Numerical Methods in Physics Week 1

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1 Nuclear Decay

We have two radioactive isotopes, A and B, with populations $N_A(t)$ and $N_B(t)$, and decay times τ_A and τ_B. Type A decays into type B, while B decays into something else we don't track. The relevant differential equations are

$$\frac{\mathrm{d}N_A}{\mathrm{d}t} = -\frac{N_A}{\tau_A}, \quad \frac{\mathrm{d}N_B}{\mathrm{d}t} = \frac{N_A}{\tau_A} - \frac{N_B}{\tau_B}. \tag{1.1}$$

The purpose of this assignment is to solve this problem numerically for a number of different conditions, using Euler integration. This method is used for its simplicity and ease of implementation. For all of the following simulations and plots, the values $\Delta t = 0.1$ is used.

1.1 Compare the numerical and analytical solutions

The analytical solutions are given in the assignment, and are as follows:

$$N_A(t) = N_A(0)\exp\left(-\frac{t}{\tau_A}\right), \tag{1.2}$$

$$N_B(t) = \begin{cases} N_B(0)\exp\left(-\frac{t}{\tau_A}\right) + t\frac{N_A(0)}{\tau_A}\exp\left(-\frac{t}{\tau_A}\right), & \tau_A = \tau_B, \\ N_B(0)\exp\left(-\frac{t}{\tau_B}\right) + \frac{N_A(0)}{\frac{\tau_A}{\tau_B}-1}\left[\exp\left(-\frac{t}{\tau_A}\right) - \exp\left(-\frac{t}{\tau_B}\right)\right], & \tau_A \neq \tau_B. \end{cases} \tag{1.3}$$

The results, along with residual plots, for $N_A(0) = N_B(0) = 1000$ and $\tau_A = 5, \tau_B = 10$ are shown below, with the analytical solution for $\tau_A \neq \tau_B$:

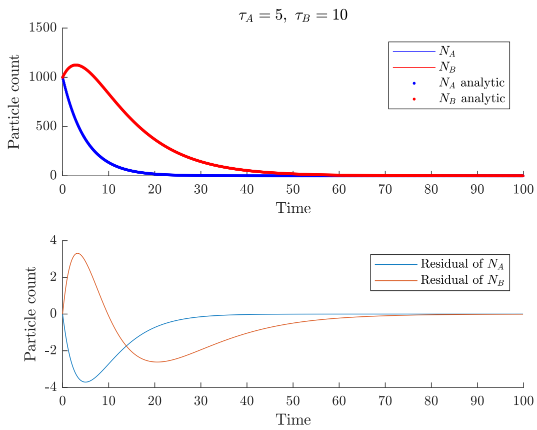

Figure 1: **INSERT PURDY CAPTION PL0X**

COMMENT ON RESULT, YAS, V V V GUT To demonstrate the solution for $\tau_A = \tau_B$, the initial conditions of $N_A = N_B = 1000$ and $\tau_A = \tau_B = 10$ are shown below. Again with accompanying residual plots:

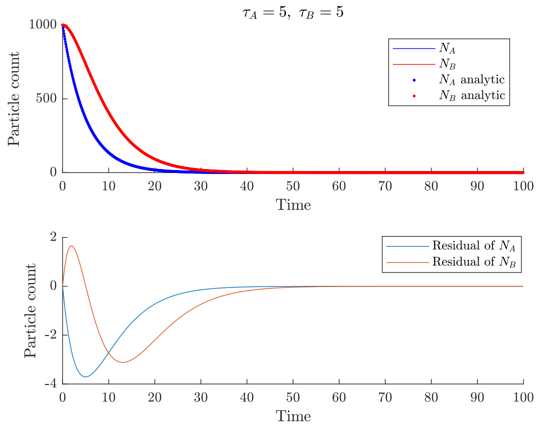

Figure 2: **INSERT PURDY CAPTION PL0X**

1.2 Explain the limit of $\tau_A/\tau_B \gg 1$

In this limit, the decay of type B is much faster than that of type A. This means, that on any appreciable time scale of change for N_A, the population of B reaches a steady-state solution, where $\mathrm{d}N_B/\,\mathrm{d}t = 0$. As such the differential equation for N_B becomes

$$\frac{\mathrm{d}N_B}{\mathrm{d}t} = \frac{N_A}{\tau_A} - \frac{N_B}{\tau_B} = 0, \quad \Rightarrow \quad N_B(t) = N_A\frac{\tau_B}{\tau_A}. \tag{1.4}$$

This is also seen in a simulation, where $\tau_A = 3600, \tau_B = 5$:

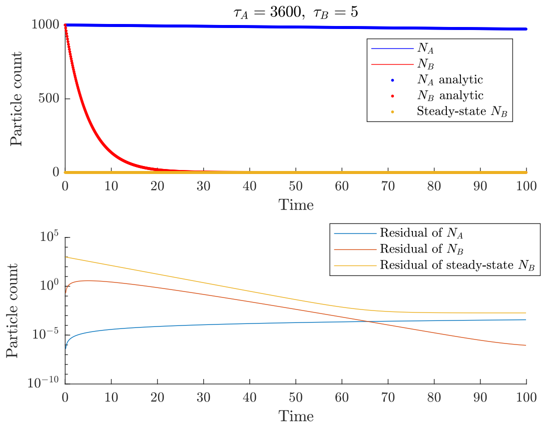

Figure 3: **INSERT PURDY CAPTION PL0X**

as seen, the steady-state solution for N_B ((1.4)) does correctly predict the behaviour for N_B, when this is approximately 0. This makes sense; the change in N_A is almost constant, corresponding to when N_B is approximately 0, which is the steady state for this particular problem.

2 Projectile motion

For this assignment, projectile motion is to be simulated, again using the Euler-method. This gives another example of how to use this simple method of simulation, for a problem which (without wind resistance) has an analytical solution. After this, uncharted waters are encountered, when wind resistance is included. Here no analytical solution is given, and all reliance upon the previous methods, like residuals between the numerical and analytical solutions, are not applicable.

Furthermore, for this problem, a second derivative is used, whereby a second Euler integration is needed, to complete the time step. The relevant differential equations are

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{v}, \quad \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -g\hat{\mathbf{y}}. \tag{2.1}$$

where the analytical solution is computed by integrating the second equation twice, and using the relevant initial conditions:

$$\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 + v_{x,0}t \\ y_0 + v_{y,0}t - gt^2/2 \end{pmatrix}. \tag{2.2}$$

2.1 Without wind resistance

The numerical and analytical solutions to projectile motion, given the initial conditions of $x_0 = 0, y_0 = 2$ m, $v_0 = 4$ m/s and $\theta = 70°$, where $v_{x,0} = v_0 \cos\theta$ and $v_{y,0} = v_0 \sin\theta$. For the numerical solution a value of $\Delta t = 0.01$ s is used. The results are shown below, along with the absolute residual, as a function of time:

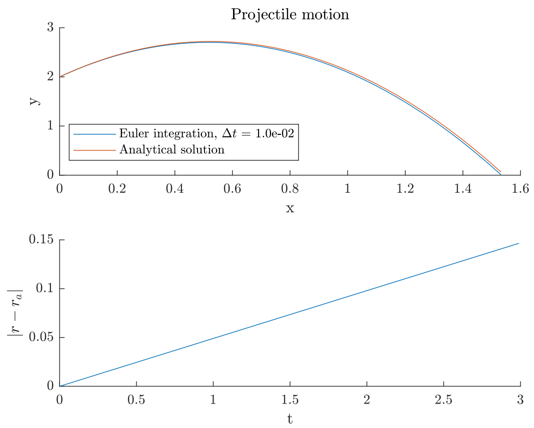

Figure 4: **INSERT PURDY CAPTION PL0X**

the absolute residual is also the global truncation error, as defined in the slides for the week 1 lectures.