Reading group notes for ALGEBRAIC TOPOLOGY by Allen Hatcher

12. August 2020

The book is available on the Math dept. page of Cornell University, here:

`https://pi.math.cornell.edu/~hatcher/AT/AT.pdf`

Definitions

Annulus

Idea: Some plaintext.

Definition: Some formal foo $y = x^2$ bar.

Antipodal

Attached along a map

A ... topological subspace of Y
$f \colon A \to X$
$X \cup_f Y = (X \amalg Y)/\sim_f$

Boundary

Cantor set

Cartesian space $\mathbb{R}^n$

Cell complex

Cell e^n

Cell structure

Component

Compact

Cone

Contractible space

CW-pair

CW complex

Deformation retraction

A homotopy $(I \times X) \to X$ between the identity $\mathrm{id}_X : X \to X$ and a retract $r : X \to A$.
I.e. a deforming of the identity by shrinking its image.
`https://en.wikipedia.org/wiki/Retraction_(topology)`

Dimension of a CW complex

Disc $\mathbb{D}^n$

Notes:
$I := D^1$ is the interval used for the definition of a homotopy.

Disjoint union

Equivalence relation

Genus

Graph

x

Homeomorphic

Homotopy

Continuous $H\colon (I \times X) \to Y$.

This can also be expressed as
• families into continuous functions $h\colon I \to (X \to Y)$ (such that $H\colon (I{\times}X) \to Y$ is also continuous.)
resp.
• neighboring paths of point $p\colon X \to (I \to Y)$ (such that $H\colon (I \times X) \to Y$ is also continuous.)

Here $I = [0,1]$ is the interval and we speak of a homotopy between the functions $h(0)$ and $h(1)$. Then $h(0)$ and $h(1)$ are homotopic.

`https://en.wikipedia.org/wiki/Homotopy`

Homotopy equivalence

A pair $f\colon X \to Y$ and $g\colon Y \to X$ such that
$g \circ f$ is homotopic to the identity on X
and also $f \circ g$ is homotopic to the identity on Y.

I.e. both composition maps are allowed to be continuous deformations of the identity.

In contrast, one gets homeomorphisms (continuous bijections) when one requires the compositions to be exactly identities.

Homotopy extension property

Homotopy Type

House with two rooms

Inclusion map

...

$A \subset X$
$\iota\colon A \to X$
$\iota(x) := x$

Is injective. In a diagram, the arrow is often written with a hook (as in $\hookrightarrow$).

Infinite sphere $\mathbb{S}^\infty$

Join

Klein bottle

Möbius band

Mapping cylinder

Neighborhood

Null-homotopic

A function is null-homotopy if it's at an end of a null-homotopy.

Null-homotopy

A homotopy with one end a constant function (mapping everything into a point)

Path-component

Product of cell complexes

Projection n-space $\mathbb{C}P^n$

Projection n-space $\mathbb{R}P^n$

Quotient map

Quotient space

TODO

Reduced suspension

Rel

Relation of homotopy among maps $X \to Y$

Retraction

A left-inverse to an inclusion.
$r\colon X \to A$
$r \circ \iota = \mathrm{id}_A$
Essentially dual to a section.
`https://en.wikipedia.org/wiki/Retraction_(topology)`

Simplex

Skeleton

Subcomplex

Topological subspace

$\langle X, T_X \rangle$
The topological subspace is the topological spaces $\langle A, T_A \rangle$ where $A \subset X$ and $T_A \equiv \{U \cap A \mid U \in T_X\}$

Topological torus $\mathbb{T}^2$

Up to homeomorphism,
$\mathbb{S}^1 \times \mathbb{S}^1$ with the box topology.

Also $\mathbb{R}^2/\sim$ with $\sim$ given via
$\langle x, y\rangle \sim \langle x+1, y\rangle \sim \langle x, y+1\rangle$

Equivalently for $\mathbb{T}^n$.

Smash product

Topological sphere $\mathbb{S}^n$

Up to homeomorphism,
$\mathbb{S}^n = \left\{x \in \mathbb{R}^{n+1} \mid ||x|| = 1\right\}$ with the subset topology induced from $\mathbb{R}^{n+1}$.

With S^n the (radius 1/2, i.e. smaller) sphere with south pole at $0 \in \mathbb{R}^{n+1}$ and north pole at $\langle 0, \dots, 0, 1\rangle$,
$P_N \colon S^n \to (\mathbb{R}^n \cup \{\mathrm{pt}\})$
$P_N(x_1 = 0, \dots, x_{n-1} = 0, x_n = 1) := \mathrm{pt}$
$P_N(a, x_n) := \frac{1}{1-x_n} \cdot a$

Suspension

Wedge sum

Exercise questions

Chapter 0

1. Construct an explicit deformation retraction of the torus with one point deleted onto a graph consisting of two circles intersecting in a point, namely, longitude and meridian circles of the torus.

2. Construct an explicit deformation retraction of $\mathbb{R}^n - \{0\}$ onto S^{n-1}.

3.

(a) Show that the composition of homotopy equivalences $X \to Y$ and $Y \to Z$ is a homotopy equivalence $X \to Z$. Deduce that homotopy equivalence is an equivalence relation.

(b) Show that the relation of homotopy among maps $X \to Y$ is an equivalence relation.

(c) Show that a map homotopic to a homotopy equivalence is a homotopy equivalence.

4. A **deformation retraction in the weak sense** of a space X to a subspace A is a homotopy $f_t\colon X \to X$ such that $f_0 = \mathbb{1}$, $f_1(X) \subset A$, and $f_t(A) \subset A$ for all t. Show that if X deformation retracts to A in this weak sense, then the inclusion $A \hookrightarrow X$ is a homotopy equivalence.

5. Show that if a space X deformation retracts toa point $x \in X$, then for each neighborhood U of x in X there exists a neighborhood $V \subset U$ of x such that the inclusion map $V \hookrightarrow U$ is nullhomotopic.

6.

(a) Let X be the subspace of $\mathbb{R}^2$ consisting of the horizontal segment $[0,1]\times\{0\}$ together with all the vertical segments $\{r\}\times[0,1-r]$ for r a rational number in $[0,1]$. Show that X deformation retracts to any point in the segment $[0,1]\times\{0\}$, but not to any other point. [See the preceding problem.]

(b) Let Y be the subspace of $\mathbb{R}^2$ that is the union of an infinite number of copies of X arranged as in the figure on Hatcher, pg 18. Show that Y is contractible but does not deformation retract onto any point.

(c) Let Z be the zigzag subspace of Y homeomorphic to $\mathbb{R}$ indicated by the heavier line. Show there is a deformation retraction in the weak sense (see Exercise 4) of Y onto Z, but no true deformation retraction.

7. Fill in the details in the following construction from [Edwards 1999] of a compact space $Y \subset \mathbb{R}^3$ with the same properties as the space Y in Exercise 6, that is, Y is contractible but does not deformation retract to any point. To begin, Let X be the union of an infinite sequence of cones on the Cantor set arranged end-to-end, as in the figure on Hatcher, pg 18. Next, form the one-point compactification of $X \times \mathbb{R}$. This embeds in $\mathbb{R}^3$ as a closed disk with curved 'fins' attached along circular arcs, and with the one-point compactification of X as a cross-sectional slice. The desired space Y is then obtained from this subspace of $\mathbb{R}^3$ by wrapping one more cone on the Cantor set around the boundary of the disk.

8. For $n > 2$, construct an $n-$room analog of the house with two rooms.

9. Show that a retract of the contractible space is contractible.

10. Show that a space X is contractible iff every map $f\colon X \to Y$, for arbitrary Y, is nullhomotopic. Similarly, show X is contractible iff every map $f\colon Y \to X$ is nullhomotopic.

11. Show that $f\colon X \to Y$ is a homotopy equivalence if there exist maps $g, h\colon Y \to X$ such that $fg \simeq \mathbb{1}$ and $hf \simeq \mathbb{1}$. More generally, show that f is a homotopy equivalence if fg and hf are homotopy equivalences.

12. Show that a homotopy equivalence $f\colon X \to Y$ induces a bijection between the set of path-components of X and the set of path-components of Y, and that f restricts to a homotopy equivalence from each path-component of X to the corresponding path component of Y. Prove also the corresponding statemnts with components instead of path-components. Deduce that if the components of a space X coincide with its path-components, then the same holds for any space Y homotopy equivalent to X.