CMIS Hand-in 5: Finite Element Method 2

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1 INTRODUCTION

In this hand-in we focus on solving a linear, elastic deformation for a system, using the finite element method. The governing equation for the system is the Cauchy momentum equation:

$$\rho \ddot{\mathbf{x}} = \mathbf{b} + \nabla \cdot \sigma, \tag{1}$$

where $\mathbf{x}$ are the deformed (or spatial) coordinates of the material, ρ is the mass density of the body, $\mathbf{b}$ is the different body forces acting on the system, and σ is the Cauchy stress tensor. On the boundary we have $\sigma\mathbf{n} = \mathbf{t}$, where $\mathbf{t}$ is the surface traction.

The deformed coordinates can also be expressed as a function of the deformation field Φ, with the undeformed (or material) coordinates being the value of the field at $t = 0$:

$$\mathbf{x} = \mathbf{\Phi}(\mathbf{X}, t), \quad \mathbf{X} = \mathbf{\Phi}(\mathbf{X}, 0). \tag{2}$$

In this case we will focus on a homogeneous rectangular bar in two dimensions, whose left side is adhered to a wall, and whose right side experiences a constant traction over its area (or length, rather). We will also consider the quasistatic problem, instead of the dynamic problem. As such we set out to solve for the value $\mathbf{\Phi}(\mathbf{X}, \infty)$, where $\ddot{\mathbf{x}} = 0$. Further we neglect all body forces, such as gravity. With this, the governing equation becomes:

$$\nabla \cdot \sigma = 0 \tag{3}$$

Now we perform the regular steps of the finite element method: We multiply the equation by some appropriate trial function $\mathbf{v}$ and then integrate over the volume of the system:

$$\int_\Omega (\nabla \cdot \sigma) \cdot \mathbf{v} \, \mathrm{d}\Omega = 0 \tag{4}$$

Next we use the product rule for divergence of tensors to split the integral in two:

$$\int_\Omega (\nabla \cdot \sigma) \cdot \mathbf{v} \, \mathrm{d}\Omega = \int_\Omega \nabla \cdot (\sigma \mathbf{v}) \, \mathrm{d}\Omega - \int_\Omega \sigma : \nabla \mathbf{v}^T \, \mathrm{d}\Omega = 0 \tag{5}$$

Using Gauss' theorem for divergence on the first integral gives us:

$$\int_\Omega \nabla \cdot (\sigma \mathbf{v}) \, \mathrm{d}\Omega = \int_{\partial\Omega} (\sigma \mathbf{v}) \cdot \mathbf{n} \, \mathrm{d}S \tag{6}$$

$$= \int_{\partial\Omega} \mathbf{v} \cdot (\sigma \mathbf{n}) \, \mathrm{d}S = \int_{\partial\Omega} \mathbf{v} \cdot \mathbf{t} \, \mathrm{d}S \tag{7}$$

In the second integral we leverage the fact that σ is symmetric to write:

$$\int_\Omega \sigma : \nabla \mathbf{v}^T \, \mathrm{d}\Omega = \int_\Omega \sigma : \nabla \mathbf{v} \, \mathrm{d}\Omega \tag{8}$$

$$= \int_\Omega \sigma : \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^T) \, \mathrm{d}\Omega \tag{9}$$

Now we choose our trial function to be a virtual displacement $\delta\mathbf{u}$ of the system. This can be written, as in last week, as the product of our trusty barycentric coordinates N^e for the triangular elements, and a virtual displacement $\delta\mathbf{u}^e$ for each element:

$$\mathbf{v} = \delta\mathbf{u} = N^e \delta\mathbf{u}^e \tag{10}$$

where the barycentric coordinates are written as a 2×6 matrix and the virtual displacement is a 6-component vector:

$$N^e = [N_i^e I_2 \; N_j^e I_2 \; N_k^e I_2], \quad \delta\mathbf{u}^e = [\delta u_{i,x}^e \; \delta u_{j,x}^e \; \cdots \; \delta u_{k,y}^e] \tag{11}$$