Notes for Dynamical Systems and Chaos

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1 Introduction

In general we work with systems of differential equations of the form:

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, \ldots, x_n),\\ &\vdots\\ \dot{x}_n &= f_n(x_1, \ldots, x_n).\end{aligned}$$

Where we say that this is an n-dimensional system. Note that this does not coincide with our regular use of the word dimension, but rather this is what we would normally call n degrees of freedom. For example, the simple harmonic oscillator is a system in one dimension, but with two degrees of freedom (x and $\dot{x}$). We can write it in the form above by defining $x_1 = x$, $x_2 = \dot{x}$:

$$\ddot{x} = -\frac{k}{m}x \to \begin{cases}\dot{x}_1 = x_2,\\ \dot{x}_2 = -\frac{k}{m}x_1.\end{cases} \tag{1.1}$$

where of course $f_1 = x_2$ and $f_2 = -kx_1/m$.

One can incorporate time explicitly by defining a new coordinate $x_{n+1} = t$, with $f_{n+1} = 1$.

Further, we normally deal with systems of equations which might not be solvable analytically. However, this does not mean no useful information can be garnered from them. We will find formulas for fixed points and bifurcations, looking at how the systems of equations changes when varying one or more parameters (say k or m in the harmonic oscillator).

2 One dimensional systems

In this section we deal with systems of the form

$$\dot{x} = f(x). \tag{2.1}$$

Useful quantities that are easily understood in one dimensional systems (but of course apply to systems of many dimensions) include fixed points, linear stability analysis, potentials and bifurcations.

2.1 Fixed point and stability analysis

Fixed points x^* occur whenever the function $f(x)$ has a root. These points are called fixed since the solution to the equation is constant in time. A simple example is the system $\dot{x} = x$, which has a fixed point at $x^* = 0$. One represents the system graphically like this:

PIC

However not all fixed points are created equally. In the example above, we see that the flow is negative when x is negative, whilst it is positive when x is positive. This then means that any small perturbation from the fixed point will always increase (in magnitude) with time. We call these points *unstable*, and draw them as unfilled circles.

On the other hand, the system $\dot{x} = -x$ has a *stable* fixed point at $x^* = 0$, since all perturbations will be (exponentially) damped. These points are drawn as filled circles.

In general, the derivative of $f(x)$, with respect to x determines the stability of the point. Often, **linear stability analysis** enough. This states that x^* is a stable (unstable) fixed point, if $f'(x^*) < 0$ ($f'(x^*) > 0$). If, however, $f'(x^*) = 0$ then linear stability analysis will not be enough to determine the stability of the point, and one needs to look at the higher order derivatives of $f(x)$.

For example, if $f''(x^*) > 0$ ($f''(x^*) < 0$), the system has minimum (maximum), and the system is left (right) half stable. A left half stable fixed point is stable to perturbations pushing it to the left, but unstable to perturbations pushing it to the right. An example of this is $\dot{x} = x^2$, which has a left stable fixed point at $x^* = 0$. It is represented as a half filled circle.

If then both the first and second derivative at the fixed point is zero, then the third derivative can be used, where the point is stable if the third derivative is less than zero, and unstable if it is positive. Of course one can go as far as one likes with this, and utilize higher and higher derivatives.

2.2 Potentials

Another way to view the one dimensional system is through the use of potentials. Here we define

$$f(x) = -\frac{dV(x)}{dx} \tag{2.2}$$

Fixed points in the system are then stationary points for the potential. In this formulation, one can imagine a ball rolling down a hill (though without inertia, as if flowing through molasses). Minima correspond to stable fixes points and maxima to unstable points (and saddle points to half stable points, of course).

2.3 Bifurcations

A bifurcation is a point in parameter space, at which the stability or number of fixed points changes. In this section a number of bifurcation types appear:

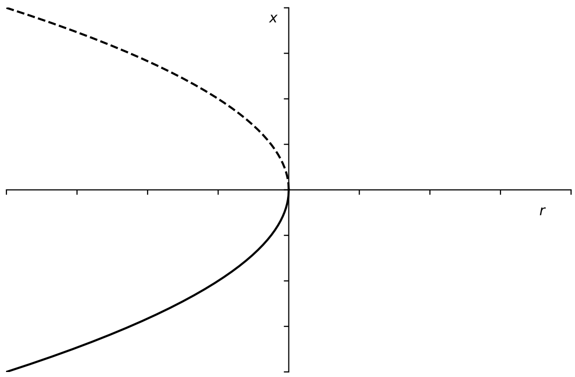
- Saddle-node bifurcations
- Transcritical bifurcations
- Pitchfork bifurcations (both subcritical and supercritical)

When introducing these types of bifurcations one often looks at normal forms, which are the simplest system that exhibit the behaviour in question. Another way of looking at this, is that the Taylor expansion for a system follows the normal form (or at least a scaled version of it) in the vicinity of the bifurcation.

A saddle-node bifurcation is a point in which two fixed points appear out of nothing. The normal form for this is

$$\dot{x} = r + x^2 \tag{2.3}$$

This system has a saddle-node bifurcation at $r_c = 0$, at which a left half stable point appears at $x^* = 0$. If r is decreased further, this point separates into a stable point at $x^* = -\sqrt{-r}$ and an unstable point at $x^* = \sqrt{-r}$. This can be drawn in a **bifurcation diagram**, where one plots the fixed points as a function of the parameter (in this case r):

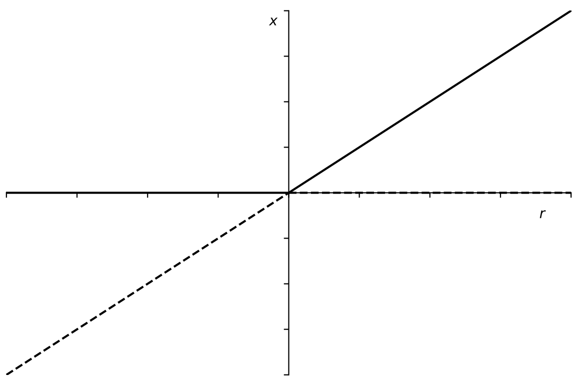


In a bifurcation diagram, stable *branches* (lines indicating stable fixed points) are drawn up fully, whilst unstable branches are drawn dashed.

A transcritical bifurcation is one in which no fixed points are created or destroyed, but in stead changes its stability. The normal form for these kinds of bifurcations is

$$\dot{x} = rx - x^2, \tag{2.4}$$

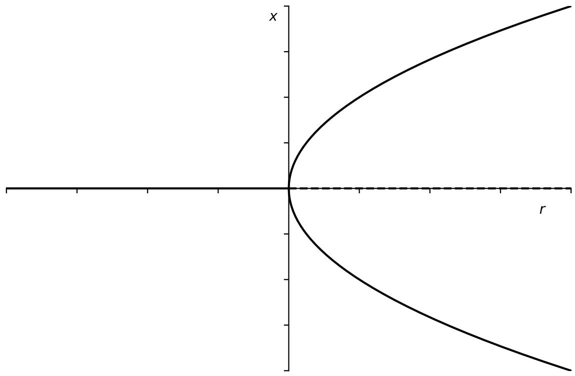
which always has a fixed point at $x^* = 0$ (and one at $x^* = r$). The bifurcation occurs at $r_c = 0$, with the fixed point being stable (unstable) for $r < r_c$ ($r > r_c$) and right half stable at $r = r_c$. Its bifurcation diagram looks like so:



A pitchfork bifurcation is one where a single fixed point turns into three fixed points, reminiscent of the shape of a pitchfork. The supercritical pitchfork starts out as a stable point, which turns into an unstable fixed point at $r = r_c$, and forms two additional stable points for $r > r_c$. Its normal form is

$$\dot{x} = rx - x^3, \quad r_c = 0, \quad x^* = 0 \vee x^* = \pm\sqrt{r}, \tag{2.5}$$

and its bifurcation diagram looks like so:



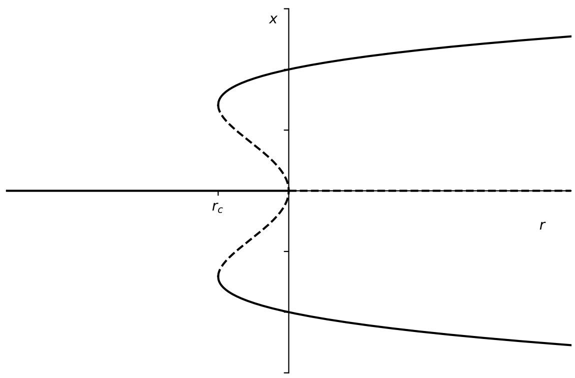
For a subcritical pitchfork bifurcation the pitchfork is inverted. The normal form and fixed points are

$$\dot{x} = rx + x^3, \quad r_c = 0, \quad x^* = 0 \vee x^* = \pm\sqrt{-r}, \tag{2.6}$$

The outer fixed points are unstable however, and the system is unphysical. Therefore one often looks at the following system:

$$\dot{x} = rx + x^3 - x^5, \quad x^* \in \left\{0, \pm\sqrt{\frac{1}{2} + \frac{\sqrt{1+4r}}{2}}, \pm\sqrt{\frac{1}{2} - \frac{\sqrt{1+4r}}{2}}\right\} \tag{2.7}$$

for $r < r_s$, there is only the fixed point at the origin. At $r_s = -1/4$ a double saddle node bifurcation occurs, and at r=0, a supercritical pitchfork bifurcation occurs, where the two inner, unstable fixed points (the last ones in the curly braces) coalesce with the stable fixed point at the origin, into a single unstable fixed point at the origin. The bifurcation diagram is:



This system can undergo catastrophes and hysteresis: if one starts at $x = 0, -1/4 < r < 0$, and varies r until $r > 0$, then x will snap to one of the other stable branches, since the origin is now an unstable fixed point. x will now stay at this point if r is returned to the initial value. This is hysteresis and catastrophe: through a (closed) walk in parameter space, the state of the system has changed!

One last "type" of bifurcations to consider is the imperfect bifurcation. This is not really a new type, but more the introduction of a parameter which affects the bifurcations of the system. Consider

$$\dot{x} = h + rx - x^3 \tag{2.8}$$

If $h = 0$, we have a supercritical pitchfork bifurcation, but for all other values $h \neq 0$, this is changed to a saddle node bifurcation at some (complicated) value of x, as seen below.

2.4 Periodic systems - flows on the circle

Lastly in one dimension, we have flows on the circle, which is really just differential equations with periodic boundary conditions. This means

$$f(x + x_0) = f(x) \tag{2.9}$$

for some x_0. Usually we identify $\theta = x$ and $x_0 = 2\pi$, corresponding to flows defined on the circle. This means that we can actually have oscillations (as long as there are no fixed points).

One cool result is that if $f(\theta) \approx 0$, but there is no fixed point, then f is at a minimum. Then $f \approx r + x^2$, for $r > 0$. The time it takes to go from $-\infty$ to ∞ (the time it takes to pass through this bottle neck) is

$$T_{\text{bottleneck}} \approx \int_{-\infty}^{+\infty} \frac{dx}{r + x^2} = \frac{\pi}{\sqrt{r}}. \tag{2.10}$$

This is called the square-root scaling law. If instead you had a maximum, then there'd be an overall minus on f, with r still larger than 0. The integral would then have to go from ∞ to $-\infty$ (since we're going the other way). Switching the limits costs us a minus sign, which gobbles up the minus on f, leaving us with the same result as before.

3 Two dimensions (or more)

In this section we mostly discuss two dimensional systems, but most of the results generalize to higher dimensions.

3.1 Linear systems

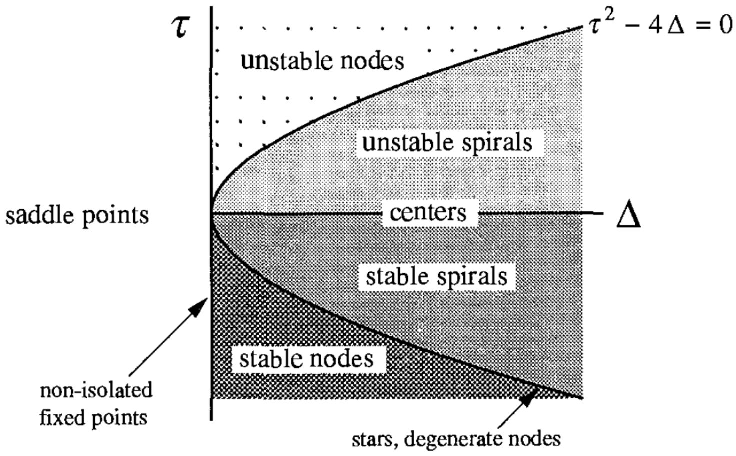
A linear system is one on the form

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}. \tag{3.1}$$

These systems always have a single fixed point at the origin, and the parameters a, b, c, d determine the behaviour around the fixed point. Actually, it is easier to classify according to the eigenvalues λ_+, λ_-, of the matrix. These can be found easily from the trace and determinant of the matrix:

$$\tau = \mathrm{Tr}(A) = \lambda_+ + \lambda_-, \quad \Delta = \det(A) = \lambda_+\lambda_-, \quad \lambda_\pm = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right). \tag{3.2}$$

The classification can be summed up in this diagram:



Now for some actual definitions of stability and explanations:

- Attracting: $\mathbf{x}^*$ is attracting, if trajectories that start near it approach it in infinite time: $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}^*$.
- Liapunov stable: $\mathbf{x}^*$ is Liapunov stable if all trajectories that start near it stay near it.
- (Asymptotically) stable: $\mathbf{x}^*$ is stable if the point is both attracting and Liapunov stable.
- Neutrally stable: $\mathbf{x}^*$ is neutrally stable if it is not attracting, but it is Liapunov stable.
- Unstable: $\mathbf{x}^*$ is unstable if it is neither attracting nor Liapunov stable.

In two dimensions there are several different kinds of fixed points, which are covered below:

- **(Un)stable nodes:** A fixed point that is attractive (repulsive), with one direction (eigenvector) attracting/repulsing quicker than the other. This happens when $\lambda_+, \lambda_- < 0$, and $\lambda_+ \neq \lambda_-$ ($\lambda_+, \lambda_- > 0, \lambda_+ \neq \lambda_-$). The most attractive (repulsive) direction, called the fast eigendirection, is the one whose eigenvalue has the largest magnitude.
- **Saddle points:** A fixed point with one attractive eigendirection (λ_-) and one repulsive (λ_+). For the saddle point, one has a *stable manifold*, which is defined as all $\mathbf{x}_0$ such that $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}^*$, and an *unstable manifold*, defined as all $\mathbf{x}_0$ such that $\lim_{t\to-\infty} \mathbf{x}(t) = \mathbf{x}^*$. A typical trajectory will asymptotically approach the *unstable* manifold as $t \to \infty$, and the *stable* manifold as $t \to -\infty$.

  We also define the *basin of attraction* for a saddle point, which is the set of all initial conditions that approach the fixed point as $t \to \infty$.

- **Stars:** A attractive (repulsive) fixed point, with equal eigenvalues $\lambda_+ = \lambda_-$ and two distinct eigenvectors. This means that all directions attract (repel) with equal strength, leading to a phase portrait in the shape of a star.
- **Centers:** A center occurs when both eigenvalues are purely imaginary, leading to an oscillating solution of ellipses.
- **(Un)stable spirals:** Occurs when the eigenvalues are complex, leading to an oscillating and an attractive (repelling) term, causing the trajectory to spiral inwards (outwards).
- **Degenerate nodes:** Occurs when $\lambda_- = \lambda_+$, but there is only one non-trivial eigenvector. Trajectories are roughly parallel to the eigenvector.
- **Non-isolated fixed points:** Occurs when one eigenvalue (λ_-) is 0, while the other is non-zero. The line of fixed points occurs along the eigenvector corresponding to λ_- (funilly enough, as it's the definition of a fixed point).

3.2 Nonlinear systems

For nonlinear systems we can still solve for fixed points, but to determine the stability we use linear stability analysis, generalized to higher dimensions. Here we calculate the Jacobian matrix for a fixed point:

$$A = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} \tag{3.3}$$

where $\dot{x} = f(x, y), \dot{y} = g(x, y)$, and the derivatives are to be evaluated at the fixed point (x^*, y^*). The eigenvalues of this matrix can then be used to determine the stability, like in linear systems. However, this only works for the robust cases, where $\Delta \neq 0, \tau \neq 0$ and $\tau^2 - 4\Delta \neq 0$ (ie, when we're dealing with nodes, spirals and saddle points).

3.2.1 Conservative systems

A conservative system is one in which there is some conserved quantity $E(\mathbf{x})$, ie $dE/dt = 0$ along trajectories. For example, if the system is of the form

$$m\ddot{x} + F(x) \tag{3.4}$$

where there is no explicit dependence of $\dot{x}$ or t for F. Then we can write $F(x) = -dV/dx$, where V is the potential energy. Then $E = 1/2m\dot{x}^2 + V(x)$ is constant along trajectories. To rule out silliness, we also demand that E is not just a constant function on every open set: $E(\mathbf{x}) \neq c$.

Conservative systems have a couple of consequences:

- A conservative system cannot have any attracting fixed points
- Centers of a conservative system are robust. The following theorem (6.5.1) states: Suppose $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, where $\mathbf{x} \in \{R\}$, and $\mathbf{f}$ is continuously differentiable. If a conserved quantity $E(\mathbf{x})$ exist, and $\mathbf{x}^*$ is an isolated fixed point, where $E(\mathbf{x}^*)$ is a local minimum (or maximum), then all trajectories sufficiently close to $\mathbf{x}^*$ are closed.

A type of trajectory that often occurs in a conservative system is that of a *homoclinic trajectories*, which is a trajectory that starts at one fixed point, and ends at the same fixed point (at $t \to \infty$). This is as opposed to a *heteroclinic trajectories*, which starts at one fixed point, but end at another. Heteroclinic trajectories occur more often in reversible systems.

3.2.2 Reversible systems

A reversible system is one that exhibits time reversal symmetry. This means that the system of equations should be unchanged under the transformation

$$t \to -t, \dot{x} = y \to -\dot{x} = -y. \tag{3.5}$$

This for example occurs when

$$\dot{x} = f(x, y), \dot{y} = g(x, y), \quad f(x, -y) = -f(x, y), \quad g(x, -y) = g(x, y). \tag{3.6}$$

A consequence of reversible systems is that all trajectories that are sufficiently close to a linear centre are closed, as per theorem 6.6.1:

Suppose the origin $\mathbf{x}^* = 0$ of the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is a linear centre, and the system is reversible. Then all trajectories sufficiently close to the origin are closed.

This can of course be generalized to linear centres that are not at the origin, by a shift in coordinates.

In higher dimensions reversibility can be defined as a mapping of the phase space $\mathbf{x} \to \mathbf{R}(\mathbf{x})$, where $\mathbf{R}^2(\mathbf{x}) = \mathbf{x}$, and the system is invariant under $t \to -t$, $\mathbf{x} \to \mathbf{R}(\mathbf{x})$.

3.2.3 Index theory

Index theory is a method by which one can get global information out of the system (as opposed to linearisation, which only gives local information). Assume we have a smooth vector field $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and a simple closed curve C (ie, not self-intersecting, and not intersecting any fixed point). Then the vector field makes an angle $\phi = \arctan(\dot{y}/\dot{x})$ with respect to the x-axis. This angle changes an integer multiple of 2π along the curve (since the angle must be the same after one trip around the curve). Let $[\phi]_C$ be the net change in angle around the curve, then the index is

$$I_C = \frac{[\phi]_C}{2\pi}. \tag{3.7}$$

The index has the following properties:

- Suppose C can be continuously deformed into another curve C', then $I_C = I_{C'}$.
- If there are no fixed points in C, then $I_C = 0$
- The index is invariant under the transformation $t \to -t$.
- Suppose C is actually a trajectory (a closed orbit), then $I_C = +1$, no matter how many fixed points the curve encloses.
- The index I of a fixed point is equal to the index of any curve C that encloses only that fixed point and no others.
- A node (stable or unstable) has index $I = 1$. A saddle has $I = -1$.
- Spirals, centres and degenerate nodes all have index $I = 1$
- The index of a curve C that encloses n fixed points with indices I_n is $I_C = \sum_{k=1}^{n} I_k$

3.3 Limit cycles

A limit cycle is an isolated, closed orbit, such that neighbouring trajectories either spiral towards it (a stable limit cycle) or spiral away from it (an unstable limit cycle). You can also have half-stable limit cycles, where on the inside they spiral towards and on the outside they spiral away (or vice versa).

Limit cycles only happen in nonlinear systems, like $\dot{r} = r(1 - r^2), \dot{\theta} = 1$, which has a stable limit cycle at $r^* = 1$.

There are a couple of ways to either rule out or establish the existence of limit cycles:

3.3.1 Ruling out orbits: Gradient systems, Liapunov functions and Dulac's criterion

Gradient Systems: Suppose a system can be written as $\dot{\mathbf{x}} = -\nabla V$ for a continuous, differentiable, single valued scalar $V(\mathbf{x})$. This is called a gradient system, and no closed orbits are possible in such a system.

Liapunov functions: Consider a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with a fixed point $\mathbf{x}^*$. Suppose one can find a function $V(\mathbf{x})$ satisfying the following conditions:

- $V(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{x}^*$ and $\mathbf{x}^*$ (V is positive definite).
- $dV/dt < 0$ for all $\mathbf{x} \neq \mathbf{x}^*$ (trajectories flow "downhill" along V).

Such a function is called a Liapunov function, and the fixed point $\mathbf{x}^*$ is globally, asymptotically stable. As such no closed orbits are possible.

There is no set recipe for finding such a function, but sometimes sums of squares ($V = bx^2 + ay^2$) work.

Dulac's criterion: Let $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ be continuously differentiable on a simply connected subset R of the plane. If there exists a continuously differentiable, real valued function $g(\mathbf{x})$ such that $\nabla \cdot (g\dot{\mathbf{x}})$ has the same sign throughout R then there are no closed orbits lying entirely in R.

Again, no recipe for g exists, but the following functions sometimes work: $g = 1, 1/x^a y^b, e^{ax}, e^{ay}$.

3.3.2 Establishing orbits: Poincaré-Bendixson theorem and trapping regions

The Poincare-Bendixson theorem states that if you have the following:

- A closed subset of the plane R
- A continuously differentiable vector field $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, on an open set containing R
- R contains no fixed points
- A trajectory C is "confined" to R, in that it stays in R for all time.

Then C is either a closed orbit, or it approaches one as $t \to \infty$. In any case R contains a closed orbit. The three first conditions are easy enough to satisfy, but the last one is a bit tougher. This is where the construction of trapping regions come in.

A Trapping Region is a closed subset R, containing no fixed points, and where the vector field on the boundary always points inward. This means that a trajectory which starts in R has no way of escaping (since if it reaches the end, it will necessarily turn inwards). Such a region then satisfy all four conditions of the above theorem.

3.3.3 Weakly non-linear oscillators, two timing and averaged equations

Many non-linear oscillators exhibit two distinct timescales. One regular, t and a slow time scale. For example, the damped oscillator $\ddot{x} + 2\varepsilon\dot{x} + x = 0$, with $x(0) = 0, \dot{x}(0) = 1$ has the solution

$$x(t) = \frac{e^{-\varepsilon}}{\sqrt{1-\varepsilon^2}} sin[(1-\varepsilon^2)^{1/2}t], \tag{3.8}$$

which exhibit (for small ε, where the damping is weak) a long time scale damping of $T = \varepsilon t$. Most often though, we cannot solve the equations explicitly, but must resort to perturbation theory. However, regular perturbation theory (like that in QM) does not work, since it doesn't incorporate the different time scales, except after an infinite amount of terms (rendering the whole technique moot). The trick then, is **two timing**, where two new variables are introduced, and assumed to be independent of each other:

$$\tau = t, \quad T = \varepsilon t. \tag{3.9}$$

ie, a regular and slow time. This means that the usual time derivative becomes

$$\dot{x} = \frac{\partial x}{\partial \tau} + \varepsilon \frac{\partial x}{\partial T} \tag{3.10}$$

If we use this on a general, weakly non-linear oscillator of the form

$$\ddot{x} + x + \varepsilon h(x, \dot{x}) = 0, \quad 0 < \varepsilon \ll 1 \tag{3.11}$$

we get the first term of the expansion to be

$$x_0 = r(T)\cos(\tau + \phi(T)) \tag{3.12}$$

where the amplitude and phase depend only on the slow time T. Substituting this into h such that

$$h = h(r\cos(\tau+\phi), -r\sin(\tau+\phi)) = h(r\cos(\theta), -r\sin(\theta)) \tag{3.13}$$

and doing some clever manipulation gives the following two **averaged equations** for r and ϕ:

$$\frac{\partial r}{\partial T} = \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\sin(\theta)\, d\theta = \langle h\sin(\theta)\rangle \tag{3.14}$$

$$r\frac{\partial \phi}{\partial T} = \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\cos(\theta)\, d\theta = \langle h\cos(\theta)\rangle \tag{3.15}$$

Some common averages are

$$\langle \sin \rangle = \langle cos \rangle = \langle \sin\cos \rangle = \langle \sin^3 \rangle = \langle \cos^3 \rangle = \langle \sin^{2n+1} \rangle = \langle \cos^{2n+1} \rangle = 0 \tag{3.16}$$

$$\langle cos^2 \rangle = \langle \sin^2 \rangle = \frac{1}{2}, \quad \langle sin^4 \rangle = \langle \cos^4 \rangle = \frac{3}{8}, \quad \langle sin^2\cos^2 \rangle = \frac{1}{8}, \tag{3.17}$$

$$\langle \cos^2\sin^4 \rangle = \frac{1}{16}, \quad \langle \sin\cos^3 \rangle = \langle \cos\sin^3 \rangle = 0 \tag{3.18}$$

$$\langle \sin^{2n} \rangle = \langle \cos^{2n} \rangle = \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}{2\cdot 4\cdot 6\cdot (2n)}. \tag{3.19}$$

The frequency is then given by

$$\omega = 1 + \varepsilon\frac{\partial\phi}{\partial T} + O(\varepsilon^2). \tag{3.20}$$

More accurate results can be obtained by including more terms in the two timing expansion, or by introducing yet another timescale $\mathscr{T} = \varepsilon^2 t$.

3.4 Bifurcations in higher dimensions - the Hopf bifurcation

The bifurcations described in the section on one-dimensional systems can of course also happen in higher dimensional systems, in the same way they do in one dimension. There are also types of bifurcations that can happen in higher order systems, but not in first order systems. One main example is the Hopf bifurcation. It comes in three variants: supercritical, subcritical and degenerate. They have the same "blueprint" in the sense that they involve a pair of complex eigenvalues, whose real part change sign. Schematically we have

$$\begin{cases} \operatorname{Re}\lambda(\mu) < 0 & \mu < \mu_c \\ \operatorname{Re}\lambda(\mu) > 0 & \mu > \mu_c \end{cases} \tag{3.21}$$

where μ is a control parameter, and μ_c is the critical value at which the bifurcation occurs. Determining whether the bifurcation is supercritical, subcritical or degenerate is often hard analytically. It's usually easiest to just to it numerically.

Supercritical Hopf bifurcation: A supercritical Hopf bifurcation happens when a stable spiral (two complex eigenvalues with negative real part) change to an unstable spiral (eigenvalues with positive real part), with a nearly elliptical, stable limit cycle appearing around the unstable spiral. A supercritical Hopf bifurcation has the following properties

- The size of the limit cycle grows from 0, and is proportional to $\sqrt{\mu - \mu_c}$, when $\mu \approx \mu_c$.
- The frequency of the limit cycle is approximately $\omega = \operatorname{Im}\lambda$, evaluated at $\mu = \mu_c$. It is exact at the bifurcation, and correct within $\mathcal{O}(\mu - \mu_c)$. The period is $T = (2\pi/\operatorname{Im}\lambda) + \mathcal{O}(\mu - \mu_c)$

Subcritical Hopf bifurcation: Before a subcritical Hopf bifurcation occurs, a stable spiral is surrounded by an unstable limit cycle, which in turn is surrounded by a stable limit cycle. As the control parameter is varied, the unstable limit cycle shrinks and engulfs the fixed point at $\mu = \mu_c$, changing the stability from a stable spiral to an unstable one.

Degenerate Hopf bifurcation: In a degenerate Hopf bifurcation a stable spiral still changes to an unstable one. But in this case no limit cycles occur. Instead there is a continuous band of closed orbits (limit cycles are isolated). This often occurs when a non-conservative system turns conservative.

4 One-dimensional maps

One-dimensional maps are a recurrence relation found by repeated iterations. They are of the form

$$x_{n+1} = f(x_n) \tag{4.1}$$

where $f(x_n)$ is some one-dimensional function. Like with differential equations we can have fixed points for maps. These occur when $x_{n+1} = x_n$, or

$$f(x_n) = x_n, \quad f(x_n) - x_n = 0. \tag{4.2}$$

4.1 Linear stability analysis

Again, like differential equations, we can employ linear stability analysis. The result is

$$\text{stability in } x^* = \begin{cases} |f'(x^*)| < 1 \;\Rightarrow\; & x^* \text{ is stable,} \\ |f'(x^*)| > 1 \;\Rightarrow\; & x^* \text{ is } \textit{un}\text{stable.} \end{cases} \tag{4.3}$$

The border case where $|f'(x^*)| = 1$ is inconclusive in linear stability analysis, and either numerics must be employed (like a cobweb diagram), or higher order stability analysis. There is also the case of **superstable** points x^*. These occur when $f'(x^*) = 0$, causing the decay to be much faster than usual.

4.2 Cycles

Cycles are a set of points which repeat upon iteration. An n-cycle consists (unsurprisingly) of n distinct points. For example, let's say we have a two cycle:

$$f(p) = q, \quad f(q) = f(f(p)) = f^2(p) = p \tag{4.4}$$

Here we see, that a point on an n-cycle is a fixed point of f^n. All points on the n-cycle will be fixed points of f^n. As will all fixed points of f (since $f^n(x^*) = f^{n-1}(x^*) = x^*$), and all points of m-cycles where n is divisible by m (ie, points of a 2-cycle will be fixed points of f^{2n}.)

4.3 Bifurcations and period doubling

Maps can undergo bifurcations much like their continuous time counterparts. However, here the map needs to cross the line $f(x_n) = x_n$ instead of $f(x) = 0$, as this is the line defining stability. The standard players are here as anticipated: pitchforks, transcritical and saddle-nodes. There is a type of bifurcation that occurs for maps, but not differential equations. This is the flip bifurcation. It occurs when $f'(x^*) = -1$.

This type of bifurcation often causes a period doubling, which means that a two cycle appears around the fixed point. This occurs if the map f is concave down about x^*. ie:

$$f''(x^*) < 0 \Rightarrow \text{stable 2-cycle appears around } x^*. \tag{4.5}$$

Flip bifurcations can also be subcritical, in which case the 2-cycle appears below (or before) the bifurcation occurs and is unstable.

4.4 Stability of cycles and Liapunov exponents

The stability of cycles can be determined in much the same way as ordinary fixed point. One utilizes the fact that any point in an n-cycle is a fixed point of f^n. As such, one needs to look at df^n/dx. If we call the points in the n-cycle x_i for $i = 0, ..., n-1$, then:

$$\lambda = \frac{df^n(x^*)}{dx} = f'(x_0)f'(x_1)\ldots f'(x_{n-1}), \quad \begin{cases} |\lambda| < 1 \;\Rightarrow\; & \text{cycle is stable} \\ |\lambda| > 1 \;\Rightarrow\; & \text{cycle is } \textit{un}\text{stable} \end{cases} \tag{4.6}$$

Here λ is called the multiplier, and is not to be confused with the Liapunov exponent (also denoted by λ in the book. I will refer to the Liapunov exponent by Λ). They are related by $\Lambda = (1/n)\ln|\lambda|$.

The Liapunov exponent for an orbit (ie, a series of points x_i) is given by

$$\Lambda = \lim_{n\to\infty} \left[\frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)| \right], \tag{4.7}$$

and if the series is a p-cycle, this reduces to

$$\Lambda = \frac{1}{p} \sum_{i=0}^{p-1} \ln |f'(x_i)|. \tag{4.8}$$

A positive Liapunov exponent corresponds to **chaos**, a negative exponent corresponds to a stable cycle or fixed point, and $\Lambda = -\infty$ corresponds to a superstable cycle/fixed point.

4.5 Unimodal maps and superstable cycles

A unimodal map is a smooth map which is concave down with a single maximum. An important result about these is that a superstable cycle for an unimodal map has the maximum x_m of the map as one of the points.

5 Fractals