Notes for Dynamical Systems and Chaos

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1 Introduction

In general we work with systems of differential equations of the form:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, \dots, x_n), \\ &\vdots \\ \dot{x}_n &= f_n(x_1, \dots, x_n). \end{aligned}$$

Where we say that this is an n-dimensional system. Note that this does not coincide with our regular use of the word dimension, but rather this is what we would normally call n degrees of freedom. For example, the simple harmonic oscillator is a system in one dimension, but with two degrees of freedom (x and $\dot{x}$). We can write it in the form above by defining $x_1 = x$, $x_2 = \dot{x}$:

$$\ddot{x} = -\frac{k}{m} x \to \begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -\frac{k}{m} x_1. \end{cases} \tag{1.1}$$

where of course $f_1 = x_2$ and $f_2 = -kx_1/m$.

One can incorporate time explicitly by defining a new coordinate $x_{n+1} = t$, with $f_{n+1} = 1$.

Further, we normally deal with systems of equations which might not be solvable analytically. However, this does not mean no useful information can be garnered from them. We will find formulas for fixed points and bifurcations, looking at how the systems of equations changes when varying one or more parameters (say k or m in the harmonic oscillator).

2 One dimensional systems

In this section we deal with systems of the form

$$\dot{x} = f(x). \tag{2.1}$$

Useful quantities that are easily understood in one dimensional systems (but of course apply to systems of many dimensions) include fixed points, linear stability analysis, potentials and bifurcations.

2.1 Fixed point and stability analysis

Fixed points x^* occur whenever the function $f(x)$ has a root. These points are called fixed since the solution to the equation is constant in time. A simple example is the system $\dot{x} = x$, which has a fixed point at $x^* = 0$. One represents the system graphically like this:

However not all fixed points are created equally. In the example above, we see that the flow is negative when x is negative, whilst it is positive when x is positive. This then means that any small perturbation from the fixed point will always increase (in magnitude) with time. We call these points *unstable*, and draw them as unfilled circles.

On the other hand, the system $\dot{x} = -x$ has a *stable* fixed point at $x^* = 0$, since all perturbations will be (exponentially) damped. These points are drawn as filled circles.

In general, the derivative of $f(x)$, with respect to x determines the stability of the point. Often, **linear stability analysis** enough. This states that x^* is a stable (unstable) fixed point, if $f'(x^*) < 0$ ($f'(x^*) > 0$). If, however, $f'(x^*) = 0$ then linear stability analysis will not be enough to determine the stability of the point, and one needs to look at the higher order derivatives of $f(x)$.

For example, if $f''(x^*) > 0$ ($f''(x^*) < 0$), the system has minimum (maximum), and the system is left (right) half stable. A left half stable fixed point is stable to perturbations pushing it to the left, but unstable to perturbations pushing it to the right. An example of this is $\dot{x} = x^2$, which has a left stable fixed point at $x^* = 0$:

If then both the first and second derivative at the fixed point is zero, then the third derivative can be used, where the point is stable if the third derivative is less than zero, and unstable if it is positive. Of course one can go as far as one likes with this, and utilize higher and higher derivatives.

2.2 Potentials

Another way to view the one dimensional system is through the use of potentials. Here we define

$$f(x) = -\frac{dV(x)}{dx} \tag{2.2}$$

Fixed points in the system are then stationary points for the potential. In this formulation, one can imagine a ball rolling down a hill (though without inertia, as if flowing through molasses). Minima correspond to stable fixes points and maxima to unstable points (and saddle points to half stable points, of course).

2.3 Bifurcations

A bifurcation is a point in parameter space, at which the stability or number of fixed points changes. In this section a number of bifurcation types appear:

- Saddle-node bifurcations
- Transcritical bifurcations
- Pitchfork bifurcations (both subcritical and supercritical)

When introducing these types of bifurcations one often looks at normal forms, which are the simplest system that exhibit the behaviour in question. Another way of looking at this, is that the Taylor expansion for a system follows the normal form (or at least a scaled version of it) in the vicinity of the bifurcation.

A saddle-node bifurcation is a point in which two fixed points appear out of nothing. The normal form for this is

$$\dot{x} = r + x^2 \tag{2.3}$$

This system has a saddle-node bifurcation at $r_c = 0$, at which a left half stable point appears at $x^* = 0$. If r is decreased further, this point separates into a stable point at $x^* = -\sqrt{-r}$ and an unstable point at $x^* = \sqrt{-r}$. This can be drawn in a **bifurcation diagram**, where one plots the fixed points as a function of the parameter (in this case r):

In a bifurcation diagram, stable *branches* (lines indicating stable fixed points) are drawn up fully, whilst unstable branches are drawn dashed.

A transcritical bifurcation is one in which no fixed points are created or destroyed, but in stead changes its stability. The normal form for these kinds of bifurcations is

$$\dot{x} = rx - x^2, \tag{2.4}$$

which always has a fixed point at $x^* = 0$ (and one at $x^* = r$). The bifurcation occurs at $r_c = 0$, with the fixed point being stable (unstable) for $r < r_c$ ($r > r_c$) and right half stable at $r = r_c$. Its bifurcation diagram looks like so:

A pitchfork bifurcation is one where a single fixed point turns into three fixed points, reminiscent of the shape of a pitchfork. The supercritical pitchfork starts out as a stable point, which turns into an unstable fixed point at $r = r_c$, and forms two additional stable points for $r > r_c$. Its normal form is

$$\dot{x} = rx - x^3, \quad r_c = 0, \quad x^* = 0 \vee x^* = \pm\sqrt{r}, \tag{2.5}$$

and its bifurcation diagram looks like so:

For a subcritical pitchfork bifurcation the pitchfork is inverted. The normal form and fixed points are

$$\dot{x} = rx + x^3, \quad r_c = 0, \quad x^* = 0 \vee x^* = \pm\sqrt{-r}, \tag{2.6}$$

The outer fixed points are unstable however, and the system is unphysical. Therefore one often looks at the following system:

$$\dot{x} = rx + x^3 - x^5, \quad x^* \in \left\{0, \pm\sqrt{\frac{1}{2} + \frac{\sqrt{1+4r}}{2}}, \pm\sqrt{\frac{1}{2} - \frac{\sqrt{1+4r}}{2}}\right\} \tag{2.7}$$

for $r < r_s$, there is only the fixed point at the origin. At $r_s = -1/4$ a double saddle node bifurcation occurs, and at r=0, a supercritical pitchfork bifurcation occurs, where the two inner, unstable fixed points (the last ones in the curly braces) coalesce with the stable fixed point at the origin, into a single unstable fixed point at the origin. The bifurcation diagram is:

This system can undergo catastrophes and hysteresis: if one starts at $x = 0, -1/4 < r < 0$, and varies r until $r > 0$, then x will snap to one of the other stable branches, since the origin is now an unstable fixed point. x will now stay at this point if r is returned to the initial value. This is hysteresis and catastrophe: through a (closed) walk in parameter space, the state of the system has changed!

One last "type" of bifurcations to consider is the imperfect bifurcation. This is not really a new type, but more the introduction of a parameter which affects the bifurcations of the system. Consider

$$\dot{x} = h + rx - x^3 \tag{2.8}$$

If $h = 0$, we have a supercritical pitchfork bifurcation, but for all other values $h \neq 0$, this is changed to a saddle node bifurcation at some (complicated) value of x, as seen below.

2.4 Periodic systems - flows on the circle

Lastly in one dimension, we have flows on the circle, which is really just differential equations with periodic boundary conditions. This means

$$f(x + x_0) = f(x) \tag{2.9}$$

for some x_0. Usually we identify $\theta = x$ and $x_0 = 2\pi$, corresponding to flows defined on the circle. This means that we can actually have oscillations (as long as there are no fixed points).

One cool result is that if $f(\theta) \approx 0$, but there is no fixed point, then f is at a minimum. Then $f \approx r + x^2$, for $r > 0$. The time it takes to go from $-\infty$ to ∞ (the time it takes to pass through this bottle neck) is

$$T_{\text{bottleneck}} \approx \int_{-\infty}^{+\infty} \frac{dx}{r + x^2} = \frac{\pi}{\sqrt{r}}. \tag{2.10}$$

This is called the square-root scaling law. If instead you had a maximum, then there'd be an overall minus on f, with r still larger than 0. The integral would then have to go from ∞ to $-\infty$ (since we're going the other way). Switching the limits costs us a minus sign, which gobbles up the minus on f, leaving us with the same result as before.

3 Two dimensions

3.1 Linear systems

A linear system is one on the form

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}. \tag{3.1}$$

These systems always have a single fixed point at the origin, and the parameters a, b, c, d determine the behaviour around the fixed point. Actually, it is easier to classify according to the eigenvalues λ_1, λ_2, of the matrix. These can be found easily from the trace and determinant of the matrix:

$$\tau = \mathrm{Tr}(A) = \lambda_1 + \lambda_2, \quad \Delta = \det(A) = \lambda_1 \lambda_2 \tag{3.2}$$