psi \rangle\!\rangle\right) && (37)\\
=& -2N(N-1)\Im\left(\langle\!\langle \Psi, p_1 q_2 \widehat{m}^a_{-1} Z^\varphi_\beta(x_1,x_2) p_1 p_2 \Psi \rangle\!\rangle\right) && (38)\\
& -N(N-1)\Im\left(\langle\!\langle \Psi, q_1 q_2 \widehat{m}^b_{-2} W_\beta(x_1 - x_2) p_1 p_2 \Psi \rangle\!\rangle\right)\\
& -2N(N-1)\Im\left(\langle\!\langle \Psi, q_1 q_2 \widehat{m}^a_{-1} Z^\varphi_\beta(x_1,x_2) p_1 q_2 \Psi \rangle\!\rangle\right) \ .
\end{aligned}$$

Lemma 6.4 *Let* $W_\beta \in \mathcal{V}_\beta$. *Let* Ψ_t *the unique solution to* $i\partial_t \Psi_t = H_{W_\beta}\Psi_t$ *with initial datum* $\Psi_0 \in L^2_s(\mathbb{R}^{2N}, \mathbb{C}) \cap H^2(\mathbb{R}^{2N}, \mathbb{C})$, $\|\Psi_0\| = 1$. *Let* φ_t *the unique solution to* $i\partial_t \varphi_t = h^{GP}_{N\|W_\beta\|_1}\varphi_t$ *with* $\varphi_t \in H^3(\mathbb{R}^2, \mathbb{C})$, $\|\varphi_0\| = 1$. *Let* $\alpha^<(\Psi_t, \varphi_t)$ *be defined as in Definition 6.1. Then*

$$\alpha^<(\Psi_t, \varphi_t) \le \alpha^<(\Psi_0, \varphi_0) + \int_0^t ds \left(\left|\gamma^<_a(\Psi_s, \varphi_s)\right| + \left|\gamma^<_b(\Psi_s, \varphi_s)\right|\right) \ . \tag{39}$$

Proof: For the proof of the Lemma we restore the upper index φ_t in order to pay respect to the time dependence of $\widehat{m}^{\varphi_t}$. The time derivative of φ_t is given by (3), i.e. $i\partial_t \varphi_t(x_j) = h^{GP}_{N\|W_\beta\|_1, j}\varphi_t(x_j)$. Here, $h^{GP}_{N\|W_\beta\|_1, j}$ denotes the operator $h^{GP}_{N\|W_\beta\|_1}$ acting on the j^{th} coordinate x_j. We then obtain

$$\begin{aligned}
&\frac{d}{dt}\langle\!\langle \Psi_t, \widehat{m}^{\varphi_t}\Psi_t \rangle\!\rangle\\
=& i\langle\!\langle H_{W_\beta}\Psi_t, \widehat{m}^{\varphi_t}\ \Psi_t \rangle\!\rangle - i\langle\!\langle \Psi_t, \widehat{m}^{\varphi_t}\ H_{W_\beta}\Psi_t \rangle\!\rangle - i\langle\!\langle \Psi_t, [\sum_{j=1}^N h^{GP}_{N\|W_\beta\|_1, j}, \widehat{m}^{\varphi_t}\ ]\Psi_t \rangle\!\rangle\\
=& i\langle\!\langle \Psi_t, [H_{W_\beta} - \sum_{j=1}^N h^{GP}_{N\|W_\beta\|_1, j}, \widehat{m}^{\varphi_t}\ ]\Psi_t \rangle\!\rangle = i\frac{N(N-1)}{2}\langle\!\langle \Psi_t, [Z^{\varphi_t}_\beta(x_1, x_2), \widehat{m}^{\varphi_t}\ ]\Psi_t \rangle\!\rangle \ ,
\end{aligned}$$

where we used symmetry of Ψ_t in the last step. Using Lemma 4.2 (d), it follows that the latter equals (dropping the explicit dependence on φ_t from now on)

$$\begin{aligned}
& i\frac{N(N-1)}{2}\langle\!\langle \Psi_t, [Z^{\varphi_t}_\beta(x_1, x_2), p_1 p_2(\widehat{m} - \widehat{m}_2)]\Psi_t \rangle\!\rangle\\
+& i\frac{N(N-1)}{2}\langle\!\langle \Psi_t, [Z^{\varphi_t}_\beta(x_1, x_2), (p_1 q_2 + q_1 p_2)(\widehat{m} - \widehat{m}_1)]\Psi_t \rangle\!\rangle \ .
\end{aligned}$$

Since $Z^{\varphi_t}_\beta$ and $p_1 p_2(\widehat{m} - \widehat{m}_2)$ as well as $p_1 q_2(\widehat{m} - \widehat{m}_1)$ are selfadjoint, we obtain

$$\begin{aligned}
&\frac{d}{dt}\langle\!\langle \Psi_t, \widehat{m}^{\varphi_t}\Psi_t \rangle\!\rangle = -N(N-1)\\
&\Im\left(\langle\!\langle \Psi_t, (p_1 p_2 + p_1 q_2 + q_1 p_2 + q_1 q_2) Z^{\varphi_t}_\beta(x_1, x_2)(\widehat{m}^b p_1 p_2 + \widehat{m}^a(p_1 q_2 + q_1 p_2))\Psi_t \rangle\!\rangle\right) \ .
\end{aligned}$$

Note that in view of Lemma 4.2 (c) $\widehat{r}Q_j Z_\beta^{\varphi_t}(x_1,x_2)Q_j = Q_j Z_\beta^{\varphi_t}(x_1,x_2)Q_j\widehat{r}$ for any $j\in\{0,1,2\}$ and any weight r. Therefore,

$$\Im\left(\langle\langle\Psi_t, p_1p_2 Z_\beta^{\varphi_t}(x_1,x_2)\widehat{m}^b p_1p_2\Psi_t\rangle\rangle\right)=0$$
$$\Im\left(\langle\langle\Psi_t,(p_1q_2+q_1p_2)Z_\beta^{\varphi_t}(x_1,x_2)\widehat{m}^a(p_1q_2+q_1p_2)\Psi_t\rangle\rangle\right)=0\ .$$

Using Symmetry and Lemma 4.2 (c), we obtain the first line (37). Furthermore,

$$\begin{aligned}\frac{d}{dt}\langle\langle\Psi_t,\widehat{m}^{\varphi_t}\Psi_t\rangle\rangle=&-2N(N-1)\Im\left(\langle\langle\Psi_t,\widehat{m}^b_{-1}p_1q_2Z_\beta^{\varphi_t}(x_1,x_2)p_1p_2\Psi_t\rangle\rangle\right)\\ &-N(N-1)\Im\left(\langle\langle\Psi_t,\widehat{m}^b_{-2}q_1q_2Z_\beta^{\varphi_t}(x_1,x_2)p_1p_2\Psi_t\rangle\rangle\right)\\ &-2N(N-1)\Im\left(\langle\langle\Psi_t,p_1p_2Z_\beta^{\varphi_t}(x_1,x_2)\widehat{m}^a p_1q_2\Psi_t\rangle\rangle\right)\\ &-2N(N-1)\Im\left(\langle\langle\Psi_t,\widehat{m}^a_{-1}q_1q_2Z_\beta^{\varphi_t}(x_1,x_2)p_1q_2\Psi_t\rangle\rangle\right)\ .\end{aligned}$$

Since $p_1p_2|\varphi_t^2|(x_1)q_1q_2=p_1p_2q_2|\varphi_t^2|(x_1)q_1=0=p_1p_2|\varphi_t^2|(x_2)q_1q_2$, we can replace $Z_\beta^{\varphi_t}(x_1,x_2)$ in the second line by $W_\beta(x_1-x_2)$. The third line equals $2N(N-1)\Im\left(\langle\langle\Psi,\widehat{m}^a p_1q_2Z_\beta^{\varphi_t}(x_1,x_2)p_1p_2\Psi\rangle\rangle\right)$. Since

$$m(k-1)-m(k+1)-(m(k)-m(k+1))=m(k-1)-m(k)$$

it follows that $\widehat{m}^b_{-1}-\widehat{m}^a=\widehat{m}^a_{-1}$ and we get

$$\begin{aligned}\frac{d}{dt}\langle\langle\Psi_t,\widehat{m}^{\varphi_t}\Psi_t\rangle\rangle=&-2N(N-1)\Im\left(\langle\langle\Psi,p_1q_2\widehat{m}^a_{-1}Z_\beta^{\varphi_t}(x_1,x_2)p_1p_2\Psi\rangle\rangle\right)\\ &-N(N-1)\Im\left(\langle\langle\Psi,q_1q_2\widehat{m}^b_{-2}W_\beta(x_1-x_2)p_1p_2\Psi\rangle\rangle\right)\\ &-2N(N-1)\Im\left(\langle\langle\Psi,q_1q_2\widehat{m}^a_{-1}Z_\beta^{\varphi_t}(x_1,x_2)p_1q_2\Psi\rangle\rangle\right)\ .\end{aligned}$$

For the second summand of $\alpha^<(\Psi_t,\varphi_t)$ we have

$$\begin{aligned}&\frac{d}{dt}\left(\mathcal{E}_{W_\beta}(\Psi_t)-\mathcal{E}^{GP}_{N\|W_\beta\|_1}(\varphi_t)\right)=\langle\langle\Psi_t,\dot{A}_t(x_1)\Psi_t\rangle\rangle-\langle\varphi_t,\dot{A}_t\varphi_t\rangle\\ &+i\left\langle\varphi_t,\left[h^{GP}_{N\|W_\beta\|_1},\left(h^{GP}_{N\|W_\beta\|_1}-\frac{N\|W_\beta\|_1}{2}|\varphi_t|^2\right)\right]\varphi_t\right\rangle+\left\langle\varphi_t,\frac{N\|W_\beta\|_1}{2}\left(\frac{d}{dt}|\varphi_t|^2\right)\varphi_t\right\rangle\\ &=\langle\langle\Psi_t,\dot{A}_t(x_1)\Psi_t\rangle\rangle-\langle\varphi_t,\dot{A}_t\varphi_t\rangle+i\left\langle\varphi_t,\left[h^{GP}_{N\|W_\beta\|_1},\frac{N\|W_\beta\|_1}{2}|\varphi_t|^2\right]\varphi_t\right\rangle\\ &-i\left\langle\varphi_t,\left[h^{GP}_{N\|W_\beta\|_1},\frac{N\|W_\beta\|_1}{2}|\varphi_t|^2\right]\varphi_t\right\rangle=\gamma_a^<(\Psi_t,\varphi_t)\ .\end{aligned}$$

The Lemma then follows using that $|f(x)|\le|f(0)|+\int_0^x dy|f'(y)|$ holds for any $f\in C^1(\mathbb{R},\mathbb{C})$.

□

6.1.2 Establishing the Grönwall estimate

Lemma 6.5 *Let $W_\beta \in \mathcal{V}_\beta$. Let Ψ_t the unique solution to $i\partial_t \Psi_t = H_{W_\beta}\Psi_t$ with initial datum $\Psi_0 \in L_s^2(\mathbb{R}^{2N}, \mathbb{C}) \cap H^2(\mathbb{R}^{2N}, \mathbb{C})$, $\|\Psi_0\| = 1$. Let φ_t the unique solution to $i\partial_t \varphi_t = h^{GP}_{N\|W_\beta\|_1}\varphi_t$ with $\varphi_t \in H^3(\mathbb{R}^2, \mathbb{C})$. Let $\mathcal{E}_{W_\beta}(\Psi_0) \leq C$. Let $\gamma_a^<(\Psi_t, \varphi_t)$ and $\gamma_b^<(\Psi_t, \varphi_t)$ be defined as in Definition* (6.3)*. Then, there exists an $\eta > 0$ such that*

$$\gamma_a^<(\Psi_t, \varphi_t) \leq C\|\dot{A}_t\|_\infty(\langle\!\langle \Psi_t, \widehat{n}^{\varphi_t}\Psi_t\rangle\!\rangle + N^{-\frac{1}{2}}) \tag{40}$$

$$\gamma_b^<(\Psi_t, \varphi_t) \leq \mathcal{K}(\varphi_t, A_t)\left(\langle\!\langle \Psi_t, \widehat{n}^{\varphi_t}\Psi_t\rangle\!\rangle + N^{-\eta} + \left|\mathcal{E}_{W_\beta}(\Psi_t) - \mathcal{E}^{GP}_{N\|W_\beta\|_1}(\varphi_t)\right|\right) \tag{41}$$

The proof of this Lemma can be found in Section 7.3. Note that

$$|\langle\!\langle \Psi_t, \widehat{n}^{\varphi_t}\Psi_t\rangle\!\rangle - \langle\!\langle \Psi_t, \widehat{m}^{\varphi_t}\Psi_t\rangle\!\rangle| \leq \|\widehat{n}^{\varphi_t} - \widehat{m}^{\varphi_t}\|_{\text{op}} = N^{-\xi}$$

Once we have proven Lemma 6.5, we obtain with Lemma 6.4, Grönwall's Lemma and the estimate above that

$$\alpha^<(\Psi_t, \varphi_t) \leq e^{\int_0^t ds\mathcal{K}(\varphi_s, A_s)}\Big(\alpha^<(\Psi_0, \varphi_0) + \int_0^t ds\mathcal{K}(\varphi_s, A_s)e^{-\int_0^s d\tau\mathcal{K}(\varphi_\tau, A_\tau)}N^{-\eta}\Big) .$$

Note that under the assumptions $\varphi_t \in H^3(\mathbb{R}^2, \mathbb{C})$ and $A_t \in L^\infty(\mathbb{R}^2, \mathbb{C})$, $\dot{A}_t \in L^\infty(\mathbb{R}^2, \mathbb{C})$ there exists a constant $C_t < \infty$, depending on t, φ_0 and A_t, such that $\int_0^t ds\mathcal{K}(\varphi_s, A_s) \leq C_t$, see Lemma 4.7. This proves, using Lemma 6.2, part (a) of Theorem 2.1. If the potential is switched off, one expects that C_t is of order t since in this case $\|\varphi_t\|_\infty$ and $\|\nabla\varphi_t\|_\infty$ are expected to decay like t^{-1}.

We want to explain on a heuristic level why $\gamma_b^<(\Psi_t, \varphi_t)$ is small. The principle argument follows the ideas and estimates of [31]. The first line in (38) is the most important one. This expression is only small if the correct coupling parameter $N\|W_\beta\|_1$ is used in the mean-field equation (3). Then,

$$Np_1W_\beta(x_1 - x_2)p_1 = Np_1W_\beta \star |\varphi|^2(x_2)p_1 \to p_1|\varphi|^2(x_2)\|W\|_1p_1$$

converges against the mean-field potential, and hence the first expression of (38) is small.

In order to estimate the second and third line of (38), one tries to bound $N^2\langle\!\langle \Psi, q_1q_2\widehat{m}^b_{-2}W_\beta(x_1-x_2)p_1p_2\Psi\rangle\!\rangle$ and $N^2\langle\!\langle \Psi, q_1q_2\widehat{m}^a_{-1}Z^\varphi_\beta(x_1-x_2)p_1q_2\Psi\rangle\!\rangle$ in terms of $\langle\!\langle \Psi, \widehat{n}\Psi\rangle\!\rangle + \mathcal{O}(N^{-\eta})$ for some $\eta > 0$. For large β, one needs to use additional smoothness properties of Ψ_t. This explains the appearance of $\left|\mathcal{E}_{W_\beta}(\Psi_t) - \mathcal{E}^{GP}_{N\|W_\beta\|_1}(\varphi_t)\right|$ on the right hand side of (41). The concise estimates are quite involved and can be found in Section 7.3.

6.2 Proof for the exponential scaling V_N

6.2.1 Adapting the weight

For the most involved scaling V_N it is necessary to modify the counting functional $\alpha^<(\Psi, \varphi)$ in order to obtain the desired Grönwall estimate. $\gamma_b^<(\Psi, \varphi)$, which was defined in (38), will not be small if we were to replace W_β by V_N. In particular, $\|V_N\| = \mathcal{O}(e^N)$ cannot be bounded by any finite polynomial in $1/N$. In order to control the dynamics of the condensate, one needs to account for the microscopic structure which is induced by V_N, as explained in Section 5. The idea we will employ is the following: For the moment, think of the most simple counting functional, namely $\langle\!\langle \Psi_t, q_1^{\varphi_t}\Psi_t\rangle\!\rangle = 1 - \langle\!\langle \Psi_t, p_1^{\varphi_t}\Psi_t\rangle\!\rangle$. This functional counts the relative number of particles which are not in the state φ_t. Instead of projecting onto φ_t, we now consider the functional

$$1 - \langle\!\langle \Psi_t, \prod_{j=2}^{N} f_\beta(x_1 - x_j) p_1^{\varphi_t} \prod_{j=2}^{N} f_\beta(x_1 - x_j)\Psi_t\rangle\!\rangle \ ,$$

which takes the short scale correlation structure into account. Neglecting all but two-particle interactions, this can be approximated by

$$\begin{aligned}
&1 - \langle\!\langle \Psi_t, \left(1 - \sum_{j=2}^{N} g_\beta(x_1 - x_j)\right) p_1^{\varphi_t} \left(1 - \sum_{j=2}^{N} g_\beta(x_1 - x_j)\right)\Psi_t\rangle\!\rangle \\
&\approx \langle\!\langle \Psi_t, q_1^{\varphi_t}\Psi_t\rangle\!\rangle + 2(N-1)\Re\left(\langle\!\langle \Psi_t, g_\beta(x_1 - x_2) p_1^{\varphi_t}\Psi_t\rangle\!\rangle\right) \ .
\end{aligned}$$

If we now take the time derivative of this new functional, one gets, among other terms, $2(N-1)\Im\langle\!\langle \Psi_t, [H_{V_N}, f_\beta(x_1-x_2)] p_1^{\varphi_t}\Psi_t\rangle\!\rangle$. The commutator equals $f_\beta(x_1-x_2)(V_N(x_1-x_2) - M_\beta(x_1-x_2)$ plus mixed derivatives and one sees, that V_N is "replaced" by M_β for the price of new terms that have to be estimated. The strategy we are going to employ is thus to estimate the time derivative of the modified functional and to show that we obtain a Grönwall estimate. Note, that, using Lemma 4.2 (e) with Lemma 5.5 (i)

$$2(N-1)|\Re\left(\langle \Psi_t, g_\beta(x_1-x_2) p_1^{\varphi_t}\Psi_t\rangle\right)| \leq CN\|\varphi_t\|_\infty \|g_\beta\| \leq C\|\varphi_t\|_\infty N^{-\beta}\ln(N)$$

holds. Hence, we obtain the a priori estimate

$$\langle\!\langle \Psi_t, q_1^{\varphi_t}\Psi_t\rangle\!\rangle \leq \langle\!\langle \Psi_t, q_1^{\varphi_t}\Psi_t\rangle\!\rangle + 2(N-1)\Re\left(\langle\!\langle \Psi_t, g_\beta(x_1-x_2) p_1^{\varphi_t}\Psi_t\rangle\!\rangle\right) + C\|\varphi_t\|_\infty N^{-\beta}\ln(N) \ .$$

which explains why the new defined functional implies convergence of the reduced density matrix $\gamma^{(1)}_{\Psi_t}$ to $|\varphi_t\rangle\langle\varphi_t|$ in trace norm. We now adapt the strategy explained above to modify the counting functional $\alpha^<(\Psi,\varphi)$.

Definition 6.6 *Let $\widehat{r} = \widehat{m}^b p_1 p_2 + \widehat{m}^a(p_1 q_2 + q_1 p_2)$.*
Let the functional $\alpha : L^2(\mathbb{R}^{2N}, \mathbb{C}) \times L^2(\mathbb{R}^2, \mathbb{C}) \to \mathbb{R}_0^+$ be defined by

$$\alpha(\Psi,\varphi) = \langle\!\langle \Psi, \widehat{m}\Psi\rangle\!\rangle + \left|\mathcal{E}_{V_N}(\Psi) - \mathcal{E}_{4\pi}^{GP}(\varphi)\right| - N(N-1)\Re\left(\langle\!\langle \Psi, g_\beta(x_1-x_2)\widehat{r}\Psi\rangle\!\rangle\right) \tag{42}$$

and the functional $\gamma : L^2(\mathbb{R}^{2N}, \mathbb{C}) \times L^2(\mathbb{R}^2, \mathbb{C}) \to \mathbb{R}$ be defined by

$$\gamma(\Psi,\varphi) = |\gamma_a(\Psi,\varphi)| + |\gamma_b(\Psi,\varphi)| + |\gamma_c(\Psi,\varphi)| + |\gamma_d(\Psi,\varphi)| + |\gamma_e(\Psi,\varphi)| + |\gamma_f(\Psi,\varphi)| \ , \tag{43}$$

where the different summands are:

(a) The change in the energy-difference

$$\gamma_a(\Psi,\varphi) = \langle\!\langle \Psi, \dot{A}_t(x_1)\Psi\rangle\!\rangle - \langle \varphi, \dot{A}_t\varphi\rangle \ .$$

(b) The new interaction term

$$\begin{aligned}\gamma_b(\Psi,\varphi) =& - N(N-1)\Im\left(\langle\!\langle \Psi, \widetilde{Z}^{\varphi}_{\beta}(x_1,x_2)\widehat{r}\,\Psi\rangle\!\rangle\right)\\ &- N(N-1)\Im\left(\langle\!\langle \Psi, g_\beta(x_1-x_2)\widehat{r}\,\mathcal{Z}^{\varphi}(x_1,x_2)\Psi\rangle\!\rangle\right),\end{aligned}$$

where, using M_β from Definition 5.4,

$$\begin{aligned}\widetilde{Z}^{\varphi}_{\beta}(x_1,x_2) &= \left(M_\beta(x_1-x_2) - 4\pi\frac{|\varphi|^2(x_1)+|\varphi|^2(x_2)}{N-1}\right) f_\beta(x_1-x_2) \\ \mathcal{Z}^{\varphi}(x_1,x_2) &= V_N(x_1-x_2) - \frac{4\pi}{N-1}|\varphi|^2(x_1) - \frac{4\pi}{N-1}|\varphi|^2(x_2)\,.\end{aligned} \tag{44}$$

(c) The mixed derivative term

$$\gamma_c(\Psi,\varphi) = - 4N(N-1)\langle\!\langle \Psi, (\nabla_1 g_\beta(x_1-x_2))\nabla_1\widehat{r}\Psi\rangle\!\rangle\,.$$

(d) Three particle interactions

$$\begin{aligned}\gamma_d(\Psi,\varphi) =& 2N(N-1)(N-2)\Im\left(\langle\!\langle \Psi, g_\beta(x_1-x_2)\left[V_N(x_1-x_3),\widehat{r}\right]\Psi\rangle\!\rangle\right)\\ &-N(N-1)(N-2)\Im\left(\langle\!\langle \Psi, g_\beta(x_1-x_2)\left[4\pi|\varphi|^2(x_3),\widehat{r}\right]\Psi\rangle\!\rangle\right)\,.\end{aligned}$$

(e) Interaction terms of the correction

$$\gamma_e(\Psi,\varphi) = \frac{1}{2}N(N-1)(N-2)(N-3)\Im\left(\langle\!\langle \Psi, g_\beta(x_1-x_2)\left[V_N(x_3-x_4),\widehat{r}\right]\Psi\rangle\!\rangle\right)\,.$$

(f) Correction terms of the mean field

$$\gamma_f(\Psi,\varphi) = -2N(N-2)\Im\left(\langle\!\langle \Psi, g_\beta(x_1-x_2)\left[4\pi|\varphi|^2(x_1),\widehat{r}\right]\Psi\rangle\!\rangle\right)\,.$$

Lemma 6.7 *Let Ψ_t the unique solution to $i\partial_t\Psi_t = H_{V_N}\Psi_t$ with initial datum $\Psi_0\in L^2_s(\mathbb{R}^{2N},\mathbb{C})\cap H^2(\mathbb{R}^{2N},\mathbb{C})$, $\|\Psi_0\|=1$. Let φ_t the unique solution to $i\partial_t\varphi_t = h^{GP}_{4\pi}\varphi_t$ with $\varphi_t\in H^3(\mathbb{R}^2,\mathbb{C})$, $\|\varphi_0\|=1$. Let $\alpha(\Psi_t,\varphi_t)$ and $\gamma(\Psi_t,\varphi_t)$ be defined as in* (42) *and* (43). *Then*

$$\alpha(\Psi_t,\varphi_t) \le \alpha(\Psi_0,\varphi_0) + \int_0^t ds\gamma(\Psi_s,\varphi_s)\,.$$

Proof: We first calculate

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\langle\!\langle \Psi,\widehat{m}\Psi\rangle\!\rangle - N(N-1)\Re\left(\langle\!\langle \Psi, g_\beta(x_1-x_2)\widehat{r}\Psi\rangle\!\rangle\right)\right)\\ &= - N(N-1)\Im\left(\langle\!\langle \Psi_t, \mathcal{Z}^{\varphi_t}(x_1,x_2)\widehat{r}\Psi_t\rangle\!\rangle\right)\\ &\quad - N(N-1)\Re\left(i\langle\!\langle \Psi_t, g_\beta(x_1-x_2)\left[H_{V_N} - \sum_{i=1}^N h^{GP}_{4\pi,i},\widehat{r}\right]\Psi_t\rangle\!\rangle\right)\\ &\quad - N(N-1)\Re\left(i\langle\!\langle \Psi_t, \left[H_{V_N}, g_\beta(x_1-x_2)\right]\widehat{r}\Psi_t\rangle\!\rangle\right)\,.\end{aligned}$$

Using symmetry and $\Re(iz) = -\Im(z)$, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\langle\!\langle \Psi, \widehat{m}\Psi\rangle\!\rangle - N(N-1)\Re\left(\langle\!\langle \Psi, g_\beta(x_1-x_2)\widehat{r}\Psi\rangle\!\rangle\right)\right)\\
&= - N(N-1)\Im\left(\langle\!\langle \Psi_t, \mathcal{Z}^{\varphi_t}(x_1,x_2)\widehat{r}\Psi_t\rangle\!\rangle\right)\\
&\quad + N(N-1)\Im\left(\langle\!\langle \Psi_t, g_\beta(x_1-x_2)\left[\mathcal{Z}^{\varphi_t}(x_1,x_2),\widehat{r}\right]\Psi_t\rangle\!\rangle\right)\\
&\quad + 2N(N-1)(N-2)\Im\left(\langle\!\langle \Psi_t, g_\beta(x_1-x_2)\left[V_N(x_1-x_3),\widehat{r}\right]\Psi_t\rangle\!\rangle\right)\\
&\quad - N(N-1)(N-2)\Im\left(\langle\!\langle \Psi_t, g_\beta(x_1-x_2)\left[4\pi|\varphi_t|^2(x_3),\widehat{r}\right]\Psi_t\rangle\!\rangle\right)\\
&\quad + \frac{1}{2}N(N-1)(N-2)(N-3)\Im\left(\langle\!\langle \Psi_t, g_\beta(x_1-x_2)\left[V_N(x_3-x_4),\widehat{r}\right]\Psi_t\rangle\!\rangle\right)\\
&\quad + N(N-1)\Im\left(\langle\!\langle \Psi_t, \left[H_{V_N}, g_\beta(x_1-x_2)\right]\widehat{r}\Psi_t\rangle\!\rangle\right) \;.\\
&\quad - 2N(N-2)\Im\left(\langle\!\langle \Psi_t, g_\beta(x_1-x_2)\left[4\pi|\varphi_t|^2(x_1),\widehat{r}\right]\Psi_t\rangle\!\rangle\right) \;.
\end{aligned}$$

The third and fourth lines equal γ_d (recall that Ψ is symmetric), the fifth line equals γ_e and the seventh line equals γ_f. Using that $(1-g_\beta(x_1-x_2))\mathcal{Z}^\varphi(x_1,x_2) = \widetilde{Z}^\varphi_\beta(x_1,x_2) + (V_N(x_1-x_2) - M_\beta(x_1-x_2))f_\beta(x_1-x_2)$ we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\langle\!\langle \Psi, \widehat{m}\Psi\rangle\!\rangle - N(N-1)\Re\left(\langle\!\langle \Psi, g_\beta(x_1-x_2)\widehat{r}\Psi\rangle\!\rangle\right)\right)\\
&\leq \gamma_d(\Psi_t,\varphi_t) + \gamma_e(\Psi_t,\varphi_t) + \gamma_f(\Psi_t,\varphi_t)\\
&\quad - N(N-1)\Im\left(\langle\!\langle \Psi_t, \widetilde{Z}^{\varphi_t}_\beta(x_1,x_2)\widehat{r}\Psi_t\rangle\!\rangle\right)\\
&\quad - N(N-1)\Im\left(\langle\!\langle \Psi_t, (V_N(x_1-x_2) - M_{\beta_1}(x_1-x_2))f_\beta(x_1-x_2)\widehat{r}\Psi_t\rangle\!\rangle\right)\\
&\quad - N(N-1)\Im\left(\langle\!\langle \Psi_t, g_\beta(x_1-x_2)\widehat{r}\mathcal{Z}^{\varphi_t}(x_1,x_2)\Psi_t\rangle\!\rangle\right)\\
&\quad + N(N-1)\Im\left(\langle\!\langle \Psi_t, \left[H_{V_N}, g_\beta(x_1-x_2)\right]\widehat{r}\Psi_t\rangle\!\rangle\right) \;.
\end{aligned} \tag{45}$$

The first, second and the fourth line give $\gamma_b + \gamma_d + \gamma_e + \gamma_f$. Using Definition (5.4) the commutator in the fifth line equals

$$\begin{aligned}
\left[H_{V_N}, g_\beta(x_1-x_2)\right] &= - \left[H_{V_N}, f_\beta(x_1-x_2)\right]\\
&= \left[\Delta_1+\Delta_2, f_\beta(x_1-x_2)\right]\\
&= (\Delta_1+\Delta_2)f_\beta(x_1-x_2)\\
&\quad + (2\nabla_1 f_\beta(x_1-x_2))\nabla_1 + (2\nabla_2 f_\beta(x_1-x_2))\nabla_2\\
&= (V_N(x_1-x_2) - M_\beta(x_1-x_2))f_\beta(x_1-x_2)\\
&\quad - (2\nabla_1 g_\beta(x_1-x_2))\nabla_1 - (2\nabla_2 g_\beta(x_1-x_2))\nabla_2 \;.
\end{aligned}$$

Using symmetry the third and fifth line in (45) give

$$-4N(N-1)\langle\!\langle \Psi_t, (\nabla_1 g_\beta(x_1-x_2))\nabla_1\widehat{r}\Psi_t\rangle\!\rangle = \gamma_c(\Psi_t,\varphi_t) \;.$$

Using

$$\frac{d}{dt}\left(\mathcal{E}_{W_\beta}(\Psi_t) - \mathcal{E}^{GP}_{N\|W_\beta\|_1}(\varphi_t)\right) = \gamma_a(\Psi_t,\varphi_t) \;,$$

we obtain the desired result.

□

6.2.2 Establishing the Grönwall estimate

Again, we will bound the time derivative of $\alpha(\Psi_t,\varphi_t)$ such that we can employ a Grönwall estimate.

Lemma 6.8 *Let Ψ_t the unique solution to $i\partial_t\Psi_t = H_{V_N}\Psi_t$ with initial datum $\Psi_0 \in L^2_s(\mathbb{R}^{2N},\mathbb{C})\cap H^2(\mathbb{R}^{2N},\mathbb{C})$, $\|\Psi_0\| = 1$. Let φ_t the unique solution to $i\partial_t\varphi_t = h^{GP}_{4\pi}\varphi_t$ with $\varphi_t \in H^3(\mathbb{R}^2,\mathbb{C})$. Let $\mathcal{E}_{V_N}(\Psi_0) \le C$. Let $\gamma(\Psi_t,\varphi_t)$ be defined as in (43). Then, there exists an $\eta > 0$ such that*

$$\gamma(\Psi_t,\varphi_t) \le\mathcal{K}(\varphi_t,A_t)\left(\langle\!\langle\Psi_t,\widehat{n}\Psi_t\rangle\!\rangle + N^{-\eta} + \left|\mathcal{E}_{V_N(\Psi_0)} - \mathcal{E}^{GP}_{b_{V_N}}(\varphi_0)\right|\right) . \tag{46}$$

A prove of the Lemma can be found in Section 7.4.
The most important estimate is the first part of γ_b, which can be estimated in the same way as $\gamma_b^<$. All other estimates are based on the smallness of the L^p-norms of g_β, see Lemma 5.5 (i). We now show that Lemma 6.8 implies convergence of the reduced density matrix $\gamma^{(1)}_{\Psi_t}$ to $|\varphi_t\rangle\langle\varphi_t|$ in trace norm. Using $\|\widehat{m}^a\|_{\text{op}} + \|\widehat{m}^b\|_{\text{op}} \le CN^{-1+\xi}$, see (57), together with Equation (11) and Lemma 5.5 (i), we obtain

$$\begin{aligned}&\|g_\beta(x_1-x_2)\widehat{r}\|_{\text{op}} \le \|g_\beta(x_1-x_2)p_1(\widehat{m}^b p_2 + \widehat{m}^a q_2)\|_{\text{op}} + \|g_\beta(x_1-x_2)p_2q_1\widehat{m}^a\|_{\text{op}}\\ \le&\mathcal{K}(\varphi,A_t)\|g_\beta\|(\|\widehat{m}^a\|_{\text{op}} + \|\widehat{m}^b\|_{\text{op}}) \le \mathcal{K}(\varphi,A_t)N^{\xi-2-\beta}\ln(N).\end{aligned}$$

Therefore, we bound

$$N(N-1)|\Re\left(\langle\Psi, g_\beta(x_1-x_2)\widehat{r}\Psi\rangle\right)| \le \mathcal{K}(\varphi,A_t)N^{-\beta+\xi}\ln(N). \tag{47}$$

For β large enough, (46) implies together with (47) that

$$\gamma(\Psi_t,\varphi_t) \le \mathcal{K}(\varphi_t,A_t)\left(\alpha(\Psi_t,\varphi_t) + N^{-\eta}\right) ,$$

for some $\eta > 0$. We get with Lemma 6.7 and Grönwall's Lemma, using (47) again, that

$$\begin{aligned}\alpha^<(\Psi_t,\varphi_t) \le e^{\int_0^t ds\mathcal{K}(\varphi_s,A_s)}\Big(&\alpha^<(\Psi_0,\varphi_0)\\ +\int_0^t &ds\mathcal{K}(\varphi_s,A_s)e^{-\int_0^s d\tau\mathcal{K}(\varphi_\tau,A_\tau)}N^{-\eta}\Big) .\end{aligned}$$

Therefore, we obtain part (b) of Theorem 2.1.

7 Rigorous estimates

7.1 Smearing out the potential W_β

In Section 5 we have defined the potential M_β to control the strongly peaked potential V_N. We will employ a similar strategy to "smear out" the potential W_β when β is large. For this, we define, for $\beta_1 < \beta$, a potential $U_{\beta_1,\beta} \in \mathcal{V}_{\beta_1}$ such that $\|W_\beta\|_1 = \|U_{\beta_1,\beta}\|_1$. Furthermore, define $h_{\beta_1,\beta}$ by $\Delta h_{\beta_1,\beta} = W_\beta - U_{\beta_1,\beta}$. The function $h_{\beta_1,\beta}$ can be thought as an electrostatic potential which is caused by the charge $W_\beta - U_{\beta_1,\beta}$. It is then possible to rewrite

$$\begin{aligned}&\langle\!\langle\chi, W_\beta(x_1-x_2)\Omega\rangle\!\rangle = \langle\!\langle\chi, U_{\beta_1,\beta}(x_1-x_2)\Omega\rangle\!\rangle\\ &-\langle\!\langle\nabla_1\chi,(\nabla_1 h_{\beta_1,\beta})(x_1-x_2)\Omega\rangle\!\rangle - \langle\!\langle\chi,(\nabla_1 h_{\beta_1,\beta})(x_1-x_2)\nabla_1\Omega\rangle\!\rangle ,\end{aligned}$$

for $\chi, \omega \in L_s^2(\mathbb{R}^{2N}, \mathbb{C})$. It is easy to verify that $h_{\beta_1,\beta}$ and $\nabla h_{\beta_1,\beta}$ are faster decaying than the potential W_β. The right hand side of the equation above is hence better to control, if one has additional control of $\nabla_1 \Omega$ and $\nabla_1 \chi$.

Definition 7.1 *For any $0 \leq \beta_1 < \beta$ and any $W_\beta \in \mathcal{V}_\beta$ we define*

$$U_{\beta_1,\beta}(x) = \begin{cases} \frac{4}{\pi}\|W_\beta\|_1 N^{2\beta_1} & \text{for } |x| < 1/2N^{-\beta_1}, \\ 0 & \text{else.} \end{cases}$$

and

$$h_{\beta_1,\beta}(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \ln|x-y|(W_\beta(y) - U_{\beta_1,\beta}(y))d^2y \; . \tag{48}$$

Lemma 7.2 *For any $0 \leq \beta_1 < \beta$ and any $W_\beta \in \mathcal{V}_\beta$, we obtain with the above definition*

(a)

$$\begin{aligned} &U_{\beta_1,\beta} \in \mathcal{V}_{\beta_1} \; , \\ &\Delta h_{\beta_1,\beta} = W_\beta - U_{\beta_1,\beta}. \end{aligned}$$

(b) Pointwise estimates

$$|h_{\beta_1,\beta}(x)| \leq C N^{-1}\ln(N), \qquad h_{\beta_1,\beta}(x) = 0 \text{ for } |x| \geq N^{-\beta_1}, \tag{49}$$

$$|\nabla h_{\beta_1,\beta}(x)| \leq C N^{-1}\left(|x|^2 + N^{-2\beta}\right)^{-\frac{1}{2}} . \tag{50}$$

(c) Norm estimates

$$\begin{aligned} \|h_{\beta_1,\beta}\|_\infty &\leq C N^{-1}\ln(N), \\ \|h_{\beta_1,\beta}\|_\lambda &\leq C N^{-1-\frac{2}{\lambda}\beta_1}\ln(N) \quad \text{for } 1 \leq \lambda \leq \infty, \\ \|\nabla h_{\beta_1,\beta}\|_\lambda &\leq C N^{-1+\beta-\frac{2}{\lambda}\beta_1} \qquad \text{for } 1 \leq \lambda \leq \infty. \end{aligned}$$

Furthermore, for $\lambda = 2$, we obtain the improved bounds

$$\|h_{0,\beta}\| \leq C N^{-1} \text{ for } \beta > 0 \; , \tag{51}$$

$$\|\nabla h_{\beta_1,\beta}\| \leq C N^{-1}(\ln(N))^{1/2} \; . \tag{52}$$

Proof:

(a) $U_{\beta_1,\beta} \in \mathcal{V}_{\beta_1}$ follows directly from the definition of $U_{\beta_1,\beta}$.
The second statement is a well known result of standard electrostatics (therefore recall that the radially symmetric Greens function of the Laplace operator in two dimensions is given by $-\frac{1}{2\pi}\ln|x-y|$). W_β can be understood as a given charge density. $-U_{\beta_1,\beta}$ then corresponds to a smeared out charge density of opposite sign such that the "total charge"

is zero. Hence, the "potential" h_{β,β_1} can be chosen to be zero outside the support of the total charge density.[2]

(b) First note that $|h_{\beta_1,\beta}(x)| = 0$ for $|x| \geq 1/2N^{-\beta_1}$, which implies the pointwise estimate

$$
\begin{aligned}
|h_{\beta_1,\beta}(x)| \leq & \frac{1}{2\pi} \int_{B_{1/2N^{-\beta_1}(0)}} d^2y \, |\ln|x-y| W_\beta(y)| \\
& + \frac{1}{2\pi} \int_{B_{1/2N^{-\beta_1}(0)}} d^2y \, |\ln|x-y| U_{\beta_1,\beta}(y)| \; .
\end{aligned}
$$

We estimate each term separately. For $RN^{-\beta} < |x|$, we obtain

$$
\int_{B_{1/2N^{-\beta_1}(0)}} d^2y |\ln|x-y||W_\beta(y) \leq C\|W_\beta\|_1 |\ln(|x| - RN^{-\beta})|,
$$

which in turn implies

$$
\int_{B_{1/2N^{-\beta_1}(0)}} d^2y |\ln|x-y||W_\beta(y) \leq C\|W_\beta\|_1 \ln N^\beta \leq CN^{-1}\ln(N)
$$

for all $2RN^{-\beta} \leq |x|$.

Let next $|x| \leq 2RN^{-\beta}$. Note that $|x-y| \leq 1$ in the integral above, using $h_{\beta_1,\beta}(x) = 0$, whenever $|x| > 1/2\beta_1$. This implies $|\ln|x-y|| = -\ln|x-y|$ in the integral. Thus,

$$
\begin{aligned}
& \int_{B_{N^{-\beta_1}(0)}} |\ln|x-y||W_\beta(y)d^2y \\
& \leq C\|W_\beta\|_\infty \int_{B_{RN^{-\beta}(0)}} -\ln|x-y|d^2y \\
& \leq CN^{-1+2\beta} \int_{B_{RN^{-\beta}(x)}} -\ln|y|d^2y \\
& \leq CN^{-1+2\beta} \int_{B_{4RN^{-\beta}(0)}} -\ln|y|d^2y \\
& = CN^{-1+2\beta} \Big[-|y|^2(2\ln|y| - 1) \Big]_0^{4RN^{-\beta}} \leq CN^{-1}\ln\left(N^\beta\right),
\end{aligned}
$$

Repeating these estimates for $U_{\beta_1,\beta}$ proves the first statement.
For the gradient, we estimate the two terms on the r.h.s. of

$$
|\nabla h_{\beta_1,\beta}(x)| \leq \frac{1}{2\pi} \int \frac{1}{|x-y|} W_\beta(y)d^2y + \frac{1}{2\pi} \int \frac{1}{|x-y|} U_{\beta,\beta_1}(y)d^2y
$$

[2] To see this, recall that the solution of $\Delta h(r) = \rho(r)$ for radially symmetric and regular enough charge density ρ is given by

$$
h(r) = \ln(r) \int_0^r r'\rho(r')dr' + \int_r^\infty \ln(r')\rho(r')r'dr' + C,
$$

where $C \in \mathbb{R}$. The r.h.s. is zero for $r \notin \mathrm{supp}(\rho)$ when the total charge vanishes $\int_0^\infty r\rho(r)dr = 0$ and C is chosen equal to zero.

separately. Let first $2RN^{-\beta} \leq |x|$. Similarly as in the previous argument, one finds

$$\int \frac{1}{|x-y|} W_\beta(y) d^2y \leq \int_{B_{RN^{-\beta}}(0)} \frac{1}{|x-y|} W_\beta(y) d^2y \leq \frac{\|W_\beta\|_1}{|x| - RN^{-\beta}}$$

for $RN^{-\beta} \leq |x|$, which implies that

$$\int \frac{1}{|x-y|} W_\beta(y) d^2y \leq \frac{C\|W_\beta\|_1}{(|x|^2 + N^{-2\beta})^{\frac{1}{2}}} \leq \frac{CN^{-1}}{(|x|^2 + N^{-2\beta})^{\frac{1}{2}}}$$

for all $2RN^{-\beta} \leq |x|$. For $|x| \leq 2RN^{-\beta}$, we make use of

$$N^\beta \leq \frac{C}{(|x|^2 + N^{-2\beta})^{1/2}}$$

and estimate

$$\begin{aligned}
\int \frac{1}{|x-y|} W_\beta(y) d^2y &\leq \|W_\beta\|_\infty \int_{B_{RN^{-\beta}(0)}} \frac{1}{|x-y|} d^2y \\
&\leq CN^{2\beta-1} \int_0^{RN^{-\beta}} d|y| = CN^{-1+\beta} \leq \frac{CN^{-1}}{(|x|^2 + N^{-2\beta})^{1/2}}.
\end{aligned}$$

Equivalently, we obtain

$$\begin{aligned}
\int \frac{1}{|x-y|} U_{\beta_1,\beta}(y) d^2y &\leq \|U_{\beta_1,\beta}\|_\infty \int_{B_{N^{-\beta_1}(0)}} \frac{1}{|x-y|} d^2y \\
&= CN^{-1+\beta_1} \leq \frac{CN^{-1}}{(|x|^2 + N^{-2\beta_1})^{1/2}} \leq \frac{CN^{-1}}{(|x|^2 + N^{-2\beta})^{\frac{1}{2}}},
\end{aligned}$$

for $|x| \leq N^{-\beta_1}$. Since $\nabla h_{\beta_1,\beta}(x) = 0$ for $|x| \geq N^{-\beta_1}$, the second statement of (b) follows.

(c) The first part of (c) follows from (b) and the fact that the support of $h_{\beta 1,\beta}$ and $\nabla h_{\beta_1,\beta}$ has radius $\leq CN^{-\beta_1}$. The bounds on the L^2-norm can be improved by

$$\begin{aligned}
\|\nabla h_{\beta_1,\beta}\|_2^2 \leq & C \int_0^{CN^{-\beta_1}} drr|\nabla h_{\beta_1,\beta}(r)|^2 \leq \frac{C}{N^2} \int_0^{CN^{-\beta_1}} dr \frac{r}{r^2 + N^{-2\beta}} \\
= & \frac{C}{N^2} \ln\left(\frac{N^{-2\beta_1} + N^{-2\beta}}{N^{-2\beta}}\right) \leq \frac{C}{N^2} \ln(N)
\end{aligned}$$

Using, for $|x| \geq 2RN^{-\beta}$, the inequality

$$|h_{0,\beta}(x)| \leq CN^{-1} |\ln(|x| - RN^{-\beta})|,$$

we obtain

$$\begin{aligned}
\|h_{0,\beta}\|_2^2 = & \int_{\mathbb{R}^2} d^2x \mathbb{1}_{B_{2RN^{-\beta}(0)}}(x) |h_{0,\beta}(x)|^2 + \int_{\mathbb{R}^2} d^2x \mathbb{1}_{B^c_{2RN^{-\beta}(0)}}(x) |h_{0,\beta}(x)|^2 \\
\leq & \|h_{0\beta}\|_\infty^2 |B_{2RN^{-\beta}(0)}| + CN^{-2} \int_{2RN^{-\beta}}^1 drr |\ln(r - RN^{-\beta})|^2 \\
\leq & C \left(N^{-2-2\beta} (\ln(N))^2 + N^{-2} \int_{RN^{-\beta}}^1 dr (r + RN^{-\beta})(\ln(r))^2 \right).
\end{aligned}$$

Using

$$
\begin{aligned}
&\int_{RN^{-\beta}}^{1} dr(r+RN^{-\beta})(\ln(r))^2 \\
=&\left(\frac{1}{4}r^2(2(\ln(r))^2-2\ln(r)+1)+RN^{-\beta}r((\ln(r))^2-2\ln(r)+2)\right)\Big|_{RN^{-\beta}}^{1} \\
\leq& C\left(1+N^{-\beta}+N^{-2\beta}(\ln(N))^2\right) ,
\end{aligned}
$$

we obtain, for any $\beta > 0$,

$$
\|h_{0,\beta}\|_2^2 \leq CN^{-2}\left(1+N^{-\beta}+N^{-2\beta}(\ln(N))^2\right) \leq CN^{-2} .
$$

□

7.2 Estimates on the cutoff

In order to smear out singular potentials as explained in the previous section and to obtain sufficient bounds, it seems at first necessary to show that $\|\nabla_1 q_1 \Psi_t\|$ decays in N. However, this term will in fact not be small for the dynamic generated by V_N. There, we rather expect that $\|\nabla_1 q_1 \Psi_t\| = \mathcal{O}(1)$ holds. It has been shown in [7] and [20] that the interaction energy is purely kinetic in the Gross-Pitaevskii regime, which implies that a relevant part of the kinetic energy is concentrated around the scattering centers. We must thus cutoff the part which is used to form the microscopic structure. For this, we define the set $\overline{\mathcal{A}}_j^{(d)}$ which includes all configurations where the distance between particle x_i and particle x_j , $j \neq i$ is smaller than N^{-d}. It is then possible to prove that the kinetic energy concentrated on the complement of $\overline{\mathcal{A}}_j^{(d)}$, i.e. $\|\mathbb{1}_{\mathcal{A}_1^{(d)}} \nabla_1 q_1 \Psi\|$, is small, see Lemma 7.9.

Definition 7.3 *For any $j,k = 1,\dots,N$ and $d > 0$ let*

$$
a_{j,k}^{(d)} = \{(x_1,x_2,\dots,x_N) \in \mathbb{R}^{2N} : |x_j - x_k| < N^{-d}\} \subseteq \mathbb{R}^{2N} \tag{53}
$$

$$
\overline{\mathcal{A}}_j^{(d)} = \bigcup_{k\neq j} a_{j,k}^{(d)} \qquad \mathcal{A}_j^{(d)} = \mathbb{R}^{2N} \backslash \overline{\mathcal{A}}_j^{(d)} \qquad \overline{\mathcal{B}}_j^{(d)} = \bigcup_{k\neq l\neq j} a_{k,l}^{(d)} \qquad \mathcal{B}_j^{(d)} = \mathbb{R}^{2N} \backslash \overline{\mathcal{B}}_j^{(d)} .
$$

Lemma 7.4 *Let $\Psi \in L_s^2(\mathbb{R}^{2N},\mathbb{C}) \cap H^1(\mathbb{R}^{2N},\mathbb{C})$ $\|\Psi\| = 1$ and let $\|\nabla_1 \Psi\|$ be uniformly bounded in N. Then, for all $j \neq k$ with $1 \leq j,k \leq N$,*

(a)

$$
\begin{aligned}
\|\mathbb{1}_{\overline{\mathcal{A}}_j^{(d)}} p_j\|_{op} &\leq C\|\varphi\|_\infty N^{1/2-d} , \\
\|\mathbb{1}_{\overline{\mathcal{A}}_j^{(d)}} \nabla_j p_j\|_{op} &\leq C\|\nabla\varphi\|_\infty N^{1/2-d} , \\
\|\mathbb{1}_{a_{j,k}^{(d)}} p_j\|_{op} &\leq C\|\varphi\|_\infty N^{-d} .
\end{aligned}
$$

(b) *For any* $1 < p < \infty$

$$\|\mathbb{1}_{\overline{\mathcal{A}}_j^{(d)}}\Psi\| \leq CN^{\frac{1-2d}{2}\frac{p-1}{p}} ,$$

which implies that

$$\|\mathbb{1}_{\overline{\mathcal{A}}_j^{(d)}}\Psi\| \leq CN^{\frac{1}{2}-d+\epsilon}$$

for any $\epsilon > 0$.

(c)

$$\|\mathbb{1}_{\overline{\mathcal{B}}_j^{(d)}}\Psi\| \leq CN^{1-d+\epsilon}$$

for any $\epsilon > 0$.

(d) *For any* $k \neq j$

$$\|[\mathbb{1}_{\overline{\mathcal{A}}_j^{(d)}}, p_k]\|_{op} = \|[\mathbb{1}_{a_{j,k}^{(d)}}, p_k]\|_{op} = \|[\mathbb{1}_{\mathcal{A}_j^{(d)}}, p_k]\|_{op} \leq C\|\varphi\|_\infty N^{-d} .$$

Proof:

(a) First note that the volume of the sets $a_{j,k}^{(d)}$ introduced in Definition 7.3 are $|a_{j,k}^{(d)}| = \pi N^{-2d}$.

$$\|\mathbb{1}_{\overline{\mathcal{A}}_j^{(d)}} p_j\|_{\text{op}} = \|\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} p_1\|_{\text{op}} = \|p_1 \mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} p_1\|_{\text{op}}^{\frac{1}{2}} \leq \left(\|\varphi\|_\infty^2 \|\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\|_{1,\infty}\right)^{1/2}$$

where we defined

$$\|f\|_{p,\infty} = \sup_{x_2,\ldots,x_N\in\mathbb{R}^2} \left(\int dx_1 |f(x_1,\ldots,x_N)|^p\right)^{\frac{1}{p}} .$$

Using $\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} \leq \sum_{k=2}^N \mathbb{1}_{a_{1,k}^{(d)}}$ as well as $\left(\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\right)^p = \mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}$, we obtain

$$\|\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\|_{p,\infty} \leq \sup_{x_2,\ldots,x_N\in\mathbb{R}^2} \left(\int dx_1 \sum_{k=2}^N \mathbb{1}_{a_{1,k}^{(d)}}\right)^{\frac{1}{p}} \leq (N|a_{1,k}|)^{\frac{1}{p}} \leq CN^{(1-2d)\frac{1}{p}} .$$

This implies

$$\|\mathbb{1}_{\overline{\mathcal{A}}_j^{(d)}} p_j\|_{\text{op}} \leq C\|\varphi\|_\infty N^{\frac{1}{2}-d} .$$

The second statement of (a) can be proven similarly. Analogously, we obtain

$$\|\mathbb{1}_{a_{j,k}^{(d)}} p_j\|_{\text{op}} \leq \|\varphi\|_\infty |a_{j,k}^{(d)}|^{1/2} \leq C\|\varphi\|_\infty N^{-d}.$$

(b) Without loss of generality, we can set $j=1$. Recall the two-dimensional Sobolev inequality, for $\varrho \in H^1(\mathbb{R}^2,\mathbb{C})$, $\|\varrho\|_m \leq C\|\nabla\varrho\|^{\frac{m-2}{m}}\|\varrho\|^{\frac{2}{m}}$ holds for any $2<m<\infty$. Using Hölder and Sobolev for the x_1-integration, we get, for $p>1$

$$\begin{aligned}
&\|\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\Psi\|^2 = \langle\!\langle \Psi, \mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\Psi\rangle\!\rangle = \int d^2x_2\dots d^2x_N \int d^2x_1 |\Psi(x_1,\dots,x_N)|^2 \mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}(x_1,\dots,x_N)\\
\leq&\|\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\|_{\frac{p}{p-1},\infty}\int d^2x_2\dots d^2x_N\left(\int d^2x_1|\Psi(x_1,\dots,x_N)|^{2p}\right)^{1/p}\\
\leq&CN^{(1-2d)\frac{p-1}{p}}\int d^2x_2\dots d^2x_N\left(\int d^2x_1|\nabla_1\Psi(x_1,\dots,x_N)|^2\right)^{\frac{p-1}{p}}\left(\int d^2\tilde{x}_1|\Psi(\tilde{x}_1,\dots,x_N)|^2\right)^{\frac{1}{p}} .
\end{aligned}$$

Using Hölder for the $x_2,\dots x_N$-integration with the conjugate pair $r=\frac{p}{p-1}$ and $s=p$, we obtain

$$\|\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\Psi\|^2 \leq CN^{(1-2d)\frac{p-1}{p}}\|\nabla_1\Psi\|^{2\frac{p-1}{p}}\|\Psi\|^{\frac{2}{p}} .$$

Using $\|\nabla_1\Psi\|<C$, (b) follows.

(c) We use that $\overline{\mathcal{B}}_j^{(d)} \subset \bigcup_{k=1}\overline{\mathcal{A}}_k^{(d)}$. Hence one can find pairwise disjoint sets $\mathcal{C}_k\subset\overline{\mathcal{A}}_k^{(d)}$, $k=1,\dots,N$ such that $\overline{\mathcal{B}}_j^{(d)}\subset\bigcup_{k=1}\mathcal{C}_k$. Since the sets $\mathcal{C}_k$ are pairwise disjoint, the $\mathbb{1}_{\mathcal{C}_k}\Psi$ are pairwise orthogonal and we get

$$\|\mathbb{1}_{\overline{\mathcal{B}}_j^{(d)}}\Psi\|^2 = \sum_{k=1}\|\mathbb{1}_{\mathcal{C}_k}\Psi\|^2 \leq \sum_{k=1}^N\|\mathbb{1}_{\overline{\mathcal{A}}_k^{(d)}}\Psi\|^2 .$$

(d)

$$\begin{aligned}
\|[\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}},p_2]\|_{\text{op}} \leq&\|[\mathbb{1}_{a_{1,2}},p_2]\|_{\text{op}} \leq \|\mathbb{1}_{a_{1,2}}p_2\|_{\text{op}}+\|p_2\mathbb{1}_{a_{1,2}}\|_{\text{op}}\\
\leq&2\|\varphi\|_\infty|a_{1,2}|^{\frac{1}{2}} \leq C\|\varphi\|_\infty N^{-d} .
\end{aligned}$$

□

7.3 Estimates for the functionals γ_a, $\gamma_a^<$ and $\gamma_b^<$

Control of γ_a and $\gamma_a^<$

Lemma 7.5 *For any multiplication operator $B: L^2(\mathbb{R}^2,\mathbb{C})\to L^2(\mathbb{R}^2,\mathbb{C})$ and any $\varphi\in L^2(\mathbb{R}^2,\mathbb{C})$ and any $\Psi\in L_s^2(\mathbb{R}^{2N},\mathbb{C})$ we have*

$$|\langle\!\langle\Psi,B(x_1)\Psi\rangle\!\rangle-\langle\varphi,B\varphi\rangle| \leq C\|B\|_\infty(\langle\!\langle\Psi,\widehat{n}^\varphi\Psi\rangle\!\rangle+N^{-\frac{1}{2}}) .$$

Proof: Using $1=p_1+q_1$,

$$\begin{aligned}
&\langle\!\langle\Psi,B(x_1)\Psi\rangle\!\rangle-\langle\varphi,B\varphi\rangle\\
=&\langle\!\langle\Psi,p_1B(x_1)p_1\Psi\rangle\!\rangle+2\Re\langle\!\langle\Psi,q_1B(x_1)p_1\Psi\rangle\!\rangle+\langle\!\langle\Psi,q_1B(x_1)q_1\Psi\rangle\!\rangle-\langle\varphi,B\varphi\rangle\\
\leq&\langle\varphi,B\varphi\rangle(\|p_1\Psi\|^2-1)+2\Re\langle\!\langle\Psi,\widehat{n}^{-1/2}q_1B(x_1)p_1\widehat{n}_1^{1/2}\Psi\rangle\!\rangle\\
&+\langle\!\langle\Psi,q_1B(x_1)q_1\Psi\rangle\!\rangle,
\end{aligned}$$

where we used Lemma 4.2 (c). Since $\|p_1\Psi\|^2 - 1 = \|q_1\Psi\|^2$ it follows that

$$\begin{aligned} |\langle\langle\Psi, B(x_1)\Psi\rangle\rangle - \langle\varphi, B\varphi\rangle| &\le C\|B\|_\infty \left(\langle\langle\Psi, \widehat{n}^2\Psi\rangle\rangle + \langle\langle\Psi, \widehat{n}_1\Psi\rangle\rangle + \langle\langle\Psi, \widehat{n}\Psi\rangle\rangle\right) \\ &\le C\|B\|_\infty(\langle\langle\Psi, \widehat{n}\Psi\rangle\rangle + N^{-\frac{1}{2}}) \ . \end{aligned} \tag{54}$$

□

Using Lemma 7.5, setting $B = \dot{A}_t$, we get

$$\gamma_a^<(\Psi_t, \varphi_t) = \gamma_a(\Psi_t, \varphi_t) \le C\|\dot{A}_t\|_\infty(\langle\langle\Psi_t, \widehat{n}^{\varphi_t}\Psi_t\rangle\rangle + N^{-\frac{1}{2}}) \ ,$$

which yields the first bound (40) in Lemma 6.5.

Control of $\gamma_b^<$ To control $\gamma_b^<$ we will first prove that $\|\nabla_1\Psi_t\|$ is uniformly bounded in N, if initially the energy per particle $\mathcal{E}_U(\Psi_0)$ is of order one.

Lemma 7.6 *Let $\Psi_0 \in L_s^2(\mathbb{R}^{2N}, \mathbb{C}) \cap H^2(\mathbb{R}^{2N}, \mathbb{C})$ with $\|\Psi_0\| = 1$. For any $U \in L^2(\mathbb{R}^2, \mathbb{R})$, $U(x) \ge 0$, let Ψ_t the unique solution to $i\partial_t\Psi_t = H_U\Psi_t$ with initial datum Ψ_0. Let $\mathcal{E}_U(\Psi_0) \le C$. Then*

$$\|\nabla_1\Psi_t\| \le \mathcal{K}(\varphi_t, A_t).$$

Proof: Using $\frac{d}{dt}\mathcal{E}_U(\Psi_t) \le \|\dot{A}_t\|_\infty$, we obtain $\mathcal{E}_U(\Psi_t) \le \mathcal{K}(\varphi_t, A_t)$. This yields

$$\|\nabla_1\Psi_t\|^2 \le \mathcal{K}(\varphi_t, A_t) - (N-1)\|\sqrt{U}\Psi_t\|^2 + \|A_t\|_\infty \le \mathcal{K}(\varphi_t, A_t).$$

□

Next, we control $\widehat{m}^a$ and $\widehat{m}^b$ which were defined in Definition 6.1. The difference $m(k) - m(k+1)$ and $m(k) - m(k+2)$ is of leading order given by the derivative of the function $m(k)$ – k understood as real variable – with respect to k. The k-derivative of $m(k)$ equals

$$m(k)' = \begin{cases} 1/(2\sqrt{kN}), & \text{for } k \ge N^{1-2\xi}; \\ 1/2(N^{-1+\xi}), & \text{else.} \end{cases} \tag{55}$$

It is then easy to show that, for any $j \in \mathbb{Z}$, there exists a $C_j < \infty$ such that

$$\widehat{m}_j^x \le C_j N^{-1}\widehat{n}^{-1} \text{ for } x \in \{a, b\} \tag{56}$$

$$\|\widehat{m}_j^x\|_{\text{op}} \le C_j N^{-1+\xi} \text{ for } x \in \{a, b\} \tag{57}$$

$$\|\widehat{n}\widehat{m}_j^x\|_{\text{op}} \le C_j N^{-1} \text{ for } x \in \{a, b\} \tag{58}$$

$$\|\widehat{r}\|_{\text{op}} \le \|\widehat{m}^a\|_{\text{op}} + \|\widehat{m}^b\|_{\text{op}} \le CN^{-1+\xi} \ . \tag{59}$$

The different terms we have to estimate for $\gamma_b^<$ are found in (38). In order to facilitate the notation, let $\widehat{w} \in \{N\widehat{m}_{-1}^a, N\widehat{m}_{-2}^b\}$. Then $w(k) < n(k)^{-1}$ and $\|\widehat{w}\|_{\text{op}} \le CN^\xi$ follows.

Lemma 7.7 *Let $\beta > 0$ and $W_\beta \in \mathcal{V}_\beta$. Let $\Psi \in L_s^2(\mathbb{R}^{2N}, \mathbb{C}) \cap H^2(\mathbb{R}^{2N}, \mathbb{C})$, $\|\Psi\| = 1$ and let $\|\nabla_1\Psi\| \le \mathcal{K}(\varphi, A_t)$. Let $w(k) < n(k)^{-1}$ and $\|\widehat{w}\|_{op} \le CN^\xi$ for some $\xi \ge 0$. Then,*

(a)

$$N\left|\langle\langle\Psi p_1 p_2 Z_\beta^\varphi(x_1, x_2) q_1 p_2 \widehat{w}\Psi\rangle\rangle\right| \le \mathcal{K}(\varphi, A_t)\left(N^{-1} + N^{-2\beta}\ln(N)\right) \ .$$

(b)

$$
\begin{aligned}
&N|\langle\!\langle\Psi, p_1p_2 W_\beta(x_1-x_2)\widehat{w}q_1q_2\Psi\rangle\!\rangle| \\
&\leq \mathcal{K}(\varphi,A_t)\left(\langle\!\langle\Psi,\widehat{n}\Psi\rangle\!\rangle + \inf_{\eta>0}\inf_{\beta>\beta_1>0}\left(N^{\eta-2\beta_1}\ln(N)^2 + \|\widehat{w}\|_{op}N^{-1+2\beta_1} + \|\widehat{w}\|_{op}^2 N^{-\eta}\right)\right) .
\end{aligned}
$$

(c)

$$
\begin{aligned}
N|\langle\!\langle\Psi p_1q_2 Z^\varphi_\beta(x_1,x_2)\widehat{w}q_1q_2\Psi\rangle\!\rangle| \leq \mathcal{K}(\varphi,A_t)\Big(\langle\!\langle\Psi,\widehat{n}\Psi\rangle\!\rangle + N^{-1/6}\ln(N) \\
+\inf\left\{\left|\mathcal{E}_{V_N}(\Psi)-\mathcal{E}^{GP}_{4\pi}(\varphi)\right|, \left|\mathcal{E}_{W_\beta}(\Psi)-\mathcal{E}^{GP}_{N\|W_\beta\|_1}(\varphi)\right| + N^{-2\beta}\ln(N)\right\}\Big) .
\end{aligned}
$$

Proof:

(a) In view of Lemma 4.4, we obtain

$$
\begin{aligned}
N\left|\langle\!\langle\Psi, p_1p_2 Z^\varphi_\beta(x_1,x_2)q_1p_2\widehat{w}\Psi\rangle\!\rangle\right| \leq& N\|p_1p_2Z^\varphi_\beta(x_1,x_2)q_1p_2\|_{\mathrm{op}}\|\widehat{n}\widehat{w}\Psi\| \\
\leq& CN\|p_1p_2Z^\varphi_\beta(x_1,x_2)q_1p_2\|_{\mathrm{op}} .
\end{aligned}
$$

$\|p_1p_2Z^\varphi_\beta(x_1,x_2)q_1p_2\|_{\mathrm{op}}$ can be estimated using $p_1q_1=0$ and (14):

$$
\begin{aligned}
&N\left\|p_1p_2\left(W_\beta(x_1-x_2)-\frac{N\|W_\beta\|_1}{N-1}|\varphi(x_1)|^2-\frac{N\|W_\beta\|_1}{N-1}|\varphi(x_2)|^2\right)q_1p_2\right\|_{\mathrm{op}} \\
&\leq \|p_1p_2(NW_\beta(x_1-x_2)-N\|W_\beta\|_1|\varphi(x_1)|^2)p_2\|_{\mathrm{op}} + C\|\varphi\|_\infty^2N^{-1} \\
&\leq \|\varphi\|_\infty\ \|N(W_\beta\star|\varphi|^2)-\|NW_\beta\|_1|\varphi|^2\| + C\|\varphi\|^2_\infty N^{-1} .
\end{aligned}
$$

Let h be given by

$$
h(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} d^2y \ln|x-y|NW_\beta(y) + \frac{1}{2\pi}\|NW_\beta\|_1\ln|x| ,
$$

which implies

$$
\Delta h(x) = NW_\beta(x) - \|NW_\beta\|_1\delta(x) .
$$

As above (see Lemma 7.2), we obtain $h(x)=0$ for $x\notin B_{RN^{-\beta}}(0)$, where $RN^{-\beta}$ is the radius of the support of W_β. Thus,

$$
\begin{aligned}
\|h\|_1 \leq& \frac{1}{2\pi}\int_{\mathbb{R}^2}d^2x\int_{\mathbb{R}^2}d^2y|\ln|x-y||\mathbb{1}_{B_{RN^{-\beta}(0)}}(x)NW_\beta(y) &(60)\\
&-\frac{1}{2\pi}N\|W_\beta\|_1\int_{\mathbb{R}^2}d^2x\ln(|x|)\mathbb{1}_{B_{RN^{-\beta}(0)}}(x) \leq CN^{-2\beta}\ln(N) &(61)
\end{aligned}
$$

Integration by parts and Young's inequality give that

$$
\begin{aligned}
&\|N(W_\beta\star|\varphi|^2)-\|NW_\beta\|_1|\varphi|^2\| = \|(\Delta h)\star|\varphi|^2\| \\
\leq&\|h\|_1\|\Delta|\varphi|^2\|_2 \leq \mathcal{K}(\varphi,A_t)N^{-2\beta}\ln(N) .
\end{aligned}
$$

Thus, we obtain the bound

$$
N\left|\langle\!\langle\Psi, p_1p_2Z^\varphi_\beta(x_1,x_2)q_1p_2\widehat{w}\Psi\rangle\!\rangle\right| \leq \mathcal{K}(\varphi,A_t)\left(N^{-1}+N^{-2\beta}\ln(N)\right) , \tag{62}
$$

which then proves (a).

(b) We will first consider $\beta < 1/2$.
Using Lemma 4.2 (c) and Lemma 4.6 with $O_{1,2} = q_2 W_\beta(x_1 - x_2) p_2$, $\Omega = N^{-1/2}(\widehat{w})^{1/2} q_1 \Psi$ and $\chi = N^{1/2} p_1 (\widehat{w}_2)^{1/2}\Psi$ we get

$$\begin{aligned}
&|\langle\langle \Psi, p_1 p_2 W_\beta(x_1 - x_2) q_1 q_2 \widehat{w} \Psi \rangle\rangle| \\
=&|\langle\langle \Psi, (\widehat{w})^{1/2} q_1 q_2 W_\beta(x_1 - x_2) p_1 p_2 (\widehat{w}_2)^{1/2} \Psi \rangle\rangle| \\
&\leq N^{-1} \left\| (\widehat{w})^{1/2} q_1 \Psi \right\|^2 + N \big| \langle\langle q_2 (\widehat{w}_2)^{1/2}\, \Psi, p_1 \sqrt{W_\beta}(x_1 - x_2) p_3 \sqrt{W_\beta}(x_1 - x_3) \\
&\qquad\qquad \sqrt{W_\beta}(x_1 - x_2) p_2 \sqrt{W_\beta}(x_1 - x_3) p_1 q_3 (\widehat{w}_2)^{1/2}\, \Psi \rangle\rangle \big| \\
&+ N(N-1)^{-1} \| q_2 W_\beta(x_1 - x_2) p_2 p_1 (\widehat{w}_2)^{1/2} \Psi \|^2 \\
&\leq N^{-1} \left\| (\widehat{w})^{1/2} q_1 \Psi \right\|^2 + N \| \sqrt{W_\beta}(x_1 - x_2) p_1 \|_{\mathrm{op}}^4 \, \| q_2 (\widehat{w}_2)^{1/2}\, \Psi \|^2 \\
&+ 2N(N-1)^{-1} \| p_1 q_2 (\widehat{w}_1)^{1/2} W_\beta(x_1 - x_2) p_2 p_1 \Psi \|^2 \\
&+ 2N(N-1)^{-1} \| q_1 q_2 (\widehat{w})^{1/2} W_\beta(x_1 - x_2) p_2 p_1 \Psi \|^2 \, .
\end{aligned}$$

With Lemma 4.2 (e) we get the bound

$$\begin{aligned}
\leq & N^{-1} \| (\widehat{w})^{1/2} \widehat{n} \Psi \|^2 + N \|\varphi\|_\infty^4 \|W_\beta\|_1^2 \, \| \widehat{n} (\widehat{w}_2)^{1/2}\, \Psi \|^2 \\
&+ 2N(N-1)^{-1} \|W_\beta\|^2 \|\varphi\|_\infty^2 \left(\|\widehat{w}_1\|_{\mathrm{op}} + \|\widehat{w}\|_{\mathrm{op}} \right) \, .
\end{aligned}$$

Note, that $\|W_\beta\|_1 \leq C N^{-1}$, $\|W_\beta\|^2 \leq C N^{-2+2\beta}$. Furthermore, using $\widehat{n} < \widehat{n}_2$, we have under the conditions on $\widehat{w}$

$$\| (\widehat{w})^{1/2} \widehat{n}_2 \Psi \| \leq \| (\widehat{w}_2)^{1/2} \widehat{n}_2 \Psi \| \leq \| (\widehat{n}_2)^{1/2} \Psi \| \leq \sqrt{\langle \Psi, \widehat{n} \Psi \rangle} + 2N^{-\frac{1}{2}} \, . \tag{63}$$

In total, we obtain

$$N |\langle\langle \Psi, p_1 p_2 W_\beta(x_1 - x_2) q_1 q_2 \widehat{w} \Psi \rangle\rangle| \leq \mathcal{K}(\varphi, A_t) \left(\langle\langle \Psi, \widehat{n} \Psi \rangle\rangle + \|\widehat{w}\|_{\mathrm{op}} N^{-1+2\beta} \right)$$

and we get (b) for the case $\beta < 1/2$.

b) for $1/2 \leq \beta$: We use $U_{\beta_1,\beta}$ from Definition 7.1 for some $0 < \beta_1 < 1/2$. We then obtain

$$\begin{aligned}
& N \langle\langle \Psi, p_1 p_2 W_\beta(x_1 - x_2) \widehat{w} q_1 q_2 \Psi \rangle\rangle \\
=& N \langle\langle \Psi, p_1 p_2 U_{\beta_1,\beta}(x_1 - x_2) \widehat{w} q_1 q_2 \Psi \rangle\rangle && (64) \\
+& N \langle\langle \Psi, p_1 p_2 \left(W_\beta(x_1 - x_2) - U_{\beta_1,\beta}(x_1 - x_2) \right) \widehat{w} q_1 q_2 \Psi \rangle\rangle && (65)
\end{aligned}$$

Term (64) has been controlled above. So we are left to control (65).
Let $\Delta h_{\beta_1,\beta} = W_\beta - U_{\beta_1,\beta}$. Integrating by parts and using that $\nabla_1 h_{\beta_1,\beta}(x_1 - x_2) = -\nabla_2 h_{\beta_1,\beta}(x_1 - x_2)$ gives

$$\begin{aligned}
& N \, |\langle\langle \Psi, p_1 p_2 \left(W_\beta(x_1 - x_2) - U_{\beta_1,\beta}(x_1 - x_2) \right) \widehat{w} q_1 q_2 \Psi \rangle\rangle| \\
& \leq N \, |\langle\langle \nabla_1 p_1 \Psi, p_2 \nabla_2 h_{\beta_1,\beta}(x_1 - x_2) \widehat{w} q_1 q_2 \Psi \rangle\rangle| && (66) \\
& + N \, |\langle\langle \Psi, p_1 p_2 \nabla_2 h_{\beta_1,\beta}(x_1 - x_2) \nabla_1 \widehat{w} q_1 q_2 \Psi \rangle\rangle| \, . && (67)
\end{aligned}$$

Let $t_1 \in \{p_1, \nabla_1 p_1\}$ and let $\Gamma \in \{\widehat{w} q_1 \Psi, \nabla_1 \widehat{w} q_1 \Psi\}$.

For both (66) and (67), we use Lemma 4.6 with $O_{1,2} = N^{1+\eta/2} q_2 \nabla_2 h_{\beta_1,\beta}(x_1 - x_2) p_2$, $\chi = t_1\Psi$ and $\Omega = N^{-\eta/2}\Gamma$. This yields

$$
\begin{aligned}
(66) + (67) \leq 2 \sup_{t_1 \in \{p_1, \nabla_1 p_1\}, \Gamma \in \{\widehat{w} q_1 \Psi, \nabla_1 \widehat{w} q_1 \Psi\}} \Big(N^{-\eta} \|\Gamma\|^2 \quad & (68) \\
+ \frac{N^{2+\eta}}{N-1} \|q_2 \nabla_2 h_{\beta_1,\beta}(x_1 - x_2) t_1 p_2 \Psi\|^2 \quad & (69) \\
+ N^{2+\eta} \left| \langle\!\langle \Psi, t_1 p_2 q_3 \nabla_2 h_{\beta_1,\beta}(x_1 - x_2) \nabla_3 h_{\beta_1,\beta}(x_1 - x_3) t_1 q_2 p_3 \Psi \rangle\!\rangle \right| \Big) . \quad & (70)
\end{aligned}
$$

The first term can be bounded using Corrolary 4.5 by

$$
\begin{aligned}
N^{-\eta} \|\nabla_1 \widehat{w} q_1 \Psi\|^2 \leq & N^{-\eta} \|\widehat{w}\|_{\text{op}}^2 \|\nabla_1 q_1 \Psi\|^2 \\
N^{-\eta} \|\widehat{w} q_1 \Psi\|^2 \leq & C N^{-\eta} .
\end{aligned}
$$

Thus (68) $\leq \mathcal{K}(\varphi, A_t) N^{-\eta} \|\widehat{w}\|_{\text{op}}^2$ using that $\|\nabla_1 q_1 \Psi\| \leq \mathcal{K}(\varphi, A_t)$. By $\|t_1\Psi\|^2 \leq \mathcal{K}(\varphi, A_t)$, we obtain

$$
\begin{aligned}
(69) \leq & \mathcal{K}(\varphi, A_t) \frac{N^{2+\eta}}{N-1} \|\nabla_2 h_{\beta_1,\beta}(x_1 - x_2) p_2\|_{\text{op}}^2 \leq \mathcal{K}(\varphi, A_t) \frac{N^{2+\eta}}{N-1} \|\varphi\|_\infty^2 \|\nabla h_{\beta_1,\beta}\|^2 \\
\leq & \mathcal{K}(\varphi, A_t) N^{\eta-1} \ln(N) ,
\end{aligned}
$$

where we used Lemma 7.2 in the last step.

Next, we estimate

$$
\begin{aligned}
(70) \leq & N^{2+\eta} \|p_2 \nabla_2 h_{\beta_1,\beta}(x_1 - x_2) t_1 q_2 \Psi\|^2 \\
\leq & 2N^{2+\eta} \|p_2 h_{\beta_1,\beta}(x_1 - x_2) t_1 \nabla_2 q_2 \Psi\|^2 \\
& + 2N^{2+\eta} \| |\varphi(x_2)\rangle \langle \nabla\varphi(x_2)| h_{\beta_1,\beta}(x_1 - x_2) t_1 q_2 \Psi\|^2 \\
\leq & 2N^{2+\eta} \|p_2 h_{\beta_1,\beta}(x_1 - x_2)\|_{\text{op}}^2 \|t_1 \nabla_2 q_2 \Psi\|^2 \\
& + 2N^{2+\eta} \| |\varphi(x_2)\rangle \langle \nabla\varphi(x_2)| h_{\beta_1,\beta}(x_1 - x_2)\|_{\text{op}}^2 \|t_1 q_2 \Psi\|^2 \\
\leq & \mathcal{K}(\varphi, A_t) N^{2+\eta} \|h_{\beta_1,\beta}\|^2 \\
\leq & \mathcal{K}(\varphi, A_t) N^{\eta - 2\beta_1} \ln(N)^2 .
\end{aligned}
$$

Thus, for all $\eta \in \mathbb{R}$

$$
\begin{aligned}
& N \langle\!\langle \Psi, p_1 p_2 \left(W_\beta(x_1 - x_2) - U_{\beta_1,\beta}(x_1 - x_2) \right) \widehat{w} q_1 q_2 \Psi \rangle\!\rangle \\
\leq & \mathcal{K}(\varphi, A_t) \left(\|\widehat{w}\|_{\text{op}}^2 N^{-\eta} + N^{\eta-1} \ln(N) + N^{\eta - 2\beta_1} \ln(N)^2 \right) .
\end{aligned}
$$

Combining both estimates for $\beta < 1/2$ and $\beta \geq 1/2$, we obtain, using $N^{\eta-1}\ln(N) < N^{\eta-2\beta_1}\ln(N)$,

$$
\begin{aligned}
& N \langle\!\langle \Psi, p_1 p_2 W_\beta(x_1 - x_2) \widehat{w} q_1 q_2 \Psi \rangle\!\rangle \\
& \leq \mathcal{K}(\varphi, A_t) \left(\langle\!\langle \Psi, \widehat{n} \Psi \rangle\!\rangle + \inf_{\eta > 0} \inf_{\beta_1 > 0} \left(N^{\eta - 2\beta_1} \ln(N)^2 + N^{-1+2\beta_1} + \|\widehat{w}\|_{\text{op}}^2 N^{-\eta} \right) \right) .
\end{aligned}
$$

and we get (b) in full generality.

(c) We first estimate, noting that $q_1 p_2 |\varphi|^2(x_1) q_1 q_2 = 0$,

$$N \left| \langle\!\langle \Psi, q_1 p_2 \frac{N \|W_\beta\|_1}{N-1} |\varphi|^2(x_2) \widehat{w} q_1 q_2 \Psi \rangle\!\rangle \right| \leq C \|\varphi\|_\infty^2 \|\widehat{w} q_2\|_{\text{op}} \|q_1 \Psi\|^2$$
$$\leq \mathcal{K}(\varphi, A_t) \langle\!\langle \Psi, \widehat{n} \Psi \rangle\!\rangle \, .$$

Is is left to estimate $N \left| \langle\!\langle \Psi, q_1 p_2 W_\beta(x_1 - x_2) \widehat{w} q_1 q_2 \Psi \rangle\!\rangle \right|$. Let $U_{0,\beta}$ be given as in Definition 7.1. Using Lemma 4.2 (c) and integrating by parts we get

$$\begin{aligned}
& N \left| \langle\!\langle \Psi, q_1 p_2 V_\beta(x_1 - x_2) \widehat{w} q_1 q_2 \Psi \rangle\!\rangle \right| \\
\leq & N \left| \langle\!\langle \Psi, q_1 p_2 U_{0,\beta}(x_1 - x_2) q_1 q_2 \widehat{w} \Psi \rangle\!\rangle \right| + N \left| \langle\!\langle \Psi, q_1 p_2 (\Delta_1 h_{0,\beta}(x_1 - x_2)) q_1 q_2 \widehat{w} \Psi \rangle\!\rangle \right| \\
\leq & \|U_{0,\beta}\|_\infty N \|q_1 \Psi\| \, \|\widehat{w} q_1 q_2 \Psi\| \\
& + N \left| \langle\!\langle \nabla_1 q_1 p_2 \Psi, (\nabla_1 h_{0,\beta}(x_1 - x_2)) \widehat{w} q_1 q_2 \Psi \rangle\!\rangle \right| \\
& + N \left| \langle\!\langle \Psi, \widehat{w}_1 q_1 p_2 (\nabla_1 h_{0,\beta}(x_1 - x_2)) \nabla_1 q_1 q_2 \Psi \rangle\!\rangle \right| \\
\leq & N \|U_{0,\beta}\|_\infty \|q_1 \Psi\| \, \|\widehat{w} q_1 q_2 \Psi\| & (71) \\
& + N \left| \langle\!\langle \mathbb{1}_{\mathcal{A}_1^{(d)}} \nabla_1 q_1 \Psi, p_2 (\nabla_1 h_{0,\beta}(x_1 - x_2)) \widehat{w} q_1 q_2 \Psi \rangle\!\rangle \right| & (72) \\
& + N \left| \langle\!\langle \nabla_1 q_1 \Psi, \mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} p_2 (\nabla_1 h_{0,\beta}(x_1 - x_2)) q_1 q_2 \widehat{w} \Psi \rangle\!\rangle \right| & (73) \\
& + N \left| \langle\!\langle \Psi, \widehat{w}_1 q_1 p_2 (\nabla_1 h_{0,\beta}(x_1 - x_2)) q_2 \mathbb{1}_{\mathcal{A}_1^{(d)}} \nabla_1 q_1 \Psi \rangle\!\rangle \right| & (74) \\
& + N \left| \langle\!\langle \Psi, \widehat{w}_1 q_1 p_2 (\nabla_1 h_{0,\beta}(x_1 - x_2)) q_2 \mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} \nabla_1 q_1 \Psi \rangle\!\rangle \right| \, . & (75)
\end{aligned}$$

Lemma 4.4 and Lemma 7.2 (a) yields the bound

$$(71) \leq C \langle\!\langle \Psi, \widehat{n} \Psi \rangle\!\rangle \, .$$

For (73) and (75) we use Cauchy Schwarz and then Sobolev inequality as in Lemma 7.4 to get, for any $p > 1$,

$$\begin{aligned}
& (73) + (75) \leq N \|\nabla_1 q_1 \Psi\| \left\| \mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} p_2 (\nabla_1 h_{0,\beta}(x_1 - x_2)) q_1 q_2 \widehat{w} \Psi \right\| \\
& \quad + N \|\nabla_1 q_1 \Psi\| \left\| \mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} q_2 (\nabla_1 h_{0,\beta}(x_1 - x_2)) q_1 p_2 \widehat{w}_1 \Psi \right\| \\
\leq & C N \|\nabla_1 q_1 \Psi\| \, N^{\frac{1-2d}{2} \frac{p-1}{p}} \|\nabla_1 p_2 (\nabla_1 h_{0,\beta}(x_1 - x_2)) q_1 q_2 \widehat{w} \Psi\|^{\frac{p-1}{p}} \|p_2 (\nabla_1 h_{0,\beta}(x_1 - x_2)) q_1 q_2 \widehat{w} \Psi\|^{1/p} \\
+ & C N \|\nabla_1 q_1 \Psi\| \, N^{\frac{1-2d}{2} \frac{p-1}{p}} \|\nabla_1 q_2 (\nabla_1 h_{0,\beta}(x_1 - x_2)) q_1 p_2 \widehat{w}_1 \Psi\|^{\frac{p-1}{p}} \|q_2 (\nabla_1 h_{0,\beta}(x_1 - x_2)) q_1 p_2 \widehat{w}_1 \Psi\|^{1/p} \, .
\end{aligned}$$

Using Lemma 4.2, Lemma 4.4, Corollary 4.5 and Lemma 7.2, we obtain

$$\begin{aligned}
\|\nabla_1 p_2 (\nabla_1 h_{0,\beta}(x_1 - x_2)) q_1 q_2 \widehat{w} \Psi\| \leq & \|p_2 (\Delta_1 h_{0,\beta}(x_1 - x_2)) q_1 q_2 \widehat{w} \Psi\| \\
& + \|p_2 (\nabla_1 h_{0,\beta}(x_1 - x_2)) \nabla_1 q_1 q_2 \widehat{w} \Psi\| \\
\leq & C \left(\|p_2 (W_\beta - U_{0,\beta})(x_1 - x_2)\|_{\text{op}} + \|p_2 \nabla_1 h_{0,\beta}(x_1 - x_2))\|_{\text{op}} \right) \\
\leq & C \|\varphi\|_\infty \left(N^{-1+\beta} + N^{-1} (\ln(N))^{1/2} \right) \, ,
\end{aligned}$$

and similarly

$$
\begin{aligned}
\|\nabla_1 q_2(\nabla_1 h_{0,\beta}(x_1-x_2))q_1p_2\widehat{w}_1\Psi\| \leq& \|q_2(\Delta_1 h_{0,\beta}(x_1-x_2))q_1p_2\widehat{w}_1\Psi\| \\
&+\|q_2(\nabla_1 h_{0,\beta}(x_1-x_2))\nabla_1 q_1p_2\widehat{w}_1\Psi\| \\
\leq& C\left(\|p_2(W_\beta-U_{0,\beta})(x_1-x_2)\|_{\mathrm{op}}+\|\widehat{w}_1\|_{\mathrm{op}}\|p_2\nabla_1 h_{0,\beta}(x_1-x_2))\|_{\mathrm{op}}\right) \\
\leq& C\|\varphi\|_\infty\left(N^{-1+\beta}+\|\widehat{w}\|_{\mathrm{op}}N^{-1}(\ln(N))^{1/2}\right) .
\end{aligned}
$$

Moreover, we estimate

$$
\begin{aligned}
\|p_2(\nabla_1 h_{0,\beta}(x_1-x_2))q_1q_2\widehat{w}\Psi\| &\leq C\|\varphi\|_\infty\|\nabla_1 h_{0,\beta}\|_2 \leq C\|\varphi\|_\infty N^{-1}(\ln(N))^{1/2} \\
\|q_2(\nabla_1 h_{0,\beta}(x_1-x_2))q_1p_2\widehat{w}\Psi\| &\leq C\|\varphi\|_\infty\|\nabla_1 h_{0,\beta}\|_2 \leq C\|\varphi\|_\infty N^{-1}(\ln(N))^{1/2} .
\end{aligned}
$$

Hence, we obtain, for any $p>1$,

$$
(73)+(75) \leq C\|\varphi\|_\infty N^{1+\frac{1-2d}{2}\frac{p-1}{p}}\left(N^{-1+\beta}+\|\widehat{w}\|_{\mathrm{op}}N^{-1}(\ln(N))^{1/2}\right)^{\frac{p-1}{p}}\left(N^{-1}(\ln(N))^{1/2}\right)^{1/p} .
$$

For d large enough, the right hand side can be bounded by N^{-1}, that is

$$
(73)+(75) \leq C\|\varphi\|_\infty N^{-1} .
$$

For (72) we use that $\nabla_2 h_{0,\beta}(x_1-x_2)=-\nabla_1 h_{0,\beta}(x_1-x_2)$, Cauchy Schwarz and $ab\leq a^2+b^2$ and get

$$
(72) \leq \|\mathbb{1}_{\mathcal{A}_1^{(d)}}\nabla_1 q_1\Psi\|^2+N^2\|p_2(\nabla_2 h_{0,\beta}(x_1-x_2))\widehat{w}q_1q_2\Psi\|^2 . \tag{76}
$$

$\|\mathbb{1}_{\mathcal{A}_1^{(d)}}\nabla_1 q_1\Psi\|^2$ can be bounded using Lemma 7.9.

Integration by parts and Lemma 4.2 (c) as well as $(a+b)^2\leq 2a^2+2b^2$ gives for the second summand

$$
\begin{aligned}
&N^2\|p_1(\nabla_1 h_{0,\beta}(x_1-x_2))q_1q_2\widehat{w}\Psi\|^2 \leq 2N^2\|p_1 h_{0,\beta}(x_1-x_2)\nabla_1 q_1q_2\widehat{w}\Psi\|^2 \\
&+2N^2\||\varphi(x_1)\rangle\langle\nabla_1\varphi(x_1)|h_{0,\beta}(x_1-x_2)q_1q_2\widehat{w}\Psi\|^2 \\
\leq& 2N^2\|p_1 h_{0,\beta}(x_1-x_2)q_2(p_1\widehat{w}_1+q_1\widehat{w})\mathbb{1}_{\mathcal{A}_1^{(d)}}\nabla_1 q_1\Psi\|^2 && (77)\\
&+2N^2\|p_1 h_{0,\beta}(x_1-x_2)q_2p_1\widehat{w}_1\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\nabla_1 q_1\Psi\|^2 && (78)\\
&+2N^2\|p_1 h_{0,\beta}(x_1-x_2)q_2q_1\widehat{w}\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\nabla_1 q_1\Psi\|^2 && (79)\\
&+2N^2\||\varphi(x_1)\rangle\langle\nabla_1\varphi(x_1)|h_{0,\beta}(x_1-x_2)q_1q_2\widehat{w}\Psi\|^2 . && (80)
\end{aligned}
$$

For (77) we use Lemma 4.4, Lemma 4.2 (e) with Lemma 7.2 (c) and then Lemma 7.9.

$$
\begin{aligned}
(77) \leq& CN^2\|p_1 h_{0,\beta}(x_1-x_2)\|_{\mathrm{op}}^2\|\mathbb{1}_{\mathcal{A}_1^{(d)}}\nabla_1 q_1\Psi\|^2 \\
\leq& \mathcal{K}(\varphi,A_t)\Big(\langle\Psi,\widehat{n}^\varphi\Psi\rangle+N^{-1/6}\ln(N) \\
&+\inf\left\{\left|\mathcal{E}_{V_N}(\Psi)-\mathcal{E}_{4\pi}^{GP}(\varphi)\right|,\left|\mathcal{E}_{W_\beta}(\Psi)-\mathcal{E}_{N\|W_\beta\|_1}^{GP}(\varphi)\right|+N^{-2\beta}\ln(N)\right\}\Big) .
\end{aligned}
$$

Let $s_1 \in \{p_1, q_1\}$ and let $\widehat{d} \in \{\widehat{w}, \widehat{w}_1\}$. Note that $\|\widehat{d}\|_{\text{op}} = \|\widehat{w}\|_{\text{op}}$. Then, (78) and (79) can be estimated as

$$\begin{aligned}
&(78),(79) \leq 2N^2 \|\nabla_1 q_1 \Psi\| \|\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} \widehat{d} s_1 q_2 h_{0,\beta}(x_1 - x_2) p_1 h_{0,\beta}(x_1 - x_2) q_2 s_1 \widehat{d} \mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} \nabla_1 q_1 \Psi\| \\
&\leq C N^{2 + \frac{1-2d}{2}\frac{p-1}{p}} \|\nabla_1 q_1 \Psi\| \|\nabla_1 \widehat{d} s_1 q_2 h_{0,\beta}(x_1 - x_2) p_1 h_{0,\beta}(x_1 - x_2) q_2 s_1 \widehat{d} \mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} \nabla_1 q_1 \Psi\|^{\frac{p-1}{p}} \\
&\times \|\widehat{d} s_1 q_2 h_{0,\beta}(x_1 - x_2) p_1 h_{0,\beta}(x_1 - x_2) q_2 s_1 \widehat{d} \mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} \nabla_1 q_1 \Psi\|^{\frac{1}{p}} \\
&\leq C N^{2 + \frac{1-2d}{2}\frac{p-1}{p}} \|\nabla_1 q_1 \Psi\| \|\widehat{w}\|_{\text{op}} \|p_1 h_{0,\beta}(x_1 - x_2)\|_{\text{op}} \|\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} \nabla_1 q_1 \Psi\| \\
&\times \|\nabla_1 \widehat{d} s_1 q_2 h_{0,\beta}(x_1 - x_2) p_1\|_{\text{op}}^{\frac{p-1}{p}} \|\widehat{d} s_1 q_2 h_{0,\beta}(x_1 - x_2) p_1\|_{\text{op}}^{\frac{1}{p}} \\
&\leq \mathcal{K}(\varphi, A_t) N^{1 + \frac{1-2d}{2}\frac{p-1}{p}} \|\widehat{w}\|_{\text{op}}^2 \|\nabla_1 s_1 h_{0,\beta}(x_1 - x_2) p_1\|_{\text{op}}^{\frac{p-1}{p}} \|h_{0,\beta}(x_1 - x_2) p_1\|_{\text{op}}^{\frac{1}{p}} \\
&\leq \mathcal{K}(\varphi, A_t) N^{1 + \frac{1-2d}{2}\frac{p-1}{p}} \|\widehat{w}\|_{\text{op}}^2 \left(\|\nabla\varphi\| \|\nabla_1 h_{0,\beta}\| + \|h_{0,\beta}\|\right)^{\frac{p-1}{p}} \|h_{0,\beta}\|^{\frac{1}{p}} \\
&\leq \mathcal{K}(\varphi, A_t) \|\widehat{w}\|_{\text{op}}^2 (1 + \ln(N))^{\frac{p-1}{2p}} N^{\frac{1-2d}{2}\frac{p-1}{p}} \; .
\end{aligned}$$

Here, we used, for $s_1 \in \{p_1, 1 - p_1\}$,

$$\begin{aligned}
&\|\nabla_1 s_1 h_{0,\beta}(x_1 - x_2) p_1\|_{\text{op}} \leq \|\nabla_1 p_1 h_{0,\beta}(x_1 - x_2) p_1\|_{\text{op}} + \|\nabla_1 h_{0,\beta}(x_1 - x_2) p_1\|_{\text{op}} \\
\leq &\|\varphi\|_\infty \left(\|\nabla\varphi\| \|h_{0,\beta}\| + \|\nabla h_{0,\beta}\|\right)
\end{aligned}$$

and then applied Lemma 4.2 (e).

For d large enough, we obtain

$$(78) + (79) \leq \mathcal{K}(\varphi, A_t) N^{-2} \; .$$

Line (80) can be bounded by

$$\begin{aligned}
(80) \leq &N^2 \|h_{0,\beta}(x_1 - x_2) \nabla_1 p_1\|_{\text{op}}^2 \; \|q_1 q_2 \widehat{w} \Psi\|^2 \leq N^2 \|h_{0,\beta}\|^2 \|\nabla\varphi\|_\infty^2 \|q_1 \widehat{w}\|_{\text{op}}^2 \|q_1 \Psi\|^2 \\
\leq &C \|\nabla\varphi\|_\infty^2 \langle\!\langle \Psi \widehat{n} \Psi \rangle\!\rangle \; .
\end{aligned}$$

For (74) we use Lemma 4.6 with $\Omega = \mathbb{1}_{\mathcal{A}_1^{(d)}} \nabla_1 q_1 \Psi$,
$O_{1,2} = N q_2 (\nabla_2 h_{0,\beta}(x_1 - x_2)) p_2$ and $\chi = \widehat{w} q_1 \Psi$.

$$(74) \leq \|\mathbb{1}_{\mathcal{A}_1^{(d)}} \nabla_1 q_1 \Psi\|^2 \tag{81}$$
$$+ 2N \|q_2 (\nabla_2 h_{0,\beta}(x_1 - x_2)) \widehat{w} q_1 p_2 \Psi\|^2 \tag{82}$$
$$+ N^2 \left| \langle\!\langle \Psi, q_1 q_3 \widehat{w} (\nabla_2 h_{0,\beta}(x_1 - x_2)) p_2 p_3 (\nabla_3 h_{0,\beta}(x_1 - x_3)) \widehat{w} q_1 q_2 \Psi \rangle\!\rangle \right| \; . \tag{83}$$

Line (82) is bounded by

$$\begin{aligned}
(82) \leq &C \|\varphi\|_\infty^2 N \|(\nabla_2 h_{0,\beta}(x_1 - x_2)) p_2\|_{\text{op}}^2 \|\widehat{w} q_1\|_{\text{op}}^2 \\
\leq &C \|\varphi\|_\infty^2 N \|\nabla_2 h_{0,\beta}(x_1 - x_2)\|^2 \leq C \|\varphi\|_\infty^2 N^{-1} \ln(N) \; .
\end{aligned}$$

(81)+(83) is bounded by

$$\|\mathbb{1}_{\mathcal{A}_1^{(d)}} \nabla_1 q_1 \Psi\|^2 + N^2 \|p_2 (\nabla_2 h_{0,\beta}(x_1 - x_2)) \widehat{w} q_1 q_2 \Psi\|^2 \; .$$

Both terms have been controlled above (see (76)). In total, we obtain

$$
\begin{aligned}
N|\langle\!\langle \Psi p_1 q_2 Z_\beta^\varphi(x_1,x_2)\widehat{w} q_1 q_2 \Psi\rangle\!\rangle| \le \mathcal{K}(\varphi, A_t)\Big(\langle\!\langle \Psi, \widehat{n}\Psi\rangle\!\rangle + N^{-1/6}\ln(N) \\
+ \inf\left\{\left|\mathcal{E}_{V_N}(\Psi) - \mathcal{E}^{GP}_{4\pi}(\varphi)\right|, \left|\mathcal{E}_{W_\beta}(\Psi) - \mathcal{E}^{GP}_{N\|W_\beta\|_1}(\varphi)\right| + N^{-2\beta}\ln(N)\right\}\Big) .
\end{aligned}
$$

□

Using this Lemma, it follows that there exists an $\eta > 0$ such that

$$
\gamma_b^<(\Psi_t, \varphi_t) \le \mathcal{K}(\varphi_t, A_t)\left(\langle\!\langle \Psi_t, \widehat{n}^{\varphi_t}\Psi_t\rangle\!\rangle + N^{-\eta} + \left|\mathcal{E}_{W_\beta}(\Psi_0) - \mathcal{E}^{GP}_{N\|W_\beta\|_1}(\varphi_0)\right|\right) .
$$

This proves Lemma 6.5.

7.4 Estimates for the functional γ

For the most involved scaling which is induced by V_N, we need to control $\|p_1 V_N \Psi\|$.

Lemma 7.8 *Let $\Psi \in L^2_s(\mathbb{R}^{2N}, \mathbb{C})$ and let $\mathcal{E}_{V_N}(\Psi) \le C$. Then*

$$
\|p_1 V_N \Psi\| \le \mathcal{K}(\varphi, A_t) N^{-\frac{1}{2}} . \tag{84}
$$

Proof: We estimate

$$
\begin{aligned}
&\|p_1 V_N(x_1 - x_2)\Psi\| = \|p_1 \mathbb{1}_{\mathrm{supp}(V_N)}(x_1 - x_2) V_N(x_1 - x_2)\Psi\| \\
\le &\|p_1 \mathbb{1}_{\mathrm{supp}(V_N)}(x_1 - x_2)\|_{\mathrm{op}} \|V_N(x_1 - x_2)\Psi\| .
\end{aligned}
$$

We have

$$
\|p_1 \mathbb{1}_{\mathrm{supp}(V_N)}(x_1 - x_2)\|^2_{\mathrm{op}} \le \|\varphi\|^2_\infty \|\mathbb{1}_{\mathrm{supp}(V_N)}\|_1 \le C\|\varphi\|^2_\infty e^{-2N} .
$$

Using

$$
C \ge \mathcal{E}_{V_N}(\Psi) = \|\nabla\Psi\|^2 + (N-1)\|\sqrt{V_N}(x_1 - x_2)\Psi\|^2 + \langle\!\langle \Psi, A_t(x_1)\Psi\rangle\!\rangle
$$

as well as

$$
\begin{aligned}
\|V_N(x_1 - x_2)\Psi\|^2 = &\|\sqrt{V_N}(x_1 - x_2)\sqrt{V_N}(x_1 - x_2)\Psi\|^2 \le \|\sqrt{V_N}\|^2_\infty \|\sqrt{V_N}(x_1 - x_2)\Psi\|^2 \\
\le &Ce^{2N}\frac{\mathcal{E}_{V_N}(\Psi) + \|A_t\|_\infty}{N} \le C(1 + \|A_t\|_\infty)\frac{e^{2N}}{N} ,
\end{aligned}
$$

we obtain

$$
\|p_1 V_N \Psi\| \le \mathcal{K}(\varphi, A_t) N^{-\frac{1}{2}} .
$$

□

Control of γ_b Recall that

$$\gamma_b(\Psi,\varphi) = -N(N-1)\Im\left(\langle\!\langle \Psi, \widetilde{Z}^{\varphi}_{\beta}(x_1,x_2)\widehat{r}\,\Psi\rangle\!\rangle\right)$$
$$- N(N-1)\Im\left(\langle\!\langle \Psi, g_\beta(x_1-x_2)\widehat{r}\,\mathcal{Z}^{\varphi}(x_1,x_2)\Psi\rangle\!\rangle\right) \ .$$

Estimate (84) yields to the bound $\|p_1\mathcal{Z}^{\varphi}(x_1,x_2)\Psi\| \leq \mathcal{K}(\varphi,A_t)N^{-1/2}$. Therefore, the second line of γ_b is controlled by

$$\begin{aligned}&N^2\|g_\beta(x_1-x_2)p_1\|_{\mathrm{op}}\|\widehat{r}\|_{\mathrm{op}}\|p_1\mathcal{Z}^{\varphi}(x_1,x_2)\Psi\| \\ \leq &\mathcal{K}(\varphi,A_t)N^{3/2}\|g_\beta\|\|\widehat{r}\|_{\mathrm{op}} \leq \mathcal{K}(\varphi,A_t)N^{\xi-1/2-\beta/2}\ln(N) \ .\end{aligned}$$

The first line of γ_b can be bounded with (44) and $f_\beta = 1-g_\beta$ by

$$N(N-1)|\Im\left(\langle\!\langle \Psi, \widetilde{Z}^{\varphi}_{\beta}(x_1,x_2)\widehat{r}\,\Psi\rangle\!\rangle\right)|$$
$$\leq N^2|\Im\left(\langle\!\langle \Psi, \left(M_\beta(x_1-x_2)f_\beta(x_1-x_2) - \frac{N}{N-1}\left(\|M_\beta f_\beta\|_1|\varphi(x_1)|^2 + \|M_\beta f_\beta\|_1|\varphi(x_2)|^2\right)\right)\widehat{r}\Psi\rangle\!\rangle\right)| \tag{85}$$
$$+\frac{N^2}{N-1}|\langle\!\langle \Psi, \left(\|NM_\beta f_\beta\|_1 - 4\pi\right)\left(|\varphi(x_1)|^2 + |\varphi(x_2)|^2\right)\widehat{r}\Psi\rangle\!\rangle| \tag{86}$$
$$+\frac{N^2}{N-1}|\langle\!\langle \Psi, \left(4\pi|\varphi(x_1)|^2 + 4\pi|\varphi(x_2)|^2\right)g_\beta(x_1-x_2)\widehat{r}\Psi\rangle\!\rangle| \ . \tag{87}$$

Since $M_\beta f_\beta \in \mathcal{V}_\beta$, (85) is of the same form as $\gamma_b^<(\Psi,\varphi)$. Using Lemma 5.5 (h), the second term is controlled by

$$(86) \leq C\|\varphi\|_\infty^2 N\left(N\|M_\beta f_\beta\|_1 - 4\pi\right)\|\hat{r}\|_{\mathrm{op}} \leq C\|\varphi\|_\infty^2 N^{-1+\xi}\ln(N) \ .$$

The last term is controlled by

$$(87) \leq CN\|\varphi\|_\infty^2\|g_\beta(x_1-x_2)p_1\|_{\mathrm{op}}\|\widehat{r}\|_{\mathrm{op}} \leq C\|\varphi\|_\infty^3 N^{-1-\beta+\xi}\ln(N) \ .$$

and we get

$$|\gamma_b(\Psi,\varphi)| \leq \mathcal{K}(\varphi,A_t)\left(\langle\!\langle \Psi,\widehat{m}\Psi\rangle\!\rangle + |\mathcal{E}_{V_N}(\Psi) - \mathcal{E}^{GP}_{4\pi}(\varphi)| + N^{-\eta}\right)$$

for some $\eta > 0$.

Control of γ_c Recall that

$$\gamma_c(\Psi,\varphi) = -\,4N(N-1)\langle\!\langle \Psi, (\nabla_1 g_\beta(x_1-x_2))\nabla_1\widehat{r}\Psi\rangle\!\rangle \ .$$

Using $\widehat{r} = (p_2+q_2)\widehat{r} = p_2\widehat{r} + p_1q_2\widehat{m}^a$ and $\nabla_1 g_\beta(x_1-x_2) = -\nabla_2 g_\beta(x_1-x_2)$, integration by parts yields to

$$|\gamma_c(\Psi,\varphi)| \leq 4N^2|\langle\!\langle \Psi, g_\beta(x_1-x_2)\nabla_1\nabla_2(p_2\widehat{r} + p_1q_2\widehat{m}^a)\Psi\rangle\!\rangle| \tag{88}$$
$$+ 4N^2|\langle\!\langle \nabla_2\Psi, g_\beta(x_1-x_2)\nabla_1 p_2\widehat{r}\Psi\rangle\!\rangle| \tag{89}$$
$$+ 4N^2|\langle\!\langle \nabla_2\Psi, g_\beta(x_1-x_2)\nabla_1 p_1q_2\widehat{m}^a\Psi\rangle\!\rangle| \ . \tag{90}$$

We begin with

$$\begin{aligned}(88) \leq & CN^2\|g_\beta\|\|\nabla\varphi\|_\infty \left(\|\nabla_1\widehat{r}\psi\| + \|\nabla_2 q_2\widehat{m}^a\Psi\|\right)\\ \leq & CN^{1-\beta}\ln(N)\|\nabla\varphi\|_\infty \left(\|\nabla_1\widehat{r}\psi\| + \|\nabla_2 q_2\widehat{m}^a\Psi\|\right) \ .\end{aligned}$$

Let $s_1, t_1 \in \{p_1, q_1\}$, $s_2, t_2 \in \{p_2, q_2\}$. Inserting the identity $1 = (p_1+q_1)(p_2+q_2)$, we obtain, for $a \in \{-2,-1,0,1,2\}$,

$$\begin{aligned}\|\nabla_1\widehat{r}\Psi\| \leq & C \sup_{s_1,s_2,t_1,t_2,a} \|\widehat{r}_a s_1 s_2 \nabla_1 t_1 t_2 \Psi\| \leq C \sup_{t_1,a} \|\widehat{r}_a\|_{\mathrm{op}}\|\nabla_1 t_1\Psi\|\\ \leq & CN^{-1+\xi} \ .\end{aligned}$$

In analogy $\|\nabla_2 q_2 \widehat{m}^a\Psi\| \leq C\|\widehat{m}^a\|_{\mathrm{op}} \leq CN^{-1+\xi}$. This yields the bound

$$(88) \leq \mathcal{K}(\varphi, A_t)N^{-\beta+\xi}\ln(N) \ .$$

Furthermore, (89) is bounded by

$$(89) \leq 4N^2\|\nabla_2\Psi\| \ \|g_\beta\| \ \|\nabla\varphi\|_\infty\|\nabla_1\widehat{r}\Psi\| \leq C\|\nabla\varphi\|_\infty \ N^{\xi-\beta}\ln(N) \ . \tag{91}$$

Similarly, we obtain

$$(90) \leq 4N^2\|\nabla_2\Psi\| \ \|g_\beta\| \ \|\nabla\varphi\|_\infty\|q_2\widehat{m}^a\Psi\| \leq C\|\nabla\varphi\|_\infty \ N^{\xi-\beta}\ln(N) \ .$$

It follows that $|\gamma_c(\Psi,\varphi)| \leq \mathcal{K}(\varphi, A_t)N^{\xi-\beta}\ln(N)$.

Control of γ_d To control γ_d and γ_e we will use the notation

$$\begin{aligned}m^c(k) = m^a(k) - m^a(k+1) \quad m^d(k) = m^a(k) - m^a(k+2)\\ m^e(k) = m^b(k) - m^b(k+1) \quad m^f(k) = m^b(k) - m^b(k+2) \ .\end{aligned} \tag{92}$$

Since the second k-derivative of m is given by (see (55) for the first derivative)

$$m(k)'' = \begin{cases} -1/(4\sqrt{k^3N}), & \text{for } k \geq N^{1-2\xi};\\ 0, & \text{else.}\end{cases}$$

it is easy to verify that

$$\|\widehat{m}^x_j\|_{\mathrm{op}} \leq CN^{-2+3\xi} \text{ for } x \in \{c,d,e,f\} \ . \tag{93}$$

Recall that

$$\begin{aligned}\gamma_d(\Psi,\varphi) = & 2N(N-1)(N-2)\Im\left(\langle\!\langle \Psi, g_\beta(x_1-x_2)\left[V_N(x_1-x_3), \widehat{r}\right]\Psi\rangle\!\rangle\right)\\ & N(N-1)(N-2)\Im\left(\langle\!\langle \Psi, g_\beta(x_1-x_2)\left[4\pi|\varphi|^2(x_3), \widehat{r}\right]\Psi\rangle\!\rangle\right) \ .\end{aligned}$$

Since $p_j + q_j = 1$, we can rewrite $\widehat{r}$ as

$$\widehat{r} = \widehat{m}^b p_1 p_2 + \widehat{m}^a(p_1 q_2 + q_1 p_2) = (\widehat{m}^b - 2\widehat{m}^a)p_1 p_2 + \widehat{m}^a(p_1+p_2) \ .$$

Thus,

$$
\begin{aligned}
|\gamma_d(\Psi,\varphi)| \leq & CN^3 \left| \langle\!\langle \Psi, g_\beta(x_1-x_2) \left[V_N(x_1-x_3), (\widehat{m}^b - 2\widehat{m}^a)p_1p_2 + \widehat{m}^a(p_1+p_2) \right] \Psi \rangle\!\rangle \right| \\
& + CN^3 \left| \langle\!\langle \Psi, g_\beta(x_1-x_2) \left[4\pi|\varphi|^2(x_3), \widehat{r} \right] \Psi \rangle\!\rangle \right| \\
\leq & CN^3 \left| \langle\!\langle \Psi, g_\beta(x_1-x_2)p_2 \left[V_N(x_1-x_3), \widehat{m}^a \right] \Psi \rangle\!\rangle \right| \qquad (94) \\
& + CN^3 \left| \langle\!\langle \Psi, g_\beta(x_1-x_2)V_N(x_1-x_3)(\widehat{m}^b - 2\widehat{m}^a)p_1p_2\Psi \rangle\!\rangle \right| \qquad (95) \\
& + CN^3 \left| \langle\!\langle \Psi, g_\beta(x_1-x_2)(\widehat{m}^b - 2\widehat{m}^a)p_1p_2V_N(x_1-x_3)\Psi \rangle\!\rangle \right| \qquad (96) \\
& + CN^3 \left| \langle\!\langle \Psi, g_\beta(x_1-x_2)\widehat{m}^a p_1 V_N(x_1-x_3)\Psi \rangle\!\rangle \right| \qquad (97) \\
& + CN^3 \left| \langle\!\langle \Psi, g_\beta(x_1-x_2)V_N(x_1-x_3)\widehat{m}^a p_1 \Psi \rangle\!\rangle \right| \qquad (98) \\
& + CN^3 \left| \langle\!\langle \Psi, g_\beta(x_1-x_2) \left[4\pi|\varphi|^2(x_3), \widehat{r} \right] \Psi \rangle\!\rangle \right| \ . \qquad (99)
\end{aligned}
$$

Using Lemma 4.2 (d), we obtain the following estimate:

$$
\begin{aligned}
(94) = & CN^3 \left| \langle\!\langle \Psi, g_\beta(x_1-x_2)p_2 \left[V_N(x_1-x_3), p_1p_3\widehat{m}^d + p_1q_3\widehat{m}^c + q_1p_3\widehat{m}^c \right] \Psi \rangle\!\rangle \right| \\
\leq & CN^3 \big| \langle\!\langle \Psi, V_N(x_1-x_3) g_\beta(x_1-x_2)p_2 \mathbb{1}_{\mathrm{supp}(V_N)}(x_1-x_3) \\
& \qquad \left(p_1p_3\widehat{m}^d + p_1q_3\widehat{m}^c + q_1p_3\widehat{m}^c \right) \Psi \rangle\!\rangle \big| \\
& + CN^3 \left| \langle\!\langle \Psi, g_\beta(x_1-x_2)p_2 \left(p_1p_3\widehat{m}^d + p_1q_3\widehat{m}^c + q_1p_3\widehat{m}^c \right) V_N(x_1-x_3)\Psi \rangle\!\rangle \right| \ .
\end{aligned}
$$

Both lines are bounded by

$$
\begin{aligned}
& CN^3 \|V_N(x_1-x_3)\Psi\| \ \|g_\beta(x_1-x_2)p_2\|_{\mathrm{op}} \\
& \left(2\|\mathbb{1}_{\mathrm{supp}(V_N)}(x_1-x_3)p_1\|_{\mathrm{op}} + \|\mathbb{1}_{\mathrm{supp}(V_N)}(x_1-x_3)p_3\|_{\mathrm{op}} \right) \left(\|\widehat{m}^d\|_{\mathrm{op}} + \|\widehat{m}^c\|_{\mathrm{op}} \right) \ .
\end{aligned}
$$

In view of Lemma 4.2 (e) with Lemma 5.5 (i), $\|g_\beta(x_1-x_2)p_2\|_{\mathrm{op}} \leq \|\varphi\|_\infty \|g_\beta\| \leq C\|\varphi\|_\infty N^{-1-\beta}\ln(N)$. Using (93), together with $\|\mathbb{1}_{\mathrm{supp}(V_N)}(x_1-x_3)p_1\|_{\mathrm{op}} \|V_N(x_1-x_3)\Psi\| \leq \mathcal{K}(\varphi, A_t)N^{-1/2}$, we obtain, using $\xi < 1/2$,

$$
(94) \leq \mathcal{K}(\varphi, A_t)N^{-1/2+3\xi-\beta}\ln(N) \leq \mathcal{K}(\varphi, A_t)N^{1/2+\xi-\beta}\ln(N) \ .
$$

We continue with

$$
\begin{aligned}
& (95) + (96) + (97) \\
\leq & CN^3 \|V_N(x_1-x_3)\Psi\| \|g_\beta(x_1-x_2)p_2\|_{\mathrm{op}} \\
& \times \|\mathbb{1}_{\mathrm{supp}(V_N)}(x_1-x_3)p_1\|_{\mathrm{op}} \|(\widehat{m}^b - 2\widehat{m}^a)\|_{\mathrm{op}} \\
& + CN^3 \|g_\beta(x_1-x_2)p_2\|_{\mathrm{op}} \|\widehat{m}^b - 2\widehat{m}^a\|_{\mathrm{op}} \|p_1 V_N(x_1-x_3)\Psi\| \\
& + CN^3 \|g_\beta(x_1-x_2)p_1\|_{\mathrm{op}} \|\widehat{m}^a\|_{\mathrm{op}} \|p_1 V_N(x_1-x_3)\Psi\| \\
\leq & \mathcal{K}(\varphi, A_t)N^{1/2+\xi-\beta}\ln(N) \ .
\end{aligned}
$$

Next, we estimate (98). The support of the function $g_\beta(x_1-x_2)V_N(x_1-x_3)$ is such that $|x_1-x_2| \leq CN^{-\beta}$, as well as $|x_1-x_3| \leq Ce^{-N}$. Therefore, $g_\beta(x_1-x_2)V_N(x_1-x_3) \neq 0$ implies

$|x_2 - x_3| \leq CN^{-\beta}$. We estimate

$$
\begin{aligned}
(98) =& CN^3 \left| \langle\!\langle \Psi, g_\beta(x_1 - x_2) V_N(x_1 - x_3) p_1 \mathbb{1}_{B_{CN^{-\beta}}(0)}(x_2 - x_3) \widehat{m}^a \Psi \rangle\!\rangle \right| \\
\leq& CN^3 \| p_1 V_N(x_1 - x_3) g_\beta(x_1 - x_2) \Psi \| \| \mathbb{1}_{B_{CN^{-\beta}}(0)}(x_2 - x_3) \widehat{m}^a \Psi \| \\
\leq& CN^3 \| p_1 \mathbb{1}_{\mathrm{supp}(V_N)}(x_1 - x_3) \|_{\mathrm{op}} \| g_\beta(x_1 - x_2) V_N(x_1 - x_3) \Psi \| \| \mathbb{1}_{B_{CN^{-\beta}}(0)}(x_2 - x_3) \widehat{m}^a \Psi \| \\
\leq& CN^{5/2} \| g_\beta \|_\infty \| \mathbb{1}_{B_{CN^{-\beta}}(0)} \|_{\frac{p}{p-1}}^{\frac{1}{2}} \| \nabla_1 \widehat{m}^a \Psi \|^{\frac{p-1}{p}} \| \widehat{m}^a \Psi \|^{\frac{1}{p}} \\
\leq& CN^{5/2} \| g_\beta \|_\infty N^{-\beta/2} \| \nabla_1 \widehat{m}^a \Psi \|^{1/2} \| \widehat{m}^a \Psi \|^{1/2} \\
\leq& CN^{3/2 + \xi - \beta/2} \, .
\end{aligned}
$$

In the fourth line, we applied Sobolev inequality as in the proof of Lemma 7.4, then setting $p = 2$. Furthermore, we used $\| \nabla_1 \widehat{m}^a \Psi \|^{1/2} \| \widehat{m}^a \Psi \|^{1/2} \leq CN^{-1+\xi}$, as well as $\| g_\beta \|_\infty \leq C$, see Lemma 5.5.
Using Lemma 4.2 (d), (99) can be bounded by

$$
\begin{aligned}
& CN^3 \left| \langle\!\langle \Psi, g_\beta(x_1 - x_2) \left[4\pi |\varphi|^2(x_3), p_1 p_2 (\widehat{r} - \widehat{r}_2) + (p_1 q_2 + q_1 p_2)(\widehat{r} - \widehat{r}_1) \right] \Psi \rangle\!\rangle \right| \\
& \leq CN^3 \| \varphi \|_\infty^2 \left(\| \widehat{r} - \widehat{r}_2 \|_{\mathrm{op}} + \| \widehat{r} - \widehat{r}_1 \|_{\mathrm{op}} \right) \| g_\beta(x_1 - x_2) p_2 \|_{\mathrm{op}} \, .
\end{aligned}
$$

Note that $\| \widehat{r} - \widehat{r}_2 \|_{\mathrm{op}} + \| \widehat{r} - \widehat{r}_1 \|_{\mathrm{op}} \leq \sum_{j \in \{c,d,e,f\}} \| \widehat{m}^j \|_{\mathrm{op}} \leq CN^{-2+3\xi}$ holds. With $\| g_\beta(x_1 - x_2) p_2 \|_{\mathrm{op}} \leq CN^{-1-\beta} \ln(N)$, it then follows that

$$
|(99)| \leq C \| \varphi \|_\infty^2 N^{3\xi - \beta} \ln(N) \, .
$$

In total, we obtain

$$
|\gamma_d(\Psi, \varphi)| \leq \mathcal{K}(\varphi, A_t) \left(N^{3/2 + \xi - \beta/2} + N^{1/2 + 3\xi - \beta} \ln(N) \right) \, .
$$

Control of γ_e Recall that

$$
\begin{aligned}
\gamma_e(\Psi, \varphi) = & -\frac{1}{2} N(N-1)(N-2)(N-3) \\
& \Im \left(\langle\!\langle \Psi, g_\beta(x_1 - x_2) \left[V_N(x_3 - x_4), \widehat{r} \right] \Psi \rangle\!\rangle \right) \, .
\end{aligned}
$$

Using symmetry, Lemma 4.2 (d) and notation (92), γ_e is bounded by

$$
\begin{aligned}
\gamma_e(\Psi, \varphi) \leq & N^4 \big| \langle\!\langle \Psi, g_\beta(x_1 - x_2) \big[V_N(x_3 - x_4), \widehat{m}^c p_1 p_2 p_3 p_4 + 2 \widehat{m}^d p_1 p_2 p_3 q_4 \\
& \qquad + 2 \widehat{m}^e p_1 q_2 p_3 p_4 + 4 \widehat{m}^f p_1 q_2 p_3 q_4 \big] \Psi \rangle\!\rangle \big| \\
\leq & 4N^4 \| V_N(x_3 - x_4) \Psi \| \| \mathbb{1}_{\mathrm{supp}(V_N)}(x_3 - x_4) p_3 \|_{\mathrm{op}} \| g_\beta(x_1 - x_2) p_1 \|_{\mathrm{op}} \\
& \qquad \left(\| \widehat{m}^c \|_{\mathrm{op}} + \| \widehat{m}^d \|_{\mathrm{op}} + \| \widehat{m}^e \|_{\mathrm{op}} + \| \widehat{m}^f \|_{\mathrm{op}} \right) \, .
\end{aligned}
$$

We get with (93), Lemma 5.5 and Lemma 4.2 that

$$
|\gamma_e(\Psi, \varphi)| \leq \mathcal{K}(\varphi, A_t) N^{1/2 + 3\xi - \beta} \ln(N) \, .
$$

Control of γ_f Recall that

$$\gamma_f(\Psi,\varphi)=2N(N-1)\frac{N-2}{N-1}\Im\left(\langle\!\langle\Psi,g_\beta(x_1-x_2)\left[4\pi|\varphi|^2(x_1),\widehat{r}\right]\Psi\rangle\!\rangle\right)\ .$$

We obtain the estimate

$$|\gamma_f(\Psi,\varphi)|\le\mathcal{K}(\varphi,A_t)N^2\|g_\beta\|\|\widehat{r}\|_{\mathrm{op}}\le\mathcal{K}(\varphi,A_t)N^{\xi-\beta}\ln(N)\ .$$

Collecting all estimates, we get with $\xi<1/2$

$$|\gamma_c(\Psi,\varphi)|+|\gamma_d(\Psi,\varphi)|+|\gamma_e(\Psi,\varphi)|+|\gamma_f(\Psi,\varphi)|\le\mathcal{K}(\varphi,A_t)N^{2-\beta/2}\ln(N)\ .$$

Choosing β sufficiently large, we obtain the desired decay and hence Lemma 6.8.

7.5 Energy estimates

Lemma 7.9 *Let $\Psi\in L^2_s(\mathbb{R}^{2N},\mathbb{C})\cap H^1(\mathbb{R}^{2N},\mathbb{C})$, $\|\Psi\|=1$ with $\|\nabla_1\Psi\|\le\mathcal{K}(\varphi,A_t)$. Let $\varphi\in H^3(\mathbb{R}^2,\mathbb{C})$, $\|\varphi\|=1$. Define the sets $\mathcal{A}_1^{(d)},\overline{\mathcal{B}}_1^{(d)}$ as in Definition 7.3. Then, for d large enough,*

$$\begin{aligned}\|\mathbb{1}_{\mathcal{A}_1^{(d)}}\nabla_1q_1\Psi\|^2+\|\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}}\nabla_1q_1\Psi\|^2\le\mathcal{K}(\varphi,A_t)\Big(\langle\!\langle\Psi,\widehat{n}^\varphi\Psi\rangle\!\rangle+N^{-1/6}\ln(N)\\+\inf\Big\{\left|\mathcal{E}_{V_N}(\Psi)-\mathcal{E}^{GP}_{4\pi}(\varphi)\right|,\left|\mathcal{E}_{W_\beta}(\Psi)-\mathcal{E}^{GP}_{N\|W_\beta\|_1}(\varphi)\right|+N^{-2\beta}\ln(N)\Big\}\Big)\ .\end{aligned}$$

Proof: We start with expanding $\mathcal{E}_{W_\beta}(\Psi)-\mathcal{E}^{GP}_{N\|W_\beta\|_1}(\varphi)$. This yields

$$\begin{aligned}\mathcal{E}_{W_\beta}(\Psi)-\mathcal{E}^{GP}_{N\|W_\beta\|_1}(\varphi)=\|\nabla_1\Psi\|^2+\frac{N-1}{2}\|\sqrt{W_\beta}(x_1-x_2)\Psi\|^2\\-\|\nabla\varphi\|^2-\frac{1}{2}N\|W_\beta\|_1\|\varphi^2\|^2+\langle\!\langle\Psi,A_t(x_1)\Psi\rangle\!\rangle-\langle\varphi,A_t\varphi\rangle\\=\|\mathbb{1}_{\mathcal{A}_1^{(d)}}\nabla_1q_1\Psi\|^2+\|\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}}\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\nabla_1\Psi\|^2+M(\Psi,\varphi)+Q_\beta(\Psi,\varphi)\ ,\end{aligned}$$

where we have defined

$$\begin{aligned}M(\Psi,\varphi)=&2\Re\left(\langle\!\langle\nabla_1q_1\Psi,\mathbb{1}_{\mathcal{A}_1^{(d)}}\nabla_1p_1\Psi\rangle\!\rangle\right) &(100)\\&+\|\mathbb{1}_{\mathcal{A}_1^{(d)}}\nabla_1p_1\Psi\|^2-\|\nabla\varphi\|^2 &(101)\\&+\langle\!\langle\Psi,A_t(x_1)\Psi\rangle\!\rangle-\langle\varphi,A_t\varphi\rangle\ , &(102)\\Q_\beta(\Psi,\varphi)=&\|\mathbb{1}_{\mathcal{B}_1^{(d)}}\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\nabla_1\Psi\|^2\\&+\frac{N-1}{2}\langle\!\langle\Psi,(1-p_1p_2)W_\beta(x_1-x_2)(1-p_1p_2)\Psi\rangle\!\rangle\\&+\frac{N-1}{2}\langle\!\langle\Psi,p_1p_2W_\beta(x_1-x_2)p_1p_2\Psi\rangle\!\rangle-\frac{1}{2}N\|W_\beta\|_1\|\varphi^2\|^2\\&+(N-1)\Re\langle\!\langle\Psi,(1-p_1p_2)W_\beta(x_1-x_2)p_1p_2\Psi\rangle\!\rangle\ .\end{aligned}$$

Notice that the first two terms in $Q_\beta(\Psi,\varphi)$ are nonnegative. This yields to the bound

$$\begin{aligned} S_\beta(\Psi,\varphi) =&(N-1)|\langle\!\langle \Psi,(1-p_1p_2)W_\beta(x_1-x_2)p_1p_2\Psi\rangle\!\rangle| &(103)\\ &+\left|\frac{N-1}{2}\langle\!\langle \Psi,p_1p_2W_\beta(x_1-x_2)p_1p_2\Psi\rangle\!\rangle-\frac{1}{2}N\|W_\beta\|_1\|\varphi^2\|^2\right| &(104)\\ \geq&-Q_\beta(\Psi,\varphi)\ . \end{aligned}$$

We therefore obtain the following bound:

$$\|\mathbb{1}_{\mathcal{A}_1^{(d)}}\nabla_1 q_1\Psi\|^2+\|\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}}\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\nabla_1\Psi\|^2\leq\left|\mathcal{E}_{W_\beta}(\Psi)-\mathcal{E}^{GP}_{N\|W_\beta\|_1}(\varphi)\right|+|M(\Psi,\varphi)|+|S_\beta(\Psi,\varphi)|\ . \tag{105}$$

Next, we split up the energy difference $\mathcal{E}_{V_N}(\Psi)-\mathcal{E}^{GP}_{4\pi}(\varphi)$,

$$\begin{aligned} \mathcal{E}_{V_N}(\Psi)-\mathcal{E}^{GP}_{4\pi}(\varphi)=\|\nabla_1\Psi\|^2+\frac{N-1}{2}\|\sqrt{V_N}(x_1-x_2)\Psi\|^2-\|\nabla\varphi\|^2\\ -2\pi\|\varphi^2\|^2+\langle\!\langle \Psi,A_t(x_1)\Psi\rangle\!\rangle-\langle\varphi,A_t\varphi\rangle\ . \end{aligned}$$

In order to better estimate the terms corresponding to the two-particle interactions, we introduce, for $\mu>d$, the potential $M_\mu(x)$, defined in Definition 5.4, and continue with

$$\begin{aligned} \mathcal{E}_{V_N}(\Psi)-\mathcal{E}^{GP}_{4\pi}(\varphi)=&\|\mathbb{1}_{\mathcal{A}_1^{(d)}}\nabla_1\Psi\|^2+\|\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}}\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\nabla_1\Psi\|^2+\|\mathbb{1}_{\mathcal{B}_1^{(d)}}\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\nabla_1\Psi\|^2\\ &+\frac{N-1}{2}\|\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}}\sqrt{V_N}(x_1-x_2)\Psi\|^2\\ &+\frac{1}{2}\langle\!\langle \Psi,\sum_{j\neq1}\mathbb{1}_{\mathcal{B}_1^{(d)}}\left(V_N-M_\mu\right)(x_1-x_j)\Psi\rangle\!\rangle\\ &+\frac{1}{2}\langle\!\langle \Psi,\sum_{j\neq1}\mathbb{1}_{\mathcal{B}_1^{(d)}}M_\mu(x_1-x_j)\Psi\rangle\!\rangle-\|\nabla\varphi\|^2-2\pi\|\varphi^2\|^2\\ &+\langle\!\langle \Psi,A_t(x_1)\Psi\rangle\!\rangle-\langle\varphi,A_t\varphi\rangle\ . \end{aligned}$$

Using that $q_1 = 1 - p_1$ and symmetry gives (after reordering)

$$\begin{aligned}
&\mathcal{E}_{V_N}(\Psi) - \mathcal{E}_{4\pi}^{GP}(\varphi) \\
=&\|\mathbb{1}_{\mathcal{A}_1^{(d)}} \nabla_1 q_1 \Psi\|^2 + \|\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}} \mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} \nabla_1 \Psi\|^2 + \frac{N-1}{2} \|\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}} \sqrt{V_N}(x_1 - x_2)\Psi\|^2 \\
&+ \frac{N-1}{2} \langle\!\langle \Psi, \mathbb{1}_{\mathcal{B}_1^{(d)}} (1 - p_1 p_2) M_\mu(x_1 - x_2)(1 - p_1 p_2) \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi \rangle\!\rangle \\
&+ \|\mathbb{1}_{\mathcal{B}_1^{(d)}} \mathbb{1}_{\overline{\mathcal{A}_1}^{(d)}} \nabla_1 \Psi\|^2 + \frac{1}{2} \langle\!\langle \Psi, \sum_{j\neq 1} \mathbb{1}_{\mathcal{B}_1^{(d)}} \left(V_N - M_\mu\right)(x_1 - x_j)\Psi \rangle\!\rangle \\
&+ \frac{N-1}{2} \langle\!\langle \Psi, \mathbb{1}_{\mathcal{B}_1^{(d)}} p_1 p_2 M_\mu(x_1 - x_2) p_1 p_2 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi \rangle\!\rangle - 2\pi \|\varphi^2\|^2 \\
&+ 2\Re\left(\langle\!\langle \nabla_1 q_1 \Psi, \mathbb{1}_{\mathcal{A}_1^{(d)}} \nabla_1 p_1 \Psi \rangle\!\rangle \right) \\
&+ (N-1)\Re \langle\!\langle \Psi, \mathbb{1}_{\mathcal{B}_1^{(d)}} (1 - p_1 p_2) M_\mu(x_1 - x_2) p_1 p_2 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi \rangle\!\rangle \\
&+ \|\mathbb{1}_{\mathcal{A}_1^{(d)}} \nabla_1 p_1 \Psi\|^2 - \|\nabla \varphi\|^2 \\
&+ \langle\!\langle \Psi, A_t(x_1)\Psi \rangle\!\rangle - \langle \varphi, A_t \varphi \rangle \\
=&\|\mathbb{1}_{\mathcal{A}_1^{(d)}} \nabla_1 q_1 \Psi\|^2 + \|\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}} \mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} \nabla_1 \Psi\|^2 + M(\Psi, \varphi) + \tilde{Q}_\mu(\Psi, \varphi) \ .
\end{aligned}$$

with

$$\begin{aligned}
&\tilde{Q}_\mu(\Psi, \varphi) = \frac{N-1}{2} \langle\!\langle \Psi, \mathbb{1}_{\mathcal{B}_1^{(d)}} (1 - p_1 p_2) M_\mu(x_1 - x_2)(1 - p_1 p_2) \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi \rangle\!\rangle \\
&+ \frac{N-1}{2} \|\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}} \sqrt{V_N}(x_1 - x_2)\Psi\|^2 \\
&+ \|\mathbb{1}_{\mathcal{B}_1^{(d)}} \mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} \nabla_1 \Psi\|^2 + \frac{1}{2} \langle\!\langle \Psi, \sum_{j\neq 1} \mathbb{1}_{\mathcal{B}_1^{(d)}} \left(V_N - M_\mu\right)(x_1 - x_j)\Psi \rangle\!\rangle \qquad (106) \\
&+ (N-1)\Re \langle\!\langle \Psi, \mathbb{1}_{\mathcal{B}_1^{(d)}} (1 - p_1 p_2) M_\mu(x_1 - x_2) p_1 p_2 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi \rangle\!\rangle \\
&+ \frac{N-1}{2} \langle\!\langle \Psi, \mathbb{1}_{\mathcal{B}_1^{(d)}} p_1 p_2 M_\mu(x_1 - x_2) p_1 p_2 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi \rangle\!\rangle - 2\pi \|\varphi^2\|^2 \ .
\end{aligned}$$

The first two terms in $\tilde{Q}_\mu(\Psi, \varphi)$ are nonnegative. For $\mu > d$ Lemma 7.10 below shows that (106) is also nonnegative. Thus, for $\mu > d$, we obtain the bound

$$\begin{aligned}
\tilde{S}_\mu(\Psi, \varphi) =&(N-1) \left| \langle\!\langle \Psi, \mathbb{1}_{\mathcal{B}_1^{(d)}} (1 - p_1 p_2) M_\mu(x_1 - x_2) p_1 p_2 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi \rangle\!\rangle \right| \qquad (107) \\
&+ \left| \frac{N-1}{2} \langle\!\langle \Psi, \mathbb{1}_{\mathcal{B}_1^{(d)}} p_1 p_2 M_\mu(x_1 - x_2) p_1 p_2 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi \rangle\!\rangle - 2\pi \|\varphi^2\|^2 \right| \qquad (108) \\
&\geq - \tilde{Q}_\mu(\Psi, \varphi) \ .
\end{aligned}$$

In total, we obtain

$$\|\mathbb{1}_{\mathcal{A}_1^{(d)}} \nabla_1 q_1 \Psi\|^2 + \|\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}} \mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} \nabla_1 \Psi\|^2 \leq |M(\Psi, \varphi)| + \tilde{S}_\mu(\Psi, \varphi) + \left| \mathcal{E}_{V_N}(\Psi) - \mathcal{E}_{4\pi}^{GP}(\varphi) \right| \ . \qquad (109)$$

Next, we will estimate $M(\Psi, \varphi), S_\beta(\Psi, \varphi)$ and $\tilde{S}_\mu(\Psi, \varphi)$.

- Estimate of $S_\beta(\Psi,\varphi)$ and $\tilde{S}_\mu(\Psi,\varphi)$.
  We first estimate (108), using the same estimate as in (60). Note that

$$\langle\!\langle \Psi, \mathbb{1}_{\mathcal{B}_1^{(d)}} p_1 p_2 M_\mu(x_1-x_2) p_1 p_2 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi \rangle\!\rangle = \langle \varphi, M_\mu \star |\varphi|^2 \varphi \rangle \langle\!\langle \Psi, \mathbb{1}_{\mathcal{B}_1^{(d)}} p_1 p_2 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi \rangle\!\rangle \ .$$

  Using $\|\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}} \Psi\| \leq C N^{1-d+\epsilon}$, for any $\epsilon > 0$, (see Lemma 7.3) we obtain, together with $\|p_1 p_2 \Psi\|^2 = 1 + 2\|p_1 q_2 \Psi\|^2 + \|q_1 q_2 \Psi\|^2$

$$\begin{aligned} |(108)| \leq & 3\|q_1\Psi\|^2 + C\left(N^{1-d+\epsilon} + N^{2-2d+2\epsilon}\right) + \frac{1}{2}|N\langle\varphi, M_\mu \star |\varphi|^2\varphi\rangle - N\|M_\mu\|_1 \|\varphi^2\|^2| \\ & + \frac{1}{2}|4\pi - N\|M_\mu\|_1| \|\varphi^2\|^2 + \frac{1}{2}\langle \varphi, M_\mu \star |\varphi|^2 \varphi\rangle \ . \end{aligned}$$

  Note that, using Young's inequality and (60)

$$\begin{aligned} & |\langle \varphi, N M_\mu \star |\varphi|^2 \varphi \rangle - N\|M_\mu\|_1 \|\varphi^2\|^2| = \left| \int_{\mathbb{R}^2} d^2x |\varphi(x)|^2 \left(N(M_\mu \star |\varphi|^2)(x) - N\|M_\mu\|_1 |\varphi(x)|^2 \right) \right| \\ \leq & \|\varphi\|_\infty^2 \|N(M_\mu \star |\varphi|^2) - \|N M_\mu\|_1 |\varphi|^2\|_1 \leq C \|\varphi\|_\infty^2 \|\Delta|\varphi|^2\|_1 N^{-2\mu} \ln(N) \\ \leq & \mathcal{K}(\varphi, A_t) N^{-2\mu} \ln(N) \ . \end{aligned}$$

  Since $|N\|M_\mu\|_1 - 4\pi| \leq C\frac{\ln(N)}{N}$ (see Lemma 5.5) and $\langle \varphi, M_\mu \star |\varphi|^2 \varphi\rangle \leq \|\varphi\|_\infty^4 \|M_\mu\|_1 \leq C\|\varphi\|_\infty^4 N^{-1}$, it follows that

$$\begin{aligned} |(108)| \leq & \mathcal{K}(\varphi, A_t) \left(\langle\!\langle \Psi, \widehat{n}^\varphi \Psi \rangle\!\rangle + N^{1-d+\epsilon} + N^{2-2d+2\epsilon} + N^{-2\mu}\ln(N) + N^{-1}\ln(N) \right) \\ \leq & \mathcal{K}(\varphi, A_t) \left(\langle\!\langle \Psi, \widehat{n}^\varphi \Psi \rangle\!\rangle + N^{-1}\ln(N) \right) \ , \end{aligned} \tag{110}$$

  where the last inequality holds for d large enough (recall that we chose $\mu > d$).
  Using the same estimates, we obtain

$$(104) \leq \mathcal{K}(\varphi, A_t) \left(\langle\!\langle \Psi, \widehat{n}^\varphi \Psi \rangle\!\rangle + N^{-2\beta}\ln(N) + N^{-1}\ln(N) \right) \ .$$

  Line (107) and line (103) are controlled by Lemma 7.11, which is stated below.

$$(103), (107) \leq \mathcal{K}(\varphi, A_t)(\langle\!\langle \Psi, \widehat{n}\Psi \rangle\!\rangle + N^{-1/6}\ln(N)) \ .$$

  In total, we obtain, for any $\mu > d \geq 1$, the bound

$$\begin{aligned} S_\beta(\Psi,\varphi) \leq & \mathcal{K}(\varphi, A_t) \left(\langle\!\langle \Psi, \widehat{n}\Psi \rangle\!\rangle + N^{-2\beta}\ln(N) + N^{-1/6}\ln(N) \right) \\ \tilde{S}_\mu(\Psi,\varphi) \leq & \mathcal{K}(\varphi, A_t) \left(\langle\!\langle \Psi, \widehat{n}\Psi \rangle\!\rangle + N^{-1/6}\ln(N) \right) \ . \end{aligned}$$

- Estimate of $M(\Psi,\varphi)$.
  First, we estimate (100).

$$\begin{aligned} |(100)| \leq & 2|\langle\!\langle \nabla_1 q_1 \Psi, \mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} \nabla_1 p_1 \Psi \rangle\!\rangle| + 2|\langle\!\langle \nabla_1 q_1 \Psi, \nabla_1 p_1 \Psi \rangle\!\rangle| \\ & \leq 2\|\nabla_1 q_1 \Psi\| \, \|\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} \nabla_1 p_1\|_{\mathrm{op}} + 2|\langle\!\langle \widehat{n}^{-1/2} q_1 \Psi, \Delta_1 p_1 \widehat{n}_1^{1/2} \Psi \rangle\!\rangle| \ . \end{aligned}$$

By Lemma 7.3, we obtain $\|\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\nabla_1 p_1\|_{\text{op}} \le C\|\nabla\varphi\|_\infty N^{1/2-d}$. Furthermore, we use $\|\nabla_1 q_1\Psi\| \le \|\nabla_1\Psi\| + \|\nabla_1 p_1\Psi\| \le \mathcal{K}(\varphi, A_t)$ (see also Lemma 7.6) and $|\langle\!\langle \widehat{n}^{-1/2}q_1\Psi, \Delta_1 p_1 \widehat{n}_1^{1/2}\Psi\rangle\!\rangle| \le \mathcal{K}(\varphi, A_t)\|\widehat{n}_1^{1/2}\Psi\|\|\widehat{n}^{1/2}\Psi\| \le \mathcal{K}(\varphi, A_t)(\langle\!\langle\Psi, \widehat{n}\Psi\rangle\!\rangle + N^{-1})$. Hence, for d large enough,

$$|(100)| \le \mathcal{K}(\varphi, A_t)(\langle\!\langle\Psi, \widehat{n}\Psi\rangle\!\rangle + N^{\frac{1}{2}-d} + N^{-1}) \le \mathcal{K}(\varphi, A_t)(\langle\!\langle\Psi, \widehat{n}\Psi\rangle\!\rangle + N^{-1})\ .$$

Line (101) is estimated for d large enough, noting that $\|\nabla_1 p_1\Psi\|^2 = \|\nabla\varphi\|^2\|p_1\Psi\|^2$, by

$$\begin{aligned}(101) =& \|\mathbb{1}_{\mathcal{A}_1^{(d)}}\nabla_1 p_1\Psi\|^2 - \|\nabla\varphi\|^2\\ \le& |\|\nabla_1 p_1\Psi\|^2 - \|\nabla\varphi\|^2| + \|\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\nabla_1 p_1\Psi\|^2\\ \le& C\left(\|\nabla\varphi\|^2\langle\!\langle\Psi, q_1\Psi\rangle\!\rangle + \|\nabla\varphi\|_\infty^2 N^{1-2d}\right)\\ \le& \mathcal{K}(\varphi, A_t)\langle\!\langle\Psi\widehat{n}, \Psi\rangle\!\rangle\ .\end{aligned}$$

For line (102), we use Lemma 7.5 to obtain

$$(102) \le C\|A_t\|_\infty\left(\langle\!\langle\Psi, \widehat{n}\Psi\rangle\!\rangle + N^{-1/2}\right)\ .$$

In total, we obtain

$$M(\Psi, \varphi) \le \mathcal{K}(\varphi, A_t)\left(\langle\!\langle\Psi, \widehat{n}\Psi\rangle\!\rangle + N^{-1/2}\right)\ .$$

Note that

$$\begin{aligned}\|\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}}\nabla_1 q_1\Psi\|^2 =& \|\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}}\nabla_1 q_1\Psi\|^2 + \|\mathbb{1}_{\mathcal{A}_1^{(d)}}\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}}\nabla_1 q_1\Psi\|^2\\ \le& \|\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}}\nabla_1(1-p_1)\Psi\|^2 + \|\mathbb{1}_{\mathcal{A}_1^{(d)}}\nabla_1 q_1\Psi\|^2\\ \le& 2\|\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}}\nabla_1\Psi\|^2 + 2\|\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}}\nabla_1 p_1\Psi\|^2 + \|\mathbb{1}_{\mathcal{A}_1^{(d)}}\nabla_1 q_1\Psi\|^2.\end{aligned}$$

Using $\|\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}}\nabla_1 p_1\|_{\text{op}} \le N\|\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\nabla_1 p_1\|_{\text{op}} \le C\|\nabla\varphi\|_\infty N^{3/2-d}$, (105), (109) and the bounds for $M(\Psi, \varphi)$, $S_\beta(\Psi, \varphi)$ and $\tilde{S}_\mu(\Psi, \varphi)$, we then obtain the Lemma for d large enough.

□

Lemma 7.10

(a) Let R_β and M_β be defined as in Lemma 5.4. Then, for any $\Psi \in H^1(\mathbb{R}^{2N}, \mathbb{C})$

$$\|\mathbb{1}_{|x_1-x_2|\le R_\beta}\nabla_1\Psi\|^2 + \frac{1}{2}\langle\!\langle\Psi, (V_N - M_\beta)(x_1 - x_2)\Psi\rangle\!\rangle \ge 0\ .$$

(b) Let M_β be defined as in Lemma 5.4. Let $\Psi \in L_s^2(\mathbb{R}^{2N}, \mathbb{C}) \cap H^1(\mathbb{R}^{2N}, \mathbb{C})$. Then, for sufficiently large N and for $\beta > d$,

$$\|\mathbb{1}_{\mathcal{B}_1^{(d)}}\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\nabla_1\Psi\|^2 + \frac{1}{2}\langle\!\langle\Psi, \sum_{j\neq 1}\mathbb{1}_{\mathcal{B}_1^{(d)}}(V_N - M_\beta)(x_1 - x_j)\Psi\rangle\!\rangle \ge 0\ .$$

Proof:

(a) We first show nonnegativity of the one-particle operator $H^{Z_n} : H^2(\mathbb{R}^2, \mathbb{C}) \to L^2(\mathbb{R}^2, \mathbb{C})$ given by

$$H^{Z_n} = -\Delta + \frac{1}{2} \sum_{z_k \in Z_n} (V_N(\cdot - z_k) - M_\beta(\cdot - z_k))$$

for any $n \in \mathbb{N}$ and any n-elemental subset $Z_n \subset \mathbb{R}^2$ which is such that the supports of the potentials $M_\beta(\cdot - z_k)$ are pairwise disjoint for any two $z_k \in Z_n$.

Since $f_\beta(\cdot - z_k)$ is the the zero energy scattering state of the potential $1/2V_N(\cdot - z_k) - 1/2W_\beta(\cdot - z_k)$, it follows that

$$F_\beta^{Z_n} = \prod_{z_k \in Z_n} f_\beta(\cdot - z_k) \; .$$

fulfills $H^{Z_n} F_\beta^{Z_n} = 0$ for any such Z_n. By construction f_β is a positive function, so is $F_\beta^{Z_n}$. Since $\frac{1}{2} \sum_{z_k \in Z_n} (V_N(\cdot - z_k) - M_\beta(\cdot - z_k)) \in L^\infty(\mathbb{R}^2, \mathbb{C})$, this potential is a infinitesimal perturbation of $-\Delta$, thus $\sigma_{\text{ess}}(H^{Z_n}) = [0, \infty)$. Assume now that H^{Z_n} is not nonnegative. Then, there exists a ground state $\Psi_G \in H^2(\mathbb{R}^2, \mathbb{C})$ of H^{Z_n} of negative energy $E < 0$. The phase of the ground state can be chosen such that the ground state is real and positive (see e.g. [35], Theorem 10.12.). Since such a ground state of negative energy decays exponentially, that is $\Psi_G(x) \leq C_1 e^{-C_2|x|}, C_1, C_2 > 0$, the following scalar product is well defined (although $F_\beta^{Z_n} \notin L^2(\mathbb{R}^2, \mathbb{C})$).

$$\langle F_\beta^{Z_n}, H^{Z_n} \Psi_G \rangle = \langle F_\beta^{Z_n}, E \Psi_G \rangle < 0 \; . \tag{111}$$

On the other hand we have since $F_{\beta_1,\beta}^{X_n}$ is the zero energy scattering state

$$\langle F_\beta^{Z_n}, H^{Z_n} \Psi_G \rangle = \langle H^{Z_n} F_\beta^{Z_n}, \Psi_G \rangle = 0 \; .$$

This contradicts (111) and the nonnegativity of H^{Z_n} follows.

Now, assume that there exists a $\psi \in H^2(\mathbb{R}^2, \mathbb{C})$ such that the quadratic form

$$Q(\psi) = \|\mathbb{1}_{|\cdot| \leq R_\beta} \nabla \psi\|^2 + \frac{1}{2} \langle \psi, (V_N(\cdot) - M_\beta(\cdot)) \psi \rangle < 0 \; .$$

Since V_{β_1} and M_{β_1} are spherically symmetric we can assume that ψ is spherically symmetric. Subsituting $\psi \to a\psi, \; a \in \mathbb{R}$, we can furthermore assume that, for all $|x| = R_\beta$, $\psi(x) = 1 - \epsilon$ for $\epsilon > 0$.

Define $\tilde{\psi}$ such that $\tilde{\psi}(x) = \psi(x)$ for $|x| \leq R_\beta$ and $\tilde{\psi}(x) = 1$ for $|x| > R_\beta + \epsilon$ and $\epsilon > 0$. Furthermore, $\tilde{\psi}$ can be constructed such that $\|\mathbb{1}_{|\cdot| \geq R_\beta} \nabla \tilde{\psi}\|^2 \leq C(\epsilon + \epsilon^2)$.

Then $Q(\tilde{\psi}) = Q(\psi) < 0$ holds, because the operator associated with the quadratic form is supported inside the ball $B_0(R_\beta)$.

Using $\tilde{\psi}$, we can construct a set of points Z_n and a $\chi \in H^2(\mathbb{R}^2, \mathbb{C})$ such that $\langle \chi, H^{Z_n} \chi \rangle < 0$, contradicting to nonnegativity of H^{Z_n}.

For $R > 1$ let

$$\xi_R(x) = \begin{cases} R^2/x^2, & \text{for } |x| > R; \\ 1, & \text{else.} \end{cases}$$

Let now Z_n be a subset $Z_n \subset \mathbb{R}^2$ with $|Z_n| = n$ which is such that the supports of the potentials $M_\beta(\cdot - z_k)$ lie within the Ball around zero with radius R and are pairwise disjoint for any two $z_k \in Z_n$. Since we are in two dimensions we can choose a n which is of order R^2.

Let now $\chi_R(x) = \xi_R(x) \prod_{z_k \in Z_n} \tilde{\psi}(x - z_k)$. By construction, there exists a $D = \mathcal{O}(1)$ such that $\chi_R(x) = \tilde{\psi}(x - z_k)$ for $|x - z_k| \leq D$. From this, we obtain

$$\begin{aligned} \langle \chi_R, H^{Z_n} \chi_R \rangle =& \|\nabla \chi_R\|^2 + n\frac{1}{2}\langle \psi, (V_N(\cdot) - M_\beta(\cdot))\psi \rangle \\ =& nQ(\psi) + \sum_{z_k \in Z_n} \|\mathbb{1}_{|\cdot - z_k| \geq R_\beta} \nabla \chi_R\|^2 \\ \leq& nQ(\psi) + Cn(\epsilon + \epsilon^2) + \|\nabla \xi_R\|^2 \\ =& nQ(\psi) + Cn(\epsilon + \epsilon^2) + C\ . \end{aligned}$$

Choosing R and hence n large enough and ϵ small, we can find a Z_n such that $\langle \chi_R, H^{Z_n} \chi_R \rangle$ is negative, contradicting nonnegativity of H^{Z_n}.
Now, we can prove that

$$\|\mathbb{1}_{|x_1 - x_2| \leq R_{\beta_1}} \nabla_1 \Psi\|^2 + \frac{1}{2}\langle\langle \Psi, (V_N - M_\beta)(x_1 - x_2)\Psi \rangle\rangle \geq 0\ . \tag{112}$$

holds for any $\Psi \in H^2(\mathbb{R}^{2N}, \mathbb{C})$. Using the coordinate transformation $\tilde{x}_1 = x_1 - x_2$, $\tilde{x}_i = x_i \ \forall i \geq 2$, we have $\nabla_{x_1} = \nabla_{\tilde{x}_1}$. Thus (112) is equivalent to $\|\mathbb{1}_{|x_1| \leq R_{\beta_1}} \nabla_1 \Psi\|^2 + \frac{1}{2}\langle\langle \Psi, (V_N - M_\beta)(x_1)\Psi \rangle\rangle \geq 0 \ \forall \Psi \in H^2(\mathbb{R}^{2N}, \mathbb{C})$ which follows directly from $Q(\psi) \geq 0$ for all $\psi \in H^2(\mathbb{R}^2, \mathbb{C})$. By a standard density argument, we can conclude that $Q(\Psi) \geq 0 \ \forall \Psi \in H^1(\mathbb{R}^{2N}, \mathbb{C})$.

(b) Define $c_k = \{(x_1, \ldots, x_N) \in \mathbb{R}^{2N} | |x_1 - x_k| \leq R_\beta\}$ and $\mathcal{C}_1 = \cup_{k=2}^N c_k$. For $(x_1, \ldots, x_N) \in \mathcal{B}_1^{(d)}$ it holds that $|x_i - x_j| \geq N^{-d}$ for $2 \leq i, j \leq N$. Let $\beta > d$. Assume that $N^{-d} > 2R_\beta$, which hold for N sufficiently large, since $R_\beta \leq CN^{-\beta}$. Then, it follows that, for $i \neq j$, $\left(c_i \cap \mathcal{B}_1^{(d)}\right) \cap \left(c_j \cap \mathcal{B}_1^{(d)}\right) = \emptyset$. Under the same conditions, we also have $\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} \geq \mathbb{1}_{\mathcal{C}_1}$. Therefore

$$\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} \mathbb{1}_{\mathcal{B}_1^{(d)}} \geq \mathbb{1}_{\mathcal{C}_1} \mathbb{1}_{\mathcal{B}_1^{(d)}} = \mathbb{1}_{\mathcal{C}_1 \cap \mathcal{B}_1^{(d)}} = \mathbb{1}_{\cup_{k=2}^N \left(c_k \cap \mathcal{B}_1^{(d)}\right)} = \sum_{k=2}^N \mathbb{1}_{c_k \cap \mathcal{B}_1^{(d)}} = \mathbb{1}_{\mathcal{B}_1^{(d)}} \sum_{k=2}^N \mathbb{1}_{c_k}\ .$$

Note that $\mathbb{1}_{\mathcal{B}_1^{(d)}}$ depends only on $x_2, \ldots, x_N$. By this

$$\|\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} \mathbb{1}_{\mathcal{B}_1^{(d)}} \nabla_1 \Psi\|^2 \geq \sum_{k=2}^N \|\mathbb{1}_{c_k} \nabla_1 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi\|^2 = (N-1)\|\mathbb{1}_{|x_1 - x_2| \leq R_\beta} \nabla_1 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi\|^2\ .$$

This yields

$$(106) \geq (N-1)\left(\|\mathbb{1}_{|x_1-x_2|\leq R_\beta} \nabla_1 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi\|^2 + \frac{1}{2} \langle\!\langle \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi, (V_N - M_\beta)(x_1-x_2) \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi \rangle\!\rangle \right) \geq 0 \, .$$

where the last inequality follows from (a)

$\square$

Lemma 7.11 *Let $W_\beta \in \mathcal{V}_\beta$. Let $\Psi \in L_s^2(\mathbb{R}^{2N}, \mathbb{C}) \cap H^1(\mathbb{R}^{2N}, \mathbb{C})$ and $\|\nabla_1 \Psi\|$ be bounded uniformly in N. Let d in Definition 7.3 of $\mathbb{1}_{\mathcal{B}_1^{(d)}}$ sufficiently large. Let $\Gamma \in \{\Psi, \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi\}$. Then, for all $\beta > 0$,*

(a)

$$N \, |\langle\!\langle \Gamma, q_1 p_2 W_\beta(x_1-x_2) p_1 p_2 \Gamma \rangle\!\rangle| \leq C \|\varphi\|_\infty^2 \langle\!\langle \Psi, \hat{n} \Psi \rangle\!\rangle \, .$$

(b)

$$N |\langle\!\langle \Gamma, p_1 p_2 W_\beta(x_1-x_2) q_1 q_2 \Gamma \rangle\!\rangle| \leq \mathcal{K}(\varphi, A_t) \left(\langle\!\langle \Psi, \widehat{n} \Psi \rangle\!\rangle + N^{-1/6} \ln(N) \right) \, .$$

(c)

$$N |\langle\!\langle \Gamma, (1-p_1 p_2) W_\beta(x_1-x_2) p_1 p_2 \Gamma \rangle\!\rangle| \leq \mathcal{K}(\varphi, A_t) \left(\langle\!\langle \Psi, \widehat{n} \Psi \rangle\!\rangle + N^{-1/6} \ln(N) \right) \, .$$

Proof:

(a) Let first $\Gamma = \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi$. Then,

$$\begin{aligned} & N \left| \langle\!\langle \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi, q_1 p_2 W_\beta(x_1-x_2) p_1 p_2 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi \rangle\!\rangle \right| \\ \leq & N \left| \langle\!\langle \mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}} \Psi, q_1 p_2 W_\beta(x_1-x_2) p_1 p_2 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi \rangle\!\rangle \right| & (113) \\ + & N \left| \langle\!\langle \Psi, q_1 p_2 W_\beta(x_1-x_2) p_1 p_2 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi \rangle\!\rangle \right| \, . & (114) \end{aligned}$$

Using Lemma 7.4 together with $\|p_2 W_\beta(x_1-x_2) p_2\|_{\text{op}} \leq \|\varphi\|_\infty^2 \|W_\beta\|_1$, the first line can be bounded, for any $\epsilon > 0$, by

$$(113) \leq \mathcal{K}(\varphi, A_t) N \|\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}} \Psi\| \|W_\beta\|_1 \leq \mathcal{K}(\varphi, A_t) N^{1-d+\epsilon}. \tag{115}$$

The second term is bounded by

$$\begin{aligned} (114) = & N \left| \langle\!\langle \sqrt{W_\beta(x_1-x_2)} q_1 p_2 (\hat{n})^{-\frac{1}{2}} \Psi, \sqrt{W_\beta(x_1-x_2)} p_1 p_2 \hat{n}_1^{\frac{1}{2}} \mathbb{1}_{\mathcal{B}^{(d)}{}_1} \Psi \rangle\!\rangle \right| \\ \leq & CN \|\sqrt{W_\beta(x_1-x_2)} p_2\|_{\text{op}}^2 \left(\|q_1 (\hat{n})^{-\frac{1}{2}} \Psi\|^2 + \|\hat{n}_1^{\frac{1}{2}} \mathbb{1}_{\mathcal{B}^{(d)}{}_1} \Psi\|^2 \right) \\ \leq & CN \|\sqrt{W_\beta(x_1-x_2)} p_2\|_{\text{op}}^2 \left(\langle\!\langle \Psi, \hat{n} \Psi \rangle\!\rangle + \|\hat{n}_1^{\frac{1}{2}} \Psi\|^2 + \|\hat{n}_1^{\frac{1}{2}} \mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}} \Psi\|^2 \right) \\ \leq & CN \|W_\beta\|_1 \|\varphi\|_\infty^2 \left(\langle\!\langle \Psi, \hat{n} \Psi \rangle\!\rangle + \|\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}} \Psi\|^2 \right) \\ \leq & C \|\varphi\|_\infty^2 \left(\langle\!\langle \Psi, \hat{n} \Psi \rangle\!\rangle + N^{1-d+\epsilon} \right) . \end{aligned}$$

Choosing d large enough, $N^{1-d+\epsilon}$ is smaller than $\langle\!\langle\Psi, \hat{n}\Psi\rangle\!\rangle$. This yields (a) in the case $\Gamma = \mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi$. The inequality (a) can be proven analogously for $\Gamma = \Psi$.

(b) Let $\Gamma = \mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi$. We first consider (b) for potentials with $\beta < 1/4$. We have to estimate

$$\begin{aligned}
&N|\langle\!\langle \mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi, p_1p_2W_\beta(x_1-x_2)q_1q_2\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi\rangle\!\rangle| \leq N|\langle\!\langle \Psi, p_1p_2W_\beta(x_1-x_2)q_1q_2\Psi\rangle\!\rangle| \\
&+N|\langle\!\langle \mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}}\Psi, p_1p_2W_\beta(x_1-x_2)q_1q_2\Psi\rangle\!\rangle| + N|\langle\!\langle \Psi, p_1p_2W_\beta(x_1-x_2)q_1q_2\mathbb{1}_{\overline{\mathcal{B}_1^{(d)}}}\Psi\rangle\!\rangle| \\
&+N|\langle\!\langle \mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}}\Psi, p_1p_2W_\beta(x_1-x_2)q_1q_2\mathbb{1}_{\overline{\mathcal{B}_1^{(d)}}}\Psi\rangle\!\rangle| \\
\leq& N|\langle\!\langle \Psi, p_1p_2W_\beta(x_1-x_2)q_1q_2\Psi\rangle\!\rangle| \qquad (116)\\
&+CN\|\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}}\Psi\|\|W_\beta\|_\infty\,. \qquad (117)
\end{aligned}$$

The last term is bounded, for any $\epsilon > 0$, by

$$(117) \leq CNN^{1-d+\epsilon}N^{-1+2\beta} \leq N^{-2}\,,$$

where the last inequality holds choosing d large enough.

Using Lemma 4.2 (c) and Lemma 4.6 with $O_{1,2} = q_2W_\beta(x_1-x_2)p_2$, $\Omega = N^{-1/2}q_1\Psi$ and $\chi = N^{1/2}p_1\Psi$ we get

$$\begin{aligned}
(116) \leq& \|q_1\Psi\|^2 + N^2\big|\langle\!\langle q_2\,\Psi, p_1\sqrt{W_\beta}(x_1-x_2)p_3\sqrt{W_\beta}(x_1-x_3) \\
&\qquad\qquad \sqrt{W_\beta}(x_1-x_2)p_2\sqrt{W_\beta}(x_1-x_3)p_1q_3\,\Psi\rangle\!\rangle\big| \\
&+ N^2(N-1)^{-1}\|q_2W_\beta(x_1-x_2)p_2p_1\Psi\|^2 \\
\leq& \|q_1\Psi\|^2 + N^2\|\sqrt{W_\beta}(x_1-x_2)p_1\|_{\mathrm{op}}^4\,\|q_2\,\Psi\|^2 \\
&+ CN\|W_\beta(x_1-x_2)p_2\|_{\mathrm{op}}^2\,.
\end{aligned}$$

With Lemma 4.2 (e) we get the bound

$$\begin{aligned}
(116) \leq& \|q_1\Psi\|^2 + N^2\|\varphi\|_\infty^4\|W_\beta\|_1^2\,\|q_1\Psi\|^2 \\
&+ CN\|W_\beta\|^2\|\varphi\|_\infty^2\,.
\end{aligned}$$

Note, that $\|W_\beta\|_1 \leq CN^{-1}$, $\|W_\beta\|^2 \leq CN^{-2+2\beta}$ Hence

$$(116) \leq C\left(\langle\!\langle\Psi, q_1\Psi\rangle\!\rangle + \mathcal{K}(\varphi)N^{-1+2\beta}\right)\,.$$

Note that, for $\beta < 1/4$, $N^{-1+2\beta} \leq N^{-1/6}\ln(N)$. Using the same bounds for $\Gamma = \Psi$, we obtain (b) for the case $\beta < 1/4$.

b) for $1/4 \leq \beta$:
We use $U_{\beta_1,\beta}$ from Definition 7.1 for some $0 < \beta_1 < 1/4$.

$Z_\beta^\varphi(x_1,x_2) - W_\beta + U_{\beta_1,\beta}$ has the form of $Z_{\beta_1}^\varphi(x_1,x_2)$ which has been controlled above. It is left to control

$$N\left|\langle\!\langle \mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi, p_1p_2\left(W_\beta(x_1-x_2) - U_{\beta_1,\beta}(x_1-x_2)\right)q_1q_2\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi\rangle\!\rangle\right|\,.$$

Let $\Delta h_{\beta_1,\beta} = W_\beta - U_{\beta_1,\beta}$. Integrating by parts and using that $\nabla_1 h_{\beta_1,\beta}(x_1 - x_2) = -\nabla_2 h_{\beta_1,\beta}(x_1 - x_2)$ gives

$$N \left| \langle\!\langle \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi, p_1 p_2 \left(W_\beta(x_1 - x_2) - U_{\beta_1,\beta}(x_1 - x_2) \right) q_1 q_2 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi \rangle\!\rangle \right|$$

$$= N \left| \langle\!\langle \nabla_1 p_1 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi, p_2 \nabla_2 h_{\beta_1,\beta}(x_1 - x_2) q_1 q_2 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi \rangle\!\rangle \right| \tag{118}$$

$$+ N \left| \langle\!\langle \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi, p_1 p_2 \nabla_2 h_{\beta_1,\beta}(x_1 - x_2) \nabla_1 q_1 q_2 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi \rangle\!\rangle \right| . \tag{119}$$

Let $(a_1, b_1) = (q_1, \nabla p_1)$ or $(a_1, b_1) = (\nabla q_1, p_1)$. Then, both terms can be estimated as follows:
We use Lemma 4.6 with $\Omega = N^{-\eta/2} a_1 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi$, $O_{1,2} = N^{1+\eta/2} q_2 \nabla_2 h_{\beta_1,\beta}(x_1 - x_2) p_2$ and $\chi = b_1 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi$. We choose $\eta < 2\beta_1$.

$$N \left| \langle\!\langle \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi, a_1 p_2 \nabla_2 h_{\beta_1,\beta}(x_1 - x_2) b_1 q_2 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi \rangle\!\rangle \right|$$

$$\leq N^{-\eta} \| a_1 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi \|^2 \tag{120}$$

$$+ \frac{N^{2+\eta}}{N-1} \| q_2 \nabla_2 h_{\beta_1,\beta}(x_1 - x_2) b_1 p_2 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi \|^2 \tag{121}$$

$$+ N^{2+\eta} \left| \langle\!\langle \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi, b_1 p_2 q_3 \nabla_2 h_{\beta_1,\beta}(x_1 - x_2) \nabla_3 h_{\beta_1,\beta}(x_1 - x_3) b_1 q_2 p_3 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi \rangle\!\rangle \right|^{1/2} . \tag{122}$$

We obtain (note that $\mathbb{1}_{\mathcal{B}_1^{(d)}}$ does not depend on x_1)

$$(120) \leq N^{-\eta} \| a_1 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi \|^2 = N^{-\eta} \| \mathbb{1}_{\mathcal{B}_1^{(d)}} a_1 \Psi \|^2 \leq \mathcal{K}(\varphi, A_t) N^{-\eta} .$$

since both $\|\nabla q_1 \Psi\|$ and $\|q_1 \Psi\|$ are bounded uniformly in N. Since q_2 is a projector it follows that

$$\begin{aligned}(121) \leq & \frac{N^{2+\eta}}{N-1} \| \nabla_2 h_{\beta_1,\beta}(x_1 - x_2) p_2 \|_{\mathrm{op}}^2 \| b_1 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi \|^2 \leq C \frac{N^{2+\eta}}{N-1} \|\varphi\|_\infty^2 \|\nabla h_{\beta_1,\beta}\|^2 \| b_1 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi \|^2 \\ \leq & \mathcal{K}(\varphi, A_t) N^{\eta - 1} \ln(N) \|\varphi\|_\infty^2 ,\end{aligned}$$

where we used Lemma 7.2 in the last step.

Next, we estimate

$$(122) \leq N^{2+\eta} \| p_2 \nabla_2 h_{\beta_1,\beta}(x_1 - x_2) b_1 q_2 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi \|^2$$

$$\leq 2N^{2+\eta} \| p_2 \nabla_2 h_{\beta_1,\beta}(x_1 - x_2) b_1 q_2 \mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}} \Psi \|^2 \tag{123}$$

$$+ 2N^{2+\eta} \| p_2 \nabla_2 h_{\beta_1,\beta}(x_1 - x_2) b_1 q_2 \Psi \|^2 . \tag{124}$$

The first term can be estimated as

$$\begin{aligned}(123) \leq & C N^{2+\eta} \| \nabla_2 h_{\beta_1,\beta}(x_1 - x_2) b_1 \|_{\mathrm{op}}^2 \| \mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}} \Psi \|^2 \\ \leq & C N^{2+\eta} \| \nabla_2 h_{\beta_1,\beta} \|^2 (\|\varphi\|_\infty^2 + \|\nabla\varphi\|_\infty^2) \| \mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}} \Psi \|^2 \\ \leq & \mathcal{K}(\varphi, A_t) N^{2+\eta} N^{-2} \ln(N) N^{2-2d+2\epsilon} = \mathcal{K}(\varphi, A_t) N^{2-2d+2\epsilon+\eta} \ln(N) ,\end{aligned}$$

for any $\epsilon > 0$. For d large enough, this term is subleading. The last term can be estimated as

$$\begin{aligned}
(124) \leq & 2N^{2+\eta}\|p_2 h_{\beta_1,\beta}(x_1-x_2)b_1\nabla_2 q_2\Psi\|^2 \\
& +2N^{2+\eta}\||\varphi(x_2)\rangle\langle\nabla\varphi(x_2)|h_{\beta_1,\beta}(x_1-x_2)b_1 q_2\Psi\|^2 \\
\leq & CN^{2+\eta}\|p_2 h_{\beta_1,\beta}(x_1-x_2)\|_{\mathrm{op}}^2\|b_1\nabla_2 q_2\Psi\|^2 \\
& +CN^{2+\eta}\||\varphi(x_2)\rangle\langle\nabla\varphi(x_2)|h_{\beta_1,\beta}(x_1-x_2)\|_{\mathrm{op}}^2\|b_1 q_2\Psi\|^2 \\
\leq & CN^{2+\eta}\left(\|\nabla\varphi\|_\infty^2+\|\varphi\|_\infty^2\right)\|h_{\beta_1,\beta}\|^2(1+\|\nabla\varphi\|^2) \\
\leq & \mathcal{K}(\varphi,A_t)N^{\eta-2\beta_1}\ln(N)^2 \,.
\end{aligned}$$

Combining both estimates we obtain, for any $\beta > 1$,

$$\begin{aligned}
& N\left|\langle\!\langle \mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi, p_1p_2W_\beta(x_1-x_2)q_1q_2\mathbb{1}_{\mathcal{B}^{(d)}{}_1}\Psi\rangle\!\rangle\right| \\
& \leq \inf_{\eta>0}\inf_{0<\mu<1/4}\left(\mathcal{K}(\varphi,A_t)\left(\langle\!\langle\Psi,\widehat{n}\Psi\rangle\!\rangle+N^{-1+2\mu}+N^{-\eta}+N^{\eta-1}\ln(N)+N^{\eta-2\mu}\ln(N)\right)\right) \\
& \leq \mathcal{K}(\varphi,A_t)\left(\langle\!\langle\Psi,\widehat{n}\Psi\rangle\!\rangle+N^{-1/6}\ln(N)\right) \,.
\end{aligned}$$

where the last inequality comes from choosing $\eta = 1/3$ and $\mu = 1/4$. For $\Gamma = \Psi$, (b) can be estimated the same way, yielding the same bound.

(c) This follows from (a) and (b), using that $1 - p_1p_2 = q_1q_2 + p_1q_2 + q_1p_2$.

□

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