Reading group notes for ALGEBRAIC TOPOLOGY by Allen Hatcher

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The book is available on the Math dept. page of Cornell University, here:

`https://pi.math.cornell.edu/~hatcher/AT/AT.pdf`

Definitions

Annulus

Idea: Some plaintext.

Definition: Some formal foo $y = x^2$ bar.

Antipodal

Idea: Some plaintext.

Definition: Some formal foo $y = x^2$ bar.

Attached along a map

x

Bijection

x

Boundary

Cantor set

Cartesian space $\mathbb{R}^n$

Cell complex

Cell e^n

Cell structure

Component

Compact

Cone

Continuous

Contractible space

CW-pair

CW complex

Deformation

Dimension of a CW complex

Disc $\mathbb{D}^n$

Disjoint union

Equivalence relation

Genus

Graph

x

Homeomorphic

Homotopy

Homotopy equivalence

Homotopy extension property

Homotopy Type

House with two rooms

Infinite sphere $\mathbb{S}^\infty$

Torus $\mathbb{T}^n$

Join

Klein bottle

Möbius band

Mapping cylinder

Neighborhood

Null-homotopic

Path-component

Product of cell complexes

Projection n-space $\mathbb{C}P^n$

Projection n-space $\mathbb{R}P^n$

Quotient map

Quotient space

x

Reduced suspension

Rel

Relation of homotopy among maps $X \to Y$

Retraction

Simplex

Skeleton

Subcomplex

Subspace of $\mathbb{R}^n$

Topology

Smash product

Sphere $\mathbb{S}^n$

Suspension

Wedge sum

Exercise questions

Chapter 0

1. Construct an explicit deformation retraction of the torus with one point deleted onto a graph consisting of two circles intersecting in a point, namely, longitude and meridian circles of the torus.

2. Construct an explicit deformation retraction of $\mathbb{R}^n - \{0\}$ onto S^{n-1}.

3.

(a) Show that the composition of homotopy equivalences $X \to Y$ and $Y \to Z$ is a homotopy equivalence $X \to Z$. Deduce that homotopy equivalence is an equivalence relation.

(b) Show that the relation of homotopy among maps $X \to Y$ is an equivalence relation.

(c) Show that a map homotopic to a homotopy equivalence is a homotopy equivalence.

4. A **deformation retraction in the weak sense** of a space X to a subspace A is a homotopy $f_t\colon X \to X$ such that $f_0 = \mathbb{1}$, $f_1(X) \subset A$, and $f_t(A) \subset A$ for all t. Show that if X deformation retracts to A in this weak sense, then the inclusion $A \hookrightarrow X$ is a homotopy of equivalence.

5. Show that if a space X deformation retracts toa point $x \in X$, then for each neighborhood U of x in X there exists a neighborhood $V \subset U$ of x such that the inclusion map $V \hookrightarrow U$ is nullhomotopic.

6.

(a) Let X be the subspace of $\mathbb{R}^2$ consisting of the horizontal segment $[0,1]\times\{0\}$ together with all the vertical segments $\{r\}\times[0,1-r]$ for r a rational number in $[0,1]$. Show that X deformation retracts to any point in the segment $[0,1] \times \{0\}$, but not to any other point. [See the preceding problem.]

(b) Let Y be the subspace of $\mathbb{R}^2$ that is the union of an infinite number of copies of X arranged as in the figure on Hatcher, pg 18. Show that Y is contractible but does not deformation retract onto any point.

(c) Let Z be the zigzag subspace of Y homeomorphic to $\mathbb{R}$ indicated by the heavier line. Show there is a deformation retraction in the weak sense (see Exercise 4) of Y onto Z, but no true deformation retraction.

7. Fill in the details in the following construction from [Edwards 1999] of a compact space $Y \subset \mathbb{R}^3$ with the same properties as the space Y in Exercise 6, that is, Y is contractible but does not deformation retract to any point. To begin, Let X be the union of an infinite sequence of cones on the Cantor set arranged end-to-end, as in the figure on Hatcher, pg 18. Next, form the one-point compactification of $X \times \mathbb{R}$. This embeds in $\mathbb{R}^3$ as a closed disk with curved 'fins' attached along circular arcs, and with the one-point compactification of X as a cross-sectional slice. The desired space Y is then obtained from this subspace of $\mathbb{R}^3$ by wrapping one more cone on the Cantor set around the boundary of the disk.

8. For $n > 2$, construct an $n-$room analog of the house with two rooms.

9. Show that a retract of the contractible space is contractible.

10. Show that a space X is contractible iff every map $f\colon X \to Y$, for arbitrary Y, is nullhomotopic. Similarly, show X is contractible iff every map $f\colon Y \to X$ is nullhomotopic.

11. Show that $f\colon X \to Y$ is a homotopy equivalence if there exist maps $g, h\colon Y \to X$ such that $fg \simeq \mathbb{1}$ and $hf \simeq \mathbb{1}$. More generally, show that f is a homotopy equivalence if fg and hf are homotopy equivalences.

12. Show that a homotopy equivalence $f\colon X \to Y$ induces a bijection between the set of path-components of X and the set of path-components of Y, and that f restricts to a homotopy equivalence from each path-component of X to the corresponding path component of Y. Prove also the corresponding statemnts with components instead of path-components. Deduce that if the components of a space X coincide with its path-components, then the same holds for any space Y homotopy equivalent to X.