

Visualisation of Concepts in Condensed Matter Physics

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Supervised by Mark Spencer Rudner

Outline

- Introduction / Background
- Lattices and crystal structure
- The reciprocal lattice and scattering
- Band structure
- Discussion



Introduction / Background

Periodic medium. Discrete translational symmetry



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Periodic medium. Discrete translational symmetry

$$[\hat{T}_{\mathbf{R}}, \hat{H}] = 0, \quad V(\mathbf{r} + \mathbf{R}) = V(\mathbf{r}),$$



Introduction / Background

Periodic medium. Discrete translational symmetry

$$[\hat{T}_{\mathbf{R}}, \hat{H}] = 0, \quad V(\mathbf{r} + \mathbf{R}) = V(\mathbf{r}),$$

Bloch's theorem:

$$\psi(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} u(\mathbf{r}), \quad u(\mathbf{r} + \mathbf{R}) = u(\mathbf{r}),$$



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Crystal momentum.



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Crystal momentum. One dimension, spacing of a :

$$k \rightarrow k + \frac{2\pi}{a}, \quad -\frac{\pi}{a} \leq k \leq \frac{\pi}{a}$$



Lattices and crystal structure

The lattice



Lattices and crystal structure

The lattice

$$\mathbf{R} = \sum_{i=1}^d n_i \mathbf{a}_i, \quad n_i \in \mathbb{Z}$$

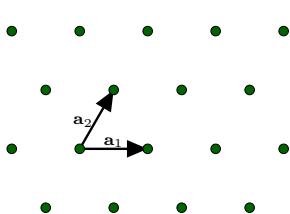


Figure: A triangular lattice. $\mathbf{a}_1 = a \cdot (1, 0)$, $\mathbf{a}_2 = a \cdot (1/2, \sqrt{3}/2)$.

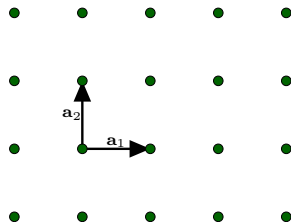


Figure: A square lattice. $\mathbf{a}_1 = a \cdot (1, 0)$, $\mathbf{a}_2 = a \cdot (0, 1)$.



Lattices and crystal structure

The unit cell

$$\mathbf{R} = \sum_{i=1}^d n_i \mathbf{a}_i, \quad n_i \in \mathbb{Z}$$

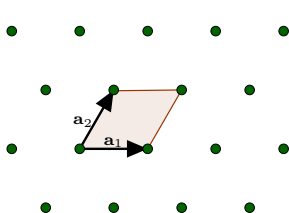


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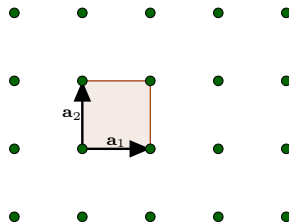


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 $\mathbf{a}_1 = a \cdot (1, 0)$, $\mathbf{a}_2 = a \cdot (0, 1)$.



Lattices and crystal structure

The basis



Lattices and crystal structure

The basis

$$\mathbf{r}_{atom,i} = \mathbf{R} + \mathbf{r}_{basis,i},$$



Lattices and crystal structure

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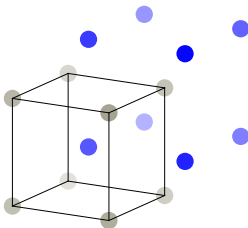


Figure: Conventional unit cell of a bcc lattice. Two atoms, one (grey) at $a \cdot (0, 0, 0)$ and one (blue) at $a \cdot (1/2, 1/2, 1/2)$.



Lattices and crystal structure

Other difficulties

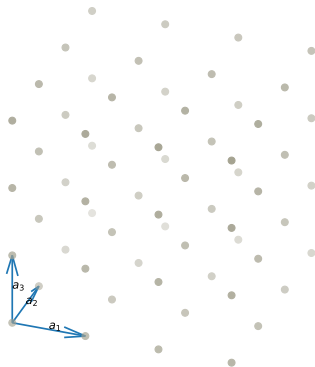


Figure: A simple cubic lattice.

$$\mathbf{a}_1 = a \cdot (1, 0, 0), \mathbf{a}_2 = a \cdot (0, 1, 0), \mathbf{a}_3 = a \cdot (0, 0, 1).$$



Lattices and crystal structure

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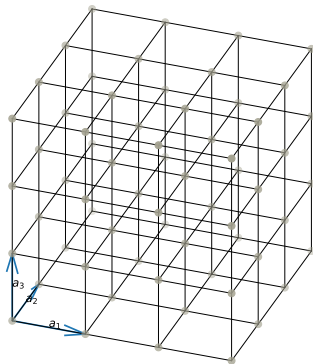


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Lattices and crystal structure



Reciprocal lattice and scattering

The reciprocal lattice



Reciprocal lattice and scattering

The reciprocal lattice

$$e^{i\mathbf{G}\cdot\mathbf{R}} = 1,$$



Reciprocal lattice and scattering

The reciprocal lattice

$$e^{i\mathbf{G}\cdot\mathbf{R}} = 1,$$

$$\mathbf{G} = h\mathbf{b}_1 + k\mathbf{b}_2 + l\mathbf{b}_3, \quad h, k, l \in \mathbb{Z}$$



Reciprocal lattice and scattering

The reciprocal lattice

$$e^{i\mathbf{G}\cdot\mathbf{R}} = 1,$$

$$\mathbf{G} = h\mathbf{b}_1 + k\mathbf{b}_2 + l\mathbf{b}_3, \quad h, k, l \in \mathbb{Z}$$

$$\mathbf{a}_i \cdot \mathbf{b}_j = 2\pi\delta_{ij},$$



Reciprocal lattice and scattering

Scattering



Reciprocal lattice and scattering

Scattering

Free particles:



Reciprocal lattice and scattering

Scattering

Free particles:

$$E_{\mathbf{k}} = \frac{\hbar^2 \mathbf{k}^2}{2m},$$



Reciprocal lattice and scattering

Scattering

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Reciprocal lattice and scattering

Scattering

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Fermi's Golden Rule:



Reciprocal lattice and scattering

Scattering

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Fermi's Golden Rule:

$$\Gamma(\mathbf{k}', \mathbf{k}) = \frac{2\pi}{\hbar} |\langle \mathbf{k}' | V | \mathbf{k} \rangle|^2 \delta(E_{\mathbf{k}'} - E_{\mathbf{k}}),$$



Reciprocal lattice and scattering

Scattering

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$$E_{\mathbf{k}} = E_{\mathbf{k}'}, \quad \Rightarrow \quad |\mathbf{k}| = |\mathbf{k}'|$$



Reciprocal lattice and scattering

Scattering

$$\langle \mathbf{k}' | V | \mathbf{k} \rangle = \int_{-\infty}^{+\infty} \frac{e^{-i\mathbf{k}' \cdot \mathbf{r}}}{\sqrt{L^3}} V(\mathbf{r}) \frac{e^{i\mathbf{k} \cdot \mathbf{r}}}{\sqrt{L^3}} d\mathbf{r},$$



Reciprocal lattice and scattering

Scattering

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Reciprocal lattice and scattering

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$$V(\mathbf{r} + \mathbf{R}) = V(\mathbf{r}), \quad \mathbf{r} = \mathbf{R} + \mathbf{x},$$



Reciprocal lattice and scattering

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Reciprocal lattice and scattering

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Reciprocal lattice and scattering

Scattering

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$$\sum_{\mathbf{R}} e^{-i(\mathbf{k}' - \mathbf{k}) \cdot \mathbf{R}} = \begin{cases} N, & \mathbf{k}' - \mathbf{k} = \mathbf{G}, \\ 0, & \mathbf{k}' - \mathbf{k} \neq \mathbf{G}. \end{cases}$$



Reciprocal lattice and scattering

Scattering conditions



Reciprocal lattice and scattering

Scattering conditions

$$|\mathbf{k}| = |\mathbf{k}'|,$$



Reciprocal lattice and scattering

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Reciprocal lattice and scattering

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Neutron scattering on cubic lattice with a basis



Reciprocal lattice and scattering

Scattering conditions

$$|\mathbf{k}| = |\mathbf{k}'|, \quad \mathbf{k}' - \mathbf{k} = \mathbf{G}.$$

Neutron scattering on cubic lattice with a basis

$$I \propto |S(\mathbf{G})|^2.$$



Reciprocal lattice and scattering

Neutron scattering



Reciprocal lattice and scattering

Neutron scattering

$$V(\mathbf{x}) = \sum_{\text{atoms } j} f_j \delta(\mathbf{x} - \mathbf{x}_j),$$



Reciprocal lattice and scattering

Neutron scattering

$$V(\mathbf{x}) = \sum_{\text{atoms } j} f_j \delta(\mathbf{x} - \mathbf{x}_j),$$

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Reciprocal lattice and scattering

Systemic absences



Reciprocal lattice and scattering

Systemic absences

$$S(\mathbf{G}) = S_{hkl} = \sum_{\text{atoms } j} f_j e^{2\pi i(hx_j + ky_j + lz_j)}$$



Reciprocal lattice and scattering

Systemic absences

$$S(\mathbf{G}) = S_{hkl} = \sum_{\text{atoms } j} f_j e^{2\pi i(hx_j + ky_j + lz_j)}$$

bcc ($h + k + l$ even)

$$S_{hkl} = f(1 + (-1)^{h+k+l})$$



Reciprocal lattice and scattering

Systemic absences

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bcc ($h + k + l$ even)

$$S_{hkl} = f(1 + (-1)^{h+k+l})$$

fcc (h, k, l all even or odd)

$$S_{hkl} = f(1 + e^{\pi i(h+k)} + e^{\pi i(k+l)} + e^{\pi i(h+l)})$$



Reciprocal lattice and scattering



Band structure

2 dimensional square lattice.

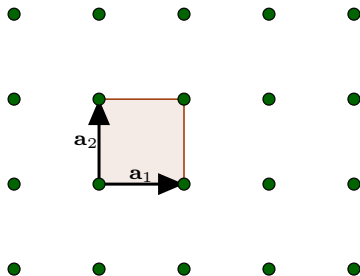


Figure: A square lattice. $\mathbf{a}_1 = a \cdot (1, 0)$, $\mathbf{a}_2 = a \cdot (0, 1)$.

First Brillouin zone: $-\pi/a \leq k_i \leq \pi/a$



Band structure

Fourier transform the Schrödinger equation.

$$V(\mathbf{r} + \mathbf{R}) = V(\mathbf{r}) \quad \Leftrightarrow \quad V(\mathbf{r}) = \sum_{\mathbf{G}} V_{\mathbf{G}} e^{i\mathbf{G} \cdot \mathbf{r}},$$

$$V_{\mathbf{G}} = \frac{1}{a^2} \int_{\text{unit-cell}} e^{-i(\mathbf{k}' - \mathbf{k}) \cdot \mathbf{x}} V(\mathbf{x}) \, d\mathbf{x},$$

$$\sum_{\mathbf{G}} \left[\frac{\hbar^2 \mathbf{k}^2}{2m} \delta_{\mathbf{G},0} + V_{\mathbf{G}} \right] \tilde{\psi}(\mathbf{k} - \mathbf{G}) = E \tilde{\psi}(\mathbf{k})$$



Band structure

Matrix equation and notation.

$$V_{\mathbf{G}} = V_{[m_1, m_2]}, \quad \mathbf{G} = m_1 \mathbf{b}_1 + m_2 \mathbf{b}_2,$$

$$|\psi\rangle = \begin{pmatrix} \vdots \\ \tilde{\psi}(\mathbf{k} - \mathbf{G}_1) \\ \tilde{\psi}(\mathbf{k}) \\ \tilde{\psi}(\mathbf{k} + \mathbf{G}_1) \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ \psi_{[0, -1]} \\ \psi_{[0, 0]} \\ \psi_{[0, 1]} \\ \vdots \end{pmatrix},$$

$$T = \frac{\hbar^2}{2m} \begin{pmatrix} \ddots & & & & \\ & (\mathbf{k} - \mathbf{G}_1)^2 & & & \\ & & \mathbf{k}^2 & & \\ & & & (\mathbf{k} + \mathbf{G}_1)^2 & \\ & & & & \ddots \end{pmatrix}.$$



Band structure

$$[m_1, m_2] \in \{[-1, -1], [-1, 0], [-1, 1], \\ [0, -1], [0, 0], [0, 1], \\ [1, -1], [1, 0], [1, 1]\}.$$

$$\begin{aligned} E\psi_{[-1,0]} &= \sum_{m'_1=-\infty}^{\infty} \sum_{m'_2=-\infty}^{\infty} V_{[m'_1, m'_2]} \psi_{[-1-m'_1, -m'_2]}, \\ &= \cdots + V_{[-2,-1]}\psi_{[1,1]} + V_{[-2,0]}\psi_{[1,0]} + V_{[-2,1]}\psi_{[1,-1]} + \cdots \\ &\quad + V_{[-1,-1]}\psi_{[-2,1]} + V_{[-1,0]}\psi_{[-2,0]} + V_{[-1,1]}\psi_{[-2,-1]} + \cdots \\ &\quad + V_{[0,-1]}\psi_{[-1,1]} + V_{[0,0]}\psi_{[-1,0]} + V_{[0,1]}\psi_{[-1,-1]} + \cdots \\ &\quad + V_{[1,-1]}\psi_{[0,1]} + V_{[1,0]}\psi_{[0,0]} + V_{[1,1]}\psi_{[0,-1]} + \cdots \end{aligned}$$



Band structure

$$\begin{aligned}[m_1, m_2] \in \{ & [-1, -1], [-1, 0], [-1, 1], \\ & [0, -1], [0, 0], [0, 1], \\ & [1, -1], [1, 0], [1, 1] \}.\end{aligned}$$

Row $[m_1, m_2]$, column $[m'_1, m'_2]$: $V_{[m_1-m'_1, m_2-m'_2]}$.



Band structure

$$V_{\text{dirac}}(\mathbf{r}) = V_0 a^2 \sum_{\mathbf{R}} \delta(\mathbf{r} - \mathbf{R}), \quad V_{\text{dirac}, \mathbf{G}} = V_0,$$

$$V_{\text{harmonic}}(\mathbf{r}) = V_0 \left[\cos\left(\frac{2\pi}{a}x\right) + \cos\left(\frac{2\pi}{a}y\right) \right],$$

$$V_{\text{harmonic}, \mathbf{G}} = \begin{cases} \frac{V_0}{2} & \text{if } [m_1, m_2] \in \{[0, 1], [0, -1], [1, 0], [-1, 0]\}, \\ 0 & \text{else.} \end{cases}$$



Band structure

$$V_{\text{harmonic}} = \frac{V_0}{2} \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$



Band structure

Define $k_0 \equiv 2\pi/a$, $E_0 \equiv \hbar^2 k_0^2/m$:

$$\sum_{\tilde{\mathbf{G}}} \left[\frac{\tilde{\mathbf{k}}^2}{2} \delta_{\tilde{\mathbf{G}},0} + \tilde{V}_{\tilde{\mathbf{G}}} \right] \tilde{\psi}(\tilde{\mathbf{k}} - \tilde{\mathbf{G}}) = \tilde{E} \tilde{\psi}(\tilde{\mathbf{k}}),$$

$$\tilde{\mathbf{G}} \equiv \mathbf{G}/k_0 = m_1 \hat{\mathbf{x}} + m_2 \hat{\mathbf{y}},$$

$$\tilde{\mathbf{k}} \equiv \mathbf{k}/k_0, \quad -\frac{1}{2} \leq \tilde{\mathbf{k}}_x \leq \frac{1}{2}$$

$$\tilde{V}_{\tilde{\mathbf{G}}} \equiv V_{\mathbf{G}}/E_0, \quad \tilde{E} \equiv E/E_0$$



Band structure



Discussion

- 3 (4) programs
 - Lattice plotting
 - (Plotting of Lattice planes)
 - Scattering simulation
 - Band structure of 2D materials
- command line interface - convert to graphical user interface
- Problems with Matplotlib

