1 Lagrangian

The Lagrangian for the system is given by

$$L = T - V \tag{1}$$

with T being the kinetic energy and V being the potential energy. These can be further split up into the kinetic and potential energy for each pendulum. We get

$$T_1 = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2), \quad T_2 = \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) \tag{2}$$

And

$$V_1 = m_1 g y_1, \quad V_2 = m_2 g y_2 \tag{3}$$

We further choose to use the angles the pendulums make with the negative y-direction as the generalized coordinates. The conversions are

$$x_1 = r_1 \sin\theta_1, \quad y_1 = -r_1\cos\theta_1, \quad x_2 = r_1\sin\theta_1 + r_2\sin\theta_2, \quad y_2 = -r_1\cos\theta_1 - r_2\cos\theta_2 \tag{4}$$

and

$$\theta_1 = \arctan(y_1/x_1) + \pi/2 \bmod 2\pi \quad \theta_2 = \arctan\frac{y_2 - y_1}{x_2 - x_1} + \pi/2 \bmod 2\pi. \tag{5}$$

In the new coordinates, the Lagrangian becomes

$$L = \frac{m_1 + m_2}{2}(\dot{\theta}_1 r_1)^2 + \frac{m_2}{2}(\dot{\theta}_2 r_2)^2 + m_2 r_1 r_2 \dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2) + g r_1(m_1 + m_2)\cos\theta_1 + m_2 g r_2\cos\theta_2 \tag{6}$$

Giving us the equations of motion:

$$\ddot{\theta}_1 = -\frac{m_2 r_2}{(m_1 + m_2)r_1}\left[\ddot{\theta}_2\cos(\theta_1 - \theta_2) + \dot{\theta}_2^2\sin(\theta_1 - \theta_2)\right] - \frac{g}{r_1}\sin\theta_1 \tag{7}$$

$$\ddot{\theta}_2 = -\frac{r_1}{r_2}\ddot{\theta}_1\cos(\theta_1 - \theta_2) + \frac{r_1}{r_2}\dot{\theta}_1^2\sin(\theta_1 - \theta_2) - \frac{g}{r_2}\sin\theta_2. \tag{8}$$

or, decoupling them:

$$\begin{aligned}
\ddot{\theta}_1 = f(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) &= \frac{m_2 r_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2)\cos(\theta_1 - \theta_2) + m_2 g \sin\theta_2\cos(\theta_1 - \theta_2)}{(m_1 + m_2)r_1 - m_2 r_1\cos^2(\theta_1 - \theta_2)} \\
&+ \frac{m_2 r_2 \dot{\theta}_2^2\sin(\theta_1 - \theta_2) - (m_1 + m_2)g\sin\theta_1}{(m_1 + m_2)r_1 - m_2 r_1 \cos^2(\theta_1 - \theta_2)} \\
\ddot{\theta}_2 = g(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) &= \frac{-m_2 r_2 \dot{\theta}_2^2\sin(\theta_1 - \theta_2)\cos(\theta_1 - \theta_2) + (m_1 + m_2)g\sin\theta_1\cos(\theta_1 - \theta_2)}{\frac{r_2}{r_1}\left[(m_1 + m_2)r_1 - m_2 r_1\cos^2(\theta_1 - \theta_2)\right]} \\
&+ \frac{-(m_1 + m_2)r_1\dot{\theta}_1^2\sin(\theta_1 - \theta_2) - (m_1 + m_2)g\sin\theta_2}{\frac{r_2}{r_1}\left[(m_1 + m_2)r_1 - m_2 r_1\cos^2(\theta_1 - \theta_2)\right]}
\end{aligned}$$

Next we use the fourth order Runge-Kutta method for advancing in time. Let $\dot{y} = f(y, t)$, then

$$y(t + \Delta t) = y(t) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), \tag{9}$$

$$k_1 = \Delta t\ f(y(t), t), \tag{10}$$

$$k_2 = \Delta t\ f(y(t) + k_1/2, t + \Delta t/2), \tag{11}$$

$$k_3 = \Delta t\ f(y(t) + k_2/2, t + \Delta t/2), \tag{12}$$

$$k_4 = \Delta t\ f(y(t) + k_3, t + \Delta t). \tag{13}$$