Z3: An Efficient SMT Solver

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Microsoft Research

Introduction

- Satisfiability Modulo Theories (SMT)
  - The next generation of verification engines.
  - SAT solvers + Theories
    - Arithmetic
    - Arrays
    - Uninterpreted Functions
  - Some problems are more naturally expressed in SMT.
  - More automation.

Example

$$x + 2 = y \Rightarrow f(\textit{read}(\textit{write}(a, x, 3), y - 2)) = f(y - x + 1)$$

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- Theory: Arithmetic

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- Theory: Arrays
- Usually used to model the memory/heap.
- read: array access.
- write: array update.

Example

$$x + 2 = y \Rightarrow f(\textit{read}(\textit{write}(a, x, 3), y - 2)) = f(y - x + 1)$$

- Theory: Free functions.
- Useful for abstracting complex operations.

Example

$$x + 2 = y \Rightarrow f(\textit{read}(\textit{write}(a, x, 3), y - 2)) = f(y - x + 1)$$

- Let's solve it.
- Substituting y.

Example

$$x+2=y \Rightarrow f(\textit{read}(\textit{write}(a,x,3), x+2-2)) = f(x+2-x+1)$$

- Simplifying (arithmetic).

Example

$$x + 2 = y \Rightarrow f(\textit{read}(\textit{write}(a, x, 3), x)) = f(3)$$

- Simplifying (array theory).
- *write*$(a, x, 3)$: update position x of array a with value 3.
- *read*(a', x): access position x of array a'.
- $\rightsquigarrow$ *read*$($*write*$(a, x, 3), x))$: is equal to 3.

Example

$$x + 2 = y \Rightarrow f(3) = f(3)$$

- Simplifying.

Example

$$x + 2 = y \Rightarrow true$$

- Simplifying.

Example

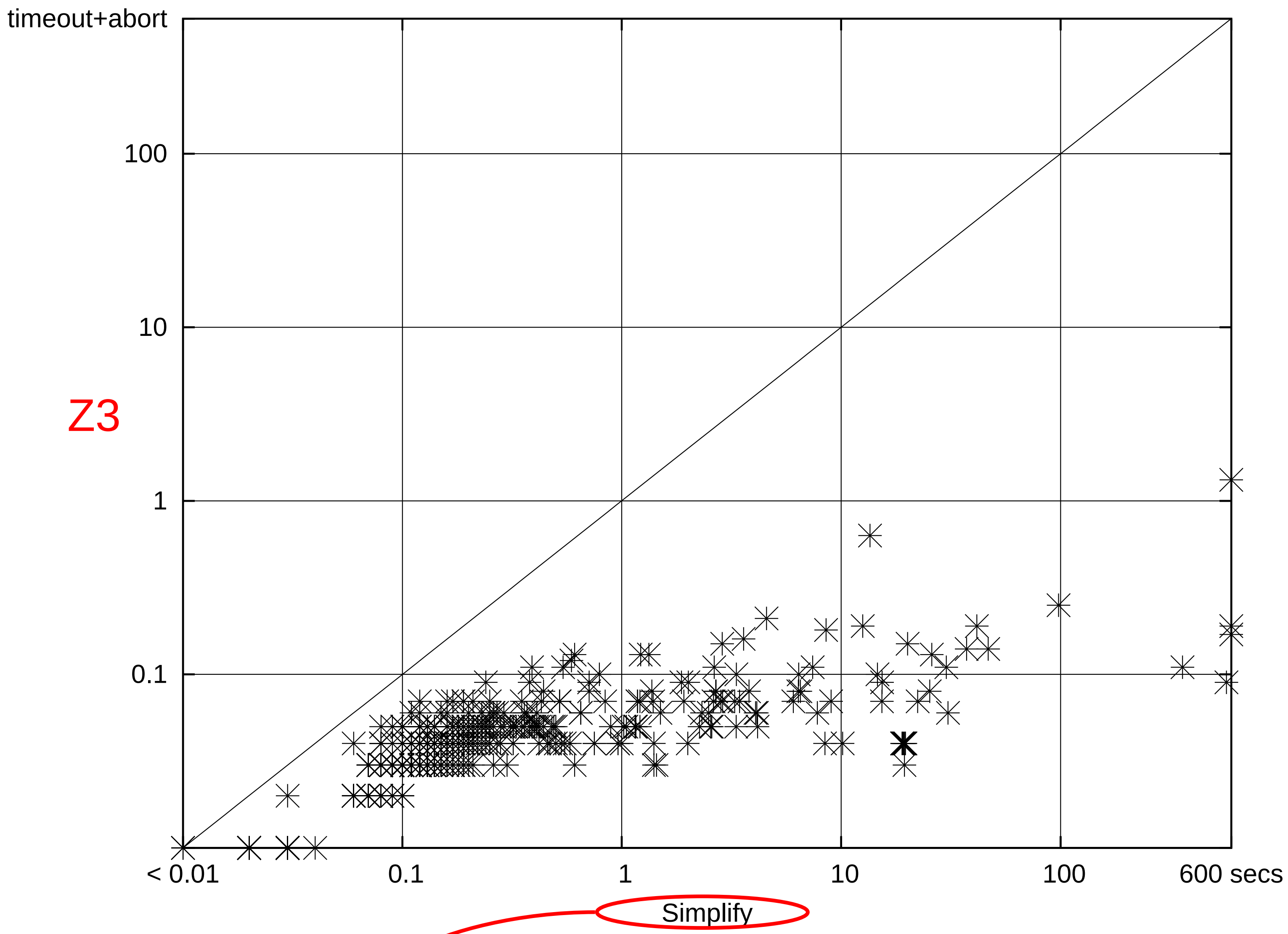
$$true$$

- Done.
- Do this when there are 100,000s of terms and formulas.

Z3: Introduction

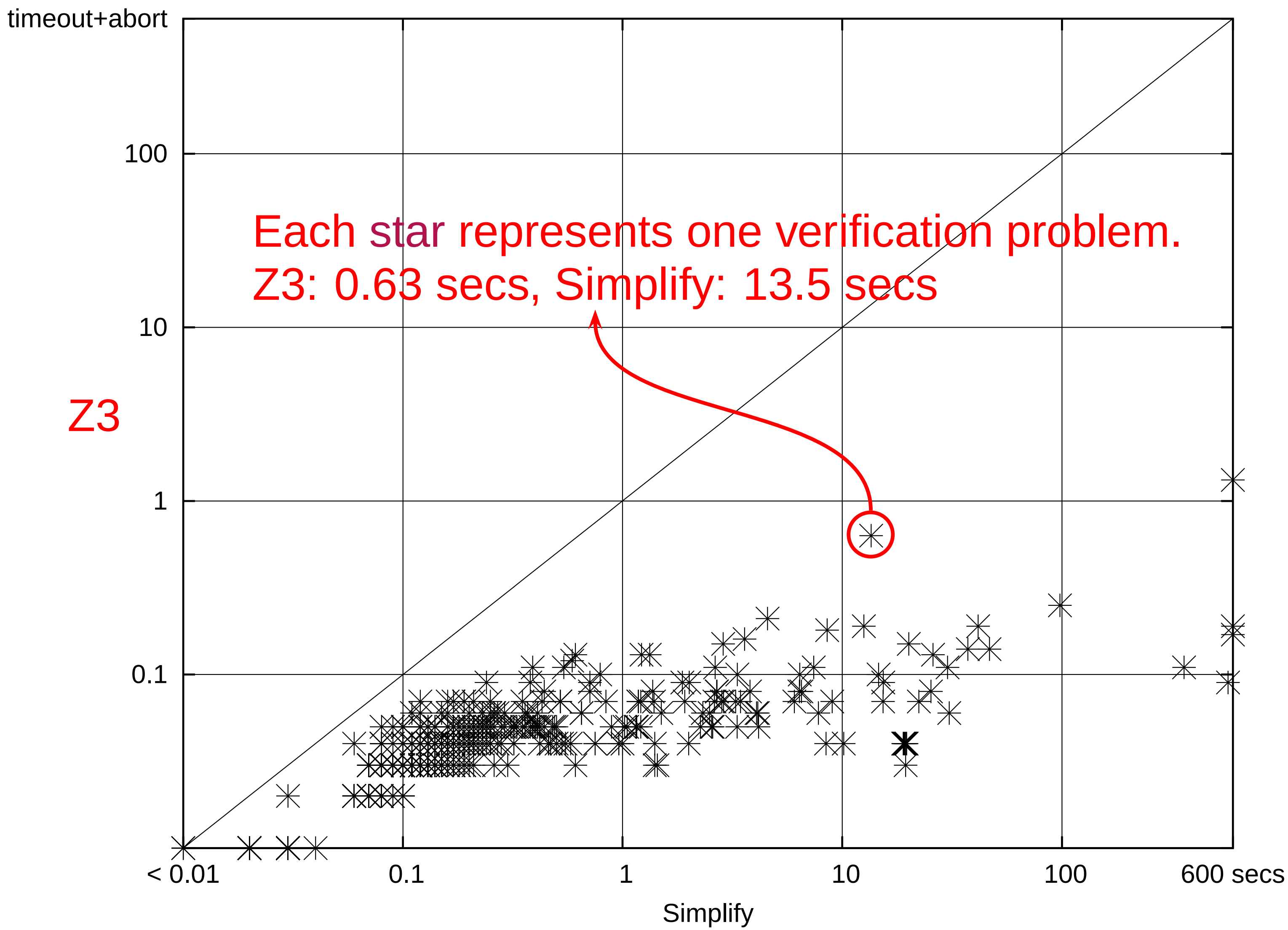
- Z3 is a new theorem prover developed at Microsoft Research.
- Version 1.1 was released last October.
- Version 0.1: 4 first places in theorem proving competition.
- Managed (.Net) & Unmanaged (C/C++) & OCaml APIs are available.
- `http://research.microsoft.com/projects/z3`

Performance (Spec#/Boogie): Z3 × Simplify



Spec#/Boogie was using Simplify from HP Labs.

Performance (Spec#/Boogie): Z3 × Simplify



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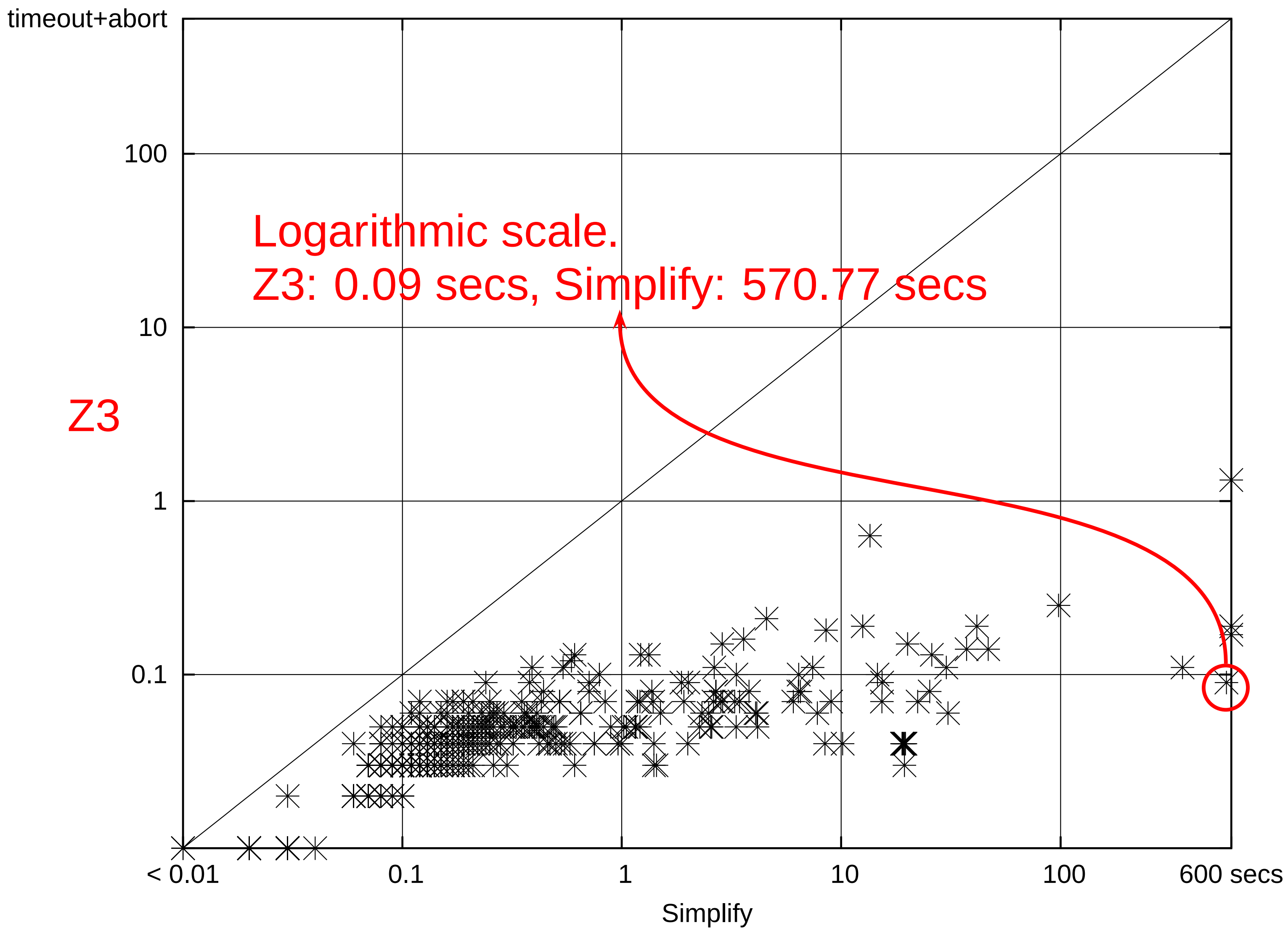

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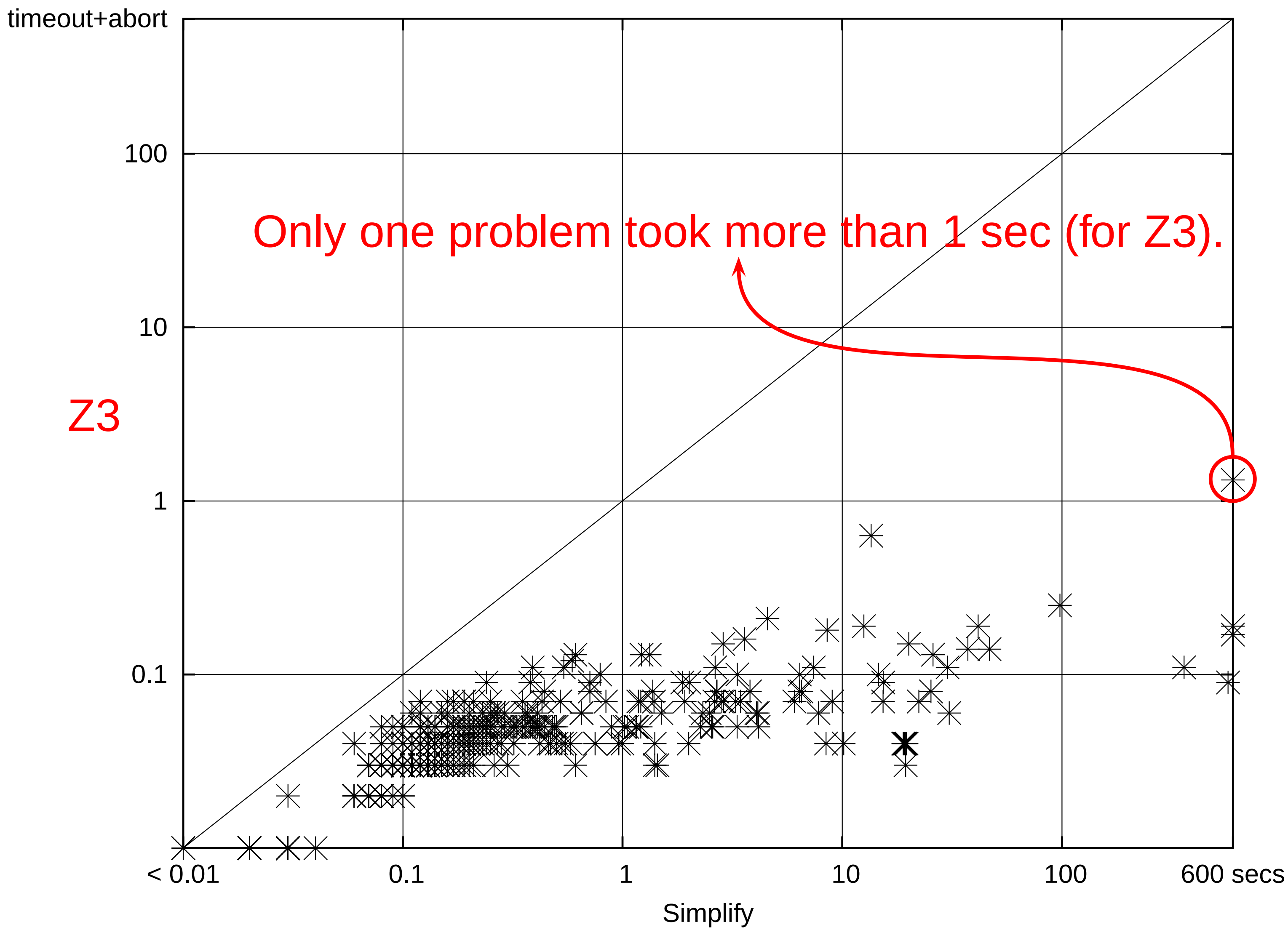

SMT-Solvers & SMT-Lib & SMT-Comp

- SMT-Solves:

  ArgoLib, Ario, Barcelogic, CVC, CVC Lite, CVC3, Ergo, ExtSAT, Fx7, Harvey, HTP, ICS (SRI), Jat, MathSAT, Sateen, Simplify, Spear, STeP (Stanford), STP, SVC, TSAT, UCLID, Yices (SRI), Zap (Microsoft), Z3 (Microsoft)

- SMT-Lib: library of benchmarks

  `http://www.smtlib.org`

- SMT-Comp: annual SMT-Solver competition.

  `http://www.smtcomp.org`

Roadmap

- Background
- Implementation Techniques

Language

- A signature Σ is a finite set of: function symbols $\Sigma_F = \{f, g, \ldots\}$, predicate symbols $\Sigma_P = \{p, q, \ldots\}$, and an arity function $\Sigma \mapsto N$.
- Function symbols with arity 0 are called constants.
- A countable set $\mathcal{V}$ of variables $\{x, y, \ldots\}$ disjoint of Σ.
- Terms:

  $$t \;:=\; f(t_1, \ldots, t_n) \mid x$$

- Formulas:

  $$\phi := p(t_1, \ldots, t_n) \mid \phi_1 \vee \phi_2 \mid \phi_1 \wedge \phi_2 \mid \neg\phi_1 \mid \exists x : \phi_1 \mid \forall x : \phi_1$$

- Free (occurrences) of variables in a formula are those not bound by a quantifier.
- A sentence is a first-order formula with no free variables.

Theories

- A (first-order) theory $\mathcal{T}$ (over a signature Σ) is a set of (deductively closed) sentences (over Σ and $\mathcal{V}$).
- Let *DC*(Γ) be the deductive closure of a set of sentences Γ.
  - For every theory $\mathcal{T}$, *DC*$(\mathcal{T}) = \mathcal{T}$.
- A theory $\mathcal{T}$ is consistent if false $\notin \mathcal{T}$.
- We can view a (first-order) theory $\mathcal{T}$ as the class of all models of $\mathcal{T}$ (due to completeness of first-order logic).

Models (Semantics)

- A model M is defined as:
  - Domain S: set of elements.
  - Interpretation $f^M : S^n \mapsto S$ for each $f \in \Sigma_F$ with *arity*$(f) = n$.
  - Interpretation $p^M \subseteq S^n$ for each $p \in \Sigma_P$ with *arity*$(p) = n$.
  - Assignment $x^M \in S$ for every variable $x \in \mathcal{V}$.
- A formula ϕ is true in a model M if it evaluates to true under the given interpretations over the domain S.
- M is a model for the theory $\mathcal{T}$ if all sentences of $\mathcal{T}$ are true in M.

Satisfiability and Validity

- A formula $\phi(\vec{x})$ is satisfiable in a theory $\mathcal{T}$ if there is a model of *DC*$(\mathcal{T} \cup \exists \vec{x}.\phi(\vec{x}))$. That is, there is a model M for $\mathcal{T}$ in which $\phi(\vec{x})$ evaluates to true, denoted by,

$$M \models_{\mathcal{T}} \phi(\vec{x})$$

- This is also called $\mathcal{T}$-satisfiability.
- A formula $\phi(\vec{x})$ is valid in a theory $\mathcal{T}$ if $\forall \vec{x}.\phi(\vec{x}) \in \mathcal{T}$. That is $\phi(\vec{x})$ evaluates to true in every model M of $\mathcal{T}$.
- $\mathcal{T}$-validity is denoted by $\models_{\mathcal{T}} \phi(\vec{x})$.
- The quantifier free $\mathcal{T}$-satisfiability problem restricts ϕ to be quantifier free.

Pure Theory of Equality (EUF)

- The theory $\mathcal{T}_{\mathcal{E}}$ of equality is the theory $DC(\emptyset)$.
- The exact set of sentences of $\mathcal{T}_{\mathcal{E}}$ depends on the signature in question.
- The theory does not restrict the possibles values of the symbols in its signature in any way. For this reason, it is sometimes called the theory of equality and uninterpreted functions.
- The satisfiability problem for $\mathcal{T}_{\mathcal{E}}$ is the satisfiability problem for first-order logic, which is undecidable.
- The satisfiability problem for conjunction of literals in $\mathcal{T}_{\mathcal{E}}$ is decidable in polynomial time using congruence closure.

Linear Integer Arithmetic

- $\Sigma_P = \{\leq\}, \Sigma_F = \{0, 1, +, -\}$.
- Let $M_{\mathcal{LIA}}$ be the standard model of integers.
- Then $\mathcal{T}_{\mathcal{LIA}}$ is defined to be the set of all Σ sentences true in the model $M_{\mathcal{LIA}}$.
- As showed by Presburger, the general satisfiability problem for $\mathcal{T}_{\mathcal{LIA}}$ is decidable, but its complexity is triply-exponential.
- The quantifier free satisfiability problem is NP-complete.
- Remark: non-linear integer arithmetic is undecidable even for the quantifier free case.

Linear Real Arithmetic

- The general satisfiability problem for $\mathcal{T}_{\mathcal{LRA}}$ is decidable, but its complexity is doubly-exponential.
- The quantifier free satisfiability problem is solvable in polynomial time, though exponential methods (Simplex) tend to perform best in practice.

Difference Logic

- Difference logic is a fragment of linear arithmetic.
- Atoms have the form: $x - y \leq c$.
- Most linear arithmetic atoms found in hardware and software verification are in this fragment.
- The quantifier free satisfiability problem is solvable in $O(nm)$.

Theory of Arrays

- $\Sigma_P = \emptyset$, $\Sigma_F = \{$*read*, *write*$\}$.
- Non-extensional arrays
  - Let $\Lambda_{\mathcal{A}}$ be the following axioms:

$$\forall a, i, v.\ read(write(a, i, v), i) = v$$

$$\forall a, i, j, v.\ i \neq j \Rightarrow read(write(a, i, v), j) = read(a, j)$$

  - $\mathcal{T}_{\mathcal{A}} = DC(\Lambda_{\mathcal{A}})$
- For extensional arrays, we need the following extra axiom:

$$\forall a, b.\ (\forall i. read(a, i) = read(b, i)) \Rightarrow a = b$$

- The satisfiability problem for $\mathcal{T}_{\mathcal{A}}$ is undecidable, the quantifier free case is NP-complete.

Other theories

- Bit-vectors
- Partial orders
- Tuples & Records
- Algebraic data types
- . . .

Combination of Theories

- In practice, we need a combination of theories.
- Example:
  - $x + 2 = y \Rightarrow f(\textit{read}(\textit{write}(a, x, 3), y - 2)) = f(y - x + 1)$
- Given

$$\begin{array}{rcl} \Sigma & = & \Sigma_1 \cup \Sigma_2 \\ \mathcal{T}_1, \mathcal{T}_2 & : & \textit{theories over } \Sigma_1, \Sigma_2 \\ \mathcal{T} & = & \textit{DC}(\mathcal{T}_1 \cup \mathcal{T}_2) \end{array}$$

- Is $\mathcal{T}$ consistent?
- Given satisfiability procedures for conjunction of literals of $\mathcal{T}_1$ and $\mathcal{T}_2$, how to decide the satisfiability of $\mathcal{T}$?

Preamble

- Disjoint signatures: $\Sigma_1 \cap \Sigma_2 = \emptyset$.
- Purification
- Stably-Infinite Theories.
- Convex Theories.

Purification

- Purification:

  $\phi \wedge P(\ldots, s[t], \ldots) \rightsquigarrow \phi \wedge P(\ldots, s[x], \ldots) \wedge x = t$,

  t is not a variable.

- Purification is satisfiability preserving and terminating.

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$$u_2 - 1 = x, f(y) + 1 = y, u_1 = x - 1, u_2 = f(u_1) \rightsquigarrow$$

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$$u_2 - 1 = x, u_3 + 1 = y, u_1 = x - 1, u_2 = f(u_1), u_3 = f(y)$$

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$$u_2 - 1 = x, u_3 + 1 = y, u_1 = x - 1, u_2 = f(u_1), u_3 = f(y)$$

After Purification

$$x = f(z), f(x) \neq f(y), 0 \leq x \leq 1, 0 \leq y \leq 1, z = y - 1$$

After Purification

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Red Model	*Blue Model*
$x^{\mathcal{E}} = *_1$	$x^{\mathcal{A}} = 0$
$y^{\mathcal{E}} = *_2$	$y^{\mathcal{A}} = 0$
$z^{\mathcal{E}} = *_3$	$z^{\mathcal{A}} = -1$
$f^{\mathcal{E}} = \{*_1 \mapsto *_4,$	
$*_2 \mapsto *_5,$	
$*_3 \mapsto *_1,$	
else $\mapsto *_6\}$	

Stably-Infinite Theories

- A theory is stably infinite if every satisfiable QFF is satisfiable in an infinite model.
- Example. Theories with only finite models are not stably infinite. $\mathcal{T}_2 = DC(\forall x, y, z. \, (x = y) \vee (x = z) \vee (y = z))$.
- The union of two consistent, disjoint, stably infinite theories is consistent.

Convexity

- A theory $\mathcal{T}$ is convex iff

  for all finite sets Γ of literals and

  for all non-empty disjunctions $\bigvee_{i \in I} x_i = y_i$ of variables,

  $\Gamma \models_{\mathcal{T}} \bigvee_{i \in I} x_i = y_i$ iff $\Gamma \models_{\mathcal{T}} x_i = y_i$ for some $i \in I$.

- Every convex theory $\mathcal{T}$ with non trivial models (i.e., $\models_T \exists x, y.\ x \neq y$) is stably infinite.

- All Horn theories are convex – this includes all (conditional) equational theories.

- Linear rational arithmetic is convex.

Convexity (cont.)

- Many theories are not convex:
  - Linear integer arithmetic.

$$y = 1, z = 2, 1 \leq x \leq 2 \models x = y \vee x = z$$

  - Nonlinear arithmetic.

$$x^2 = 1, y = 1, z = -1 \models x = y \vee x = z$$

  - Theory of Bit-vectors.
  - Theory of Arrays.

$$v_1 = \mathit{read}(\mathit{write}(a, i, v_2), j), v_3 = \mathit{read}(a, j) \models$$
$$v_1 = v_2 \vee v_1 = v_3$$

Convexity: Example

- Let $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$, where $\mathcal{T}_1$ is EUF ($O(nlog(n))$) and $\mathcal{T}_2$ is IDL ($O(nm)$).
- $\mathcal{T}_2$ is not convex.
- Satisfiability is NP-Complete for $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$.
  - Reduce 3CNF satisfiability to $\mathcal{T}$-satisfiability.
  - For each boolean variable p_i add the atomic formulas: $0 \leq x_i, x_i \leq 1$.
  - For a clause $p_1 \vee \neg p_2 \vee p_3$ add the atomic formula: $f(x_1, x_2, x_3) \neq f(0, 1, 0)$

Nelson-Oppen Combination

- Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be consistent, stably infinite theories over disjoint (countable) signatures. Assume satisfiability of conjunction of literals can decided in $O(T_1(n))$ and $O(T_2(n))$ time respectively. Then,
  1. The combined theory $\mathcal{T}$ is consistent and stably infinite.
  2. Satisfiability of quantifier free conjunction of literals in $\mathcal{T}$ can be decided in $O(2^{n^2} \times (T_1(n) + T_2(n))$.
  3. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are convex, then so is $\mathcal{T}$ and satisfiability in $\mathcal{T}$ is in $O(n^3 \times (T_1(n) + T_2(n)))$.

Nelson-Oppen Combination Procedure

- The combination procedure:

  Initial State: ϕ is a conjunction of literals over $\Sigma_1 \cup \Sigma_2$.

  Purification: Preserving satisfiability transform ϕ into $\phi_1 \wedge \phi_2$, such that, $\phi_i \in \Sigma_i$.

  Interaction: Guess a partition of $\mathcal{V}(\phi_1) \cap \mathcal{V}(\phi_2)$ into disjoint subsets. Express it as conjunction of literals ψ.
  Example. The partition $\{x_1\}, \{x_2, x_3\}, \{x_4\}$ is represented as $x_1 \neq x_2, x_1 \neq x_4, x_2 \neq x_4, x_2 = x_3$.

  Component Procedures : Use individual procedures to decide whether $\phi_i \wedge \psi$ is satisfiable.

  Return: If both return yes, return yes. No, otherwise.

NO procedure: soundness

- Each step is satisfiability preserving.
- Say ϕ is satisfiable (in the combination).
  - Purification: $\phi_1 \wedge \phi_2$ is satisfiable.

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  - Iteration: for some partition ψ, $\phi_1 \wedge \phi_2 \wedge \psi$ is satisfiable.

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  - Component procedures: $\phi_1 \wedge \psi$ and $\phi_2 \wedge \psi$ are both satisfiable in component theories.

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  - Purification: $\phi_1 \wedge \phi_2$ is satisfiable.
  - Iteration: for some partition ψ, $\phi_1 \wedge \phi_2 \wedge \psi$ is satisfiable.
  - Component procedures: $\phi_1 \wedge \psi$ and $\phi_2 \wedge \psi$ are both satisfiable in component theories.
  - Therefore, if the procedure return unsatisfiable, then ϕ is unsatisfiable.

NO procedure: correctness

- Suppose the procedure returns satisfiable.
  - Let ψ be the partition and A and B be models of $\mathcal{T}_1 \wedge \phi_1 \wedge \psi$ and $\mathcal{T}_2 \wedge \phi_2 \wedge \psi$.

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  - Let h be a bijection between S_A and S_B such that $h(x^A) = x^B$ for each shared variable.

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  - Let h be a bijection between S_A and S_B such that $h(x^A) = x^B$ for each shared variable.
  - Extend B to $\bar{B}$ by interpretations of symbols in Σ_1: $f^{\bar{B}}(b_1, \ldots, b_n) = h(f^A(h^{-1}(b_1), \ldots, h^{-1}(b_n)))$

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  - Extend B to $\bar{B}$ by interpretations of symbols in Σ_1: $f^{\bar{B}}(b_1, \ldots, b_n) = h(f^A(h^{-1}(b_1), \ldots, h^{-1}(b_n)))$
  - $\bar{B}$ is a model of: $\mathcal{T}_1 \wedge \phi_1 \wedge \mathcal{T}_2 \wedge \phi_2 \wedge \psi$

NO deterministic procedure

- Instead of guessing, we can deduce the equalities to be shared.

  Purification: no changes.

  Interaction: Deduce an equality $x = y$:

  $$\mathcal{T}_1 \vdash (\phi_1 \Rightarrow x = y)$$

  Update $\phi_2 := \phi_2 \wedge x = y$. And vice-versa. Repeat until no further changes.

  Component Procedures : Use individual procedures to decide whether ϕ_i is satisfiable.

- Remark: $\mathcal{T}_i \vdash (\phi_i \Rightarrow x = y)$ iff $\phi_i \wedge x \neq y$ is not satisfiable in $\mathcal{T}_i$.

NO deterministic procedure: correctness

- Assume the theories are convex.
    - Suppose ϕ_i is satisfiable.

NO deterministic procedure: correctness

- Assume the theories are convex.
  - Suppose ϕ_i is satisfiable.
  - Let E be the set of equalities $x_j = x_k$ ($j \neq k$) such that, $\mathcal{T}_i \not\models \phi_i \Rightarrow x_j = x_k$.

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- Assume the theories are convex.
  - Suppose ϕ_i is satisfiable.
  - Let E be the set of equalities $x_j = x_k$ ($j \neq k$) such that, $\mathcal{T}_i \not\models \phi_i \Rightarrow x_j = x_k$.
  - By convexity, $\mathcal{T}_i \not\models \phi_i \Rightarrow \bigvee_E x_j = x_k$.

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  - By convexity, $\mathcal{T}_i \not\models \phi_i \Rightarrow \bigvee_E x_j = x_k$.
  - $\phi_i \wedge \bigwedge_E x_j \neq x_k$ is satisfiable.

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  - By convexity, $\mathcal{T}_i \not\models \phi_i \Rightarrow \bigvee_E x_j = x_k$.
  - $\phi_i \wedge \bigwedge_E x_j \neq x_k$ is satisfiable.
  - The proof now is identical to the nondeterministic case.
  - Sharing equalities is sufficient, because a theory $\mathcal{T}_1$ can assume that $x^B \neq y^B$ whenever $x = y$ is not implied by $\mathcal{T}_2$ and vice versa.

NO deterministic procedure

- Deterministic procedure does not work for non convex theories.
- Example (integer arithmetic):

$$0 \leq x, y, z \leq 1, f(x) \neq f(y), f(x) \neq f(z), f(y) \neq f(z)$$

- (Expensive) solution: deduce disjunctions of equalities.

Roadmap

- Background
- Implementing SMT solvers

Architecture

- Preprocessor/Simplifier.
- SAT solver.
- Blackboard: “bus” used to connect the theories.
- Theories:
  - Arithmetic,
  - Bit-vectors,
  - Arrays,
  - etc.
- Heuristic quantifier instantiation.

Preprocessor/Simplifier

- Apply simplification rules:
  - Normalization:
    - Sort arguments of commutative operators.
    - Flat associative operators:

      or$(p_1,$ *or*$(p_2, p_3)) \leadsto$ *or*(p_1, p_2, p_3)
    - Rewrite arithmetic expressions as sums of monomials.

$$x(y+3) = 5 \leadsto 3x + xy = 5$$

  - Hash-consing.
  - Lift term if-then-else.
  - $x = t \wedge C[x] \leadsto C[t]$.
  - etc.

Preprocessor/Simplifier

- CNF translation.
- Rewrite formula to simplify atoms that are asserted during the search.
- Example:

$$x \geq 0 \wedge (x + y \leq 2 \vee x + 2y \geq 6) \wedge (x + y = 2 \vee x + 2y > 4)$$

$$\rightsquigarrow$$

$$(s_1 = x + y \wedge s_2 = x + 2y) \;\wedge$$
$$(x \geq 0 \wedge (s_1 \leq 2 \vee s_2 \geq 6) \wedge (s_1 = 2 \vee s_2 > 4))$$

- Only bounds (e.g., $s_1 \leq 2$) are asserted during the search.
- Unconstrained variables can be eliminated before the beginning of the search.

SMT solvers before SAT breakthrough

- Ad-hoc support for boolean combination of literals.
- Ad-hoc support for (non-convex) theories.
- "Case-splits" should be avoided.
- Few real benchmarks.
- Breakthrough in SAT solving changed everything.

Breakthrough in SAT solving

- Breakthrough in SAT solving influenced the way SMT solvers are implemented.
- Modern SAT solvers are based on the DPLL algorithm.
- Modern implementations add several sophisticated search techniques.
  - Backjumping
  - Learning
  - Restarts
  - Watched literals

The Original DPLL Procedure

- DPLL tries to build incrementally a satisfying truth assignment M for a CNF formula F.
- M is grown by
  - deducing the truth value of a literal from M and F, or
  - guessing a truth value.
- If a wrong guess leads to an inconsistency, the procedure backtracks and tries the opposite one.

Basic DPLL System – Example

$$\emptyset \;\|\; \overline{1} \vee 2,\;\; \overline{3} \vee 4,\;\; \overline{5} \vee \overline{6},\;\; 6 \vee \overline{5} \vee \overline{2}$$

Basic DPLL System – Example

$$\emptyset \parallel \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ 6 \vee \overline{5} \vee \overline{2} \implies (\text{Decide})$$

$$1 \parallel \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ 6 \vee \overline{5} \vee \overline{2}$$

Basic DPLL System – Example

$$\emptyset \parallel \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ 6 \vee \overline{5} \vee \overline{2} \implies (\text{Decide})$$

$$\color{red}{1} \parallel \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ 6 \vee \overline{5} \vee \overline{2} \implies (\text{UnitProp})$$

$$\color{red}{1}\ 2 \parallel \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ 6 \vee \overline{5} \vee \overline{2}$$

Basic DPLL System – Example

$$\emptyset \parallel \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ 6 \vee \overline{5} \vee \overline{2} \implies (\text{Decide})$$

$$\color{red}{1} \parallel \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ 6 \vee \overline{5} \vee \overline{2} \implies (\text{UnitProp})$$

$$\color{red}{1}\ 2 \parallel \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ 6 \vee \overline{5} \vee \overline{2} \implies (\text{Decide})$$

$$\color{red}{1}\ 2\ \color{red}{3} \parallel \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ 6 \vee \overline{5} \vee \overline{2}$$

Basic DPLL System – Example

$$
\begin{array}{rcll}
\emptyset & \| & \overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ 6 \vee \overline{5} \vee \overline{2} & \Longrightarrow\ \ (\text{Decide}) \\
\color{red}{1} & \| & \overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ 6 \vee \overline{5} \vee \overline{2} & \Longrightarrow\ \ (\text{UnitProp}) \\
\color{red}{1}\ 2 & \| & \overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ 6 \vee \overline{5} \vee \overline{2} & \Longrightarrow\ \ (\text{Decide}) \\
\color{red}{1}\ 2\ \color{red}{3} & \| & \overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ 6 \vee \overline{5} \vee \overline{2} & \Longrightarrow\ \ (\text{UnitProp}) \\
\color{red}{1}\ 2\ \color{red}{3}\ 4 & \| & \overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ 6 \vee \overline{5} \vee \overline{2} &
\end{array}
$$

Basic DPLL System – Example

$$
\begin{array}{rcll}
\emptyset & \| & \overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ 6 \vee \overline{5} \vee \overline{2} & \Longrightarrow \ (\textsf{Decide}) \\
{\color{red}1} & \| & \overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ 6 \vee \overline{5} \vee \overline{2} & \Longrightarrow \ (\textsf{UnitProp}) \\
{\color{red}1}\ 2 & \| & \overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ 6 \vee \overline{5} \vee \overline{2} & \Longrightarrow \ (\textsf{Decide}) \\
{\color{red}1}\ 2\ {\color{red}3} & \| & \overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ 6 \vee \overline{5} \vee \overline{2} & \Longrightarrow \ (\textsf{UnitProp}) \\
{\color{red}1}\ 2\ {\color{red}3}\ 4 & \| & \overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ 6 \vee \overline{5} \vee \overline{2} & \Longrightarrow \ (\textsf{Decide}) \\
{\color{red}1}\ 2\ {\color{red}3}\ 4\ {\color{red}5} & \| & \overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ 6 \vee \overline{5} \vee \overline{2} &
\end{array}
$$

Basic DPLL System – Example

$$\emptyset \;\|\; \overline{1} \vee 2,\; \overline{3} \vee 4,\; \overline{5} \vee \overline{6},\; 6 \vee \overline{5} \vee \overline{2} \;\Longrightarrow\; (\text{Decide})$$

$$1 \;\|\; \overline{1} \vee 2,\; \overline{3} \vee 4,\; \overline{5} \vee \overline{6},\; 6 \vee \overline{5} \vee \overline{2} \;\Longrightarrow\; (\text{UnitProp})$$

$$1\;2 \;\|\; \overline{1} \vee 2,\; \overline{3} \vee 4,\; \overline{5} \vee \overline{6},\; 6 \vee \overline{5} \vee \overline{2} \;\Longrightarrow\; (\text{Decide})$$

$$1\;2\;3 \;\|\; \overline{1} \vee 2,\; \overline{3} \vee 4,\; \overline{5} \vee \overline{6},\; 6 \vee \overline{5} \vee \overline{2} \;\Longrightarrow\; (\text{UnitProp})$$

$$1\;2\;3\;4 \;\|\; \overline{1} \vee 2,\; \overline{3} \vee 4,\; \overline{5} \vee \overline{6},\; 6 \vee \overline{5} \vee \overline{2} \;\Longrightarrow\; (\text{Decide})$$

$$1\;2\;3\;4\;5 \;\|\; \overline{1} \vee 2,\; \overline{3} \vee 4,\; \overline{5} \vee \overline{6},\; 6 \vee \overline{5} \vee \overline{2} \;\Longrightarrow\; (\text{UnitProp})$$

$$1\;2\;3\;4\;5\;\overline{6} \;\|\; \overline{1} \vee 2,\; \overline{3} \vee 4,\; \overline{5} \vee \overline{6},\; 6 \vee \overline{5} \vee \overline{2}$$

Basic DPLL System – Example

$$
\begin{array}{rcll}
\emptyset & \| & \overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ 6 \vee \overline{5} \vee \overline{2} & \Longrightarrow (\text{Decide}) \\
1 & \| & \overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ 6 \vee \overline{5} \vee \overline{2} & \Longrightarrow (\text{UnitProp}) \\
1\ 2 & \| & \overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ 6 \vee \overline{5} \vee \overline{2} & \Longrightarrow (\text{Decide}) \\
1\ 2\ 3 & \| & \overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ 6 \vee \overline{5} \vee \overline{2} & \Longrightarrow (\text{UnitProp}) \\
1\ 2\ 3\ 4 & \| & \overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ 6 \vee \overline{5} \vee \overline{2} & \Longrightarrow (\text{Decide}) \\
1\ 2\ 3\ 4\ 5 & \| & \overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ 6 \vee \overline{5} \vee \overline{2} & \Longrightarrow (\text{UnitProp}) \\
1\ 2\ 3\ 4\ 5\ \overline{6} & \| & \overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ 6 \vee \overline{5} \vee \overline{2} & \Longrightarrow (\text{Backjump}) \\
1\ 2\ \overline{5} & \| & \overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ 6 \vee \overline{5} \vee \overline{2} &
\end{array}
$$

Backjump with clause $\overline{1} \vee \overline{5}$

Basic DPLL System – Example

$$\ldots$$

$$1\;2\;3\;4\;5\;\overline{6} \;\|\; \overline{1} \vee 2,\;\; \overline{3} \vee 4,\;\; \overline{5} \vee \overline{6},\;\; 6 \vee \overline{5} \vee \overline{2} \quad \Longrightarrow \quad (\text{Backjump})$$

$$1\;2\;\overline{5} \;\|\; \overline{1} \vee 2,\;\; \overline{3} \vee 4,\;\; \overline{5} \vee \overline{6},\;\; 6 \vee \overline{5} \vee \overline{2}.$$

Basic DPLL System – Example

$$\ldots$$

$$1\ 2\ 3\ 4\ 5\ \overline{6} \parallel \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ 6 \vee \overline{5} \vee \overline{2} \implies \text{(Backjump)}$$

$$1\ 2\ \overline{5} \parallel \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ 6 \vee \overline{5} \vee \overline{2}.$$

$\overline{1} \vee \overline{5}$ is implied by the original set of clauses. For instance, by resolution,

$$\cfrac{\cfrac{\overline{1} \vee 2 \quad 6 \vee \overline{5} \vee \overline{2}}{\overline{1} \vee 6 \vee \overline{5}} \quad \overline{5} \vee \overline{6}}{\overline{1} \vee \overline{5}}$$

Therefore, instead deciding 3, we could have deduced $\overline{5}$.

Basic DPLL System – Example

$$\ldots$$

$$1\ 2\ 3\ 4\ 5\ \overline{6} \parallel \quad \overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ 6 \vee \overline{5} \vee \overline{2} \quad \Longrightarrow \quad (\text{Backjump})$$

$$1\ 2\ \overline{5} \parallel \quad \overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ 6 \vee \overline{5} \vee \overline{2}.$$

$\overline{1} \vee \overline{5}$ is implied by the original set of clauses. For instance, by resolution,

$$\dfrac{\dfrac{\overline{1} \vee 2 \quad 6 \vee \overline{5} \vee \overline{2}}{\overline{1} \vee 6 \vee \overline{5}} \qquad \overline{5} \vee \overline{6}}{\overline{1} \vee \overline{5}}$$

Therefore, instead deciding 3, we could have deduced $\overline{5}$.

Clauses like $\overline{1} \vee \overline{5}$ are computed by navigating the implication graph.

Lazy approach: SAT solvers + Theories

- This approach was independently developed by several groups: CVC (Stanford), ICS (SRI), MathSAT (Univ. Trento, Italy), and Verifun (HP).
- It was motivated also by the breakthroughs in SAT solving.
- SAT solver "manages" the boolean structure, and assigns truth values to the atoms in a formula.
- Efficient theory solvers are used to validate the (partial) assignment produced by the SAT solver.
- When theory solver detects unsatisfiability $\longrightarrow$ a new clause (lemma) is created.

SAT solvers + Theories (cont.)

- Example:
  - Suppose the SAT solver assigns
    $\{x = y \rightarrow \mathsf{T}, y = z \rightarrow \mathsf{T}, f(x) = f(z) \rightarrow \mathsf{F}\}$.
  - Theory solver detects the conflict, and a lemma is created
    $\neg(x = y) \vee \neg(y = z) \vee f(x) = f(z)$.
- Some theory solvers use the "proof" of the conflict to build the lemma.
- Problems in these tools:
  - The lemmas are imprecise (not minimal).
  - The theory solver is "passive": it just detects conflicts. There is no propagation step.
  - Backtracking is expensive, some tools restart from scratch when a conflict is detected.

Blackboard/Bus

- The Blackboard/Bus stores the equalities/disequalities known by the solver.
- The set of known equalities is represented as a set of equivalence classes.
  - Union-Find data structure.
- The bus is used to connect the theories.

Combining theories in practice

- Propagate all implied equalities.
  - Deterministic Nelson-Oppen.
  - Complete only for convex theories.
  - It may be expensive for some theories.
- Delayed Theory Combination.
  - Nondeterministic Nelson-Oppen.
  - Create set of interface equalities ($x = y$) between shared variables.
  - Use SAT solver to guess the partition.
  - Disadvantage: the number of additional equality literals is quadratic in the number of shared variables.

Combining theories in practice (cont.)

- Common to these methods is that they are pessimistic about which equalities are propagated.
- Model-based Theory Combination
  - Optimistic approach.
  - Use a candidate model M_i for one of the theories $\mathcal{T}_i$ and propagate all equalities implied by the candidate model, hedging that other theories will agree.

    if $M_i \models \mathcal{T}_i \cup \Gamma_i \cup \{u = v\}$ **then** propagate $u = v$.
  - If not, use backtracking to fix the model.
  - It is cheaper to enumerate equalities that are implied in a particular model than of all models.

Model based theory combination: Example

$$x = f(y - 1), f(x) \neq f(y), 0 \leq x \leq 1, 0 \leq y \leq 1$$

Purifying

Model based theory combination: Example

$$x = f(z), f(x) \neq f(y), 0 \leq x \leq 1, 0 \leq y \leq 1, z = y - 1$$

Model based theory combination: Example

$\mathcal{T}_{\mathcal{E}}$			$\mathcal{T}_{\mathcal{A}}$	
Literals	*Eq. Classes*	*Model*	*Literals*	*Model*
$x = f(z)$	$\{x, f(z)\}$	$x^{\mathcal{E}} = *_1$	$0 \leq x \leq 1$	$x^{\mathcal{A}} = 0$
$f(x) \neq f(y)$	$\{y\}$	$y^{\mathcal{E}} = *_2$	$0 \leq y \leq 1$	$y^{\mathcal{A}} = 0$
	$\{z\}$	$z^{\mathcal{E}} = *_3$	$z = y - 1$	$z^{\mathcal{A}} = -1$
	$\{f(x)\}$	$f^{\mathcal{E}} = \{*_1 \mapsto *_4,$		
	$\{f(y)\}$	$*_2 \mapsto *_5,$		
		$*_3 \mapsto *_1,$		
		else $\mapsto *_6\}$		

Assume x = y

Model based theory combination: Example

$\mathcal{T}_\mathcal{E}$			$\mathcal{T}_\mathcal{A}$	
Literals	*Eq. Classes*	*Model*	*Literals*	*Model*
$x = f(z)$	$\{x, y, f(z)\}$	$x^\mathcal{E} = *_1$	$0 \leq x \leq 1$	$x^\mathcal{A} = 0$
$f(x) \neq f(y)$	$\{z\}$	$y^\mathcal{E} = *_1$	$0 \leq y \leq 1$	$y^\mathcal{A} = 0$
$x = y$	$\{f(x), f(y)\}$	$z^\mathcal{E} = *_2$	$z = y - 1$	$z^\mathcal{A} = -1$
		$f^\mathcal{E} = \{*_1 \mapsto *_3,$	$x = y$	
		$*_2 \mapsto *_1,$		
		else $\mapsto *_4\}$		

Unsatisfiable

Model based theory combination: Example

$\mathcal{T}_{\mathcal{E}}$			$\mathcal{T}_{\mathcal{A}}$	
Literals	*Eq. Classes*	*Model*	*Literals*	*Model*
$x = f(z)$	$\{x, f(z)\}$	$x^{\mathcal{E}} = *_1$	$0 \leq x \leq 1$	$x^{\mathcal{A}} = 0$
$f(x) \neq f(y)$	$\{y\}$	$y^{\mathcal{E}} = *_2$	$0 \leq y \leq 1$	$y^{\mathcal{A}} = 0$
$x \neq y$	$\{z\}$	$z^{\mathcal{E}} = *_3$	$z = y - 1$	$z^{\mathcal{A}} = -1$
	$\{f(x)\}$	$f^{\mathcal{E}} = \{*_1 \mapsto *_4,$	$x \neq y$	
	$\{f(y)\}$	$*_2 \mapsto *_5,$		
		$*_3 \mapsto *_1,$		
		else $\mapsto *_6\}$		

Backtrack, and assert $x \neq y$.

$\mathcal{T}_{\mathcal{A}}$ model need to be fixed.

Model based theory combination: Example

$\mathcal{T}_{\mathcal{E}}$			$\mathcal{T}_{\mathcal{A}}$	
Literals	*Eq. Classes*	*Model*	*Literals*	*Model*
$x = f(z)$	$\{x, f(z)\}$	$x^{\mathcal{E}} = *_1$	$0 \leq x \leq 1$	$x^{\mathcal{A}} = 0$
$f(x) \neq f(y)$	$\{y\}$	$y^{\mathcal{E}} = *_2$	$0 \leq y \leq 1$	$y^{\mathcal{A}} = 1$
$x \neq y$	$\{z\}$	$z^{\mathcal{E}} = *_3$	$z = y - 1$	$z^{\mathcal{A}} = 0$
	$\{f(x)\}$	$f^{\mathcal{E}} = \{*_1 \mapsto *_4,$	$x \neq y$	
	$\{f(y)\}$	$*_2 \mapsto *_5,$		
		$*_3 \mapsto *_1,$		
		else $\mapsto *_6\}$		

Assume x = z

Model based theory combination: Example

$\mathcal{T}_{\mathcal{E}}$			$\mathcal{T}_{\mathcal{A}}$	
Literals	*Eq. Classes*	*Model*	*Literals*	*Model*
$x = f(z)$	$\{x, z, f(x), f(z)\}$	$x^{\mathcal{E}} = *_1$	$0 \leq x \leq 1$	$x^{\mathcal{A}} = 0$
$f(x) \neq f(y)$	$\{y\}$	$y^{\mathcal{E}} = *_2$	$0 \leq y \leq 1$	$y^{\mathcal{A}} = 1$
$x \neq y$	$\{f(y)\}$	$z^{\mathcal{E}} = *_1$	$z = y - 1$	$z^{\mathcal{A}} = 0$
$x = z$		$f^{\mathcal{E}} = \{*_1 \mapsto *_1,$	$x \neq y$	
		$*_2 \mapsto *_3,$	$x = z$	
		else $\mapsto *_4\}$		

Satisfiable

Model based theory combination: Example

$\mathcal{T}_{\mathcal{E}}$			$\mathcal{T}_{\mathcal{A}}$	
Literals	*Eq. Classes*	*Model*	*Literals*	*Model*
$x = f(z)$	$\{x, z, f(x), f(z)\}$	$x^{\mathcal{E}} = *_1$	$0 \leq x \leq 1$	$x^{\mathcal{A}} = 0$
$f(x) \neq f(y)$	$\{y\}$	$y^{\mathcal{E}} = *_2$	$0 \leq y \leq 1$	$y^{\mathcal{A}} = 1$
$x \neq y$	$\{f(y)\}$	$z^{\mathcal{E}} = *_1$	$z = y - 1$	$z^{\mathcal{A}} = 0$
$x = z$		$f^{\mathcal{E}} = \{*_1 \mapsto *_1,$	$x \neq y$	
		$*_2 \mapsto *_3,$	$x = z$	
		else $\mapsto *_4\}$		

Let h be the bijection between $S_{\mathcal{E}}$ and $S_{\mathcal{A}}$.

$$h = \{*_1 \mapsto 0, *_2 \mapsto 1, *_3 \mapsto -1, *_4 \mapsto 2, \ldots\}$$

Model based theory combination: Example

$\mathcal{T}_{\mathcal{E}}$		$\mathcal{T}_{\mathcal{A}}$	
Literals	*Model*	*Literals*	*Model*
$x = f(z)$	$x^{\mathcal{E}} = *_1$	$0 \leq x \leq 1$	$x^{\mathcal{A}} = 0$
$f(x) \neq f(y)$	$y^{\mathcal{E}} = *_2$	$0 \leq y \leq 1$	$y^{\mathcal{A}} = 1$
$x \neq y$	$z^{\mathcal{E}} = *_1$	$z = y - 1$	$z^{\mathcal{A}} = 0$
$x = z$	$f^{\mathcal{E}} = \{*_1 \mapsto *_1,$	$x \neq y$	$f^{\mathcal{A}} = \{0 \mapsto 0$
	$*_2 \mapsto *_3,$	$x = z$	$1 \mapsto -1$
	else $\mapsto *_4\}$		*else* $\mapsto 2\}$

Extending $\mathcal{A}$ using h.

$$h = \{*_1 \mapsto 0, *_2 \mapsto 1, *_3 \mapsto -1, *_4 \mapsto 2, \ldots\}$$

Non Stably-Infinite Theories in practice

- Bit-vector theory is not stably-infinite.
- How can we support it?
- Solution: add a predicate *is-bv*(x) to the bit-vector theory (intuition: *is-bv*(x) is true iff x is a bitvector).
- The result of the bit-vector operation *op*(x, y) is not specified if $\neg$*is-bv*(x) or $\neg$*is-bv*(y).
- The new bit-vector theory is stably-infinite.

Precise Lemmas

- Lemma:

  $\{a_1 = \mathit{T}, a_1 = \mathit{F}, a_3 = \mathit{F}\}$*is inconsistent* $\leadsto \neg a_1 \vee a_2 \vee a_3$

- An inconsistent A set is redundant if $A' \subset A$ is also inconsistent.
- Redundant inconsistent sets $\leadsto$ Imprecise Lemmas $\leadsto$ Ineffective pruning of the search space.
- Noise of a redundant set: $A \setminus A_{min}$.
- The imprecise lemma is useless in any context (partial assignment) where an atom in the noise has a different assignment.
- Example: suppose a_1 is in the noise, then $\neg a_1 \vee a_2 \vee a_3$ is useless when $a_1 = F$.

Precise Lemmas

- Simple approach: track dependencies.
- Record the antecedents $\psi_1, \ldots, \psi_n$ of a consequent ϕ.
- It is the same approach used in SAT solvers:

  Record the clause $C \vee l$ used to imply a literal l.

- It may be imprecise.

Precise Lemmas: simple approach

- Example: assume equations (1), (2) and (3) were asserted into the logical context.

$$x + w + 3 = 0 \quad (1)$$
$$x + z + 1 = 0 \quad (2)$$
$$x + y + 1 = 0 \quad (3)$$

Precise Lemmas: simple approach

- Example: assume equations (1), (2) and (3) were asserted into the logical context.

$$
\begin{array}{rcll}
x + w + 3 & = & 0 & (1) \\
x + z + 1 & = & 0 & (2) \\
x + y + 1 & = & 0 & (3) \\
\hline
-w + z - 2 & = & 0 & (4) = (2) - (1)
\end{array}
$$

Precise Lemmas: simple approach

- Example: assume equations (1), (2) and (3) were asserted into the logical context.

$$
\begin{array}{rcl r}
x + w + 3 & = & 0 & (1) \\
x + z + 1 & = & 0 & (2) \\
x + y + 1 & = & 0 & (3) \\
\hline
-w + z - 2 & = & 0 & (4) = (2) - (1) \\
-w + y - 2 & = & 0 & (5) = (3) - (1)
\end{array}
$$

Precise Lemmas: simple approach

- Example: assume equations (1), (2) and (3) were asserted into the logical context.

$$
\begin{array}{rcll}
x + w + 3 & = & 0 & (1) \\
x + z + 1 & = & 0 & (2) \\
x + y + 1 & = & 0 & (3) \\
\hline
-w + z - 2 & = & 0 & (4) = (2) - (1) \\
-w + y - 2 & = & 0 & (5) = (3) - (1) \\
y - z & = & 0 & (6) = (5) - (4)
\end{array}
$$

Precise Lemmas: simple approach

- Example: assume equations (1), (2) and (3) were asserted into the logical context.

$$
\begin{array}{rcll}
x + w + 3 & = & 0 & (1) \\
x + z + 1 & = & 0 & (2) \\
x + y + 1 & = & 0 & (3) \\
\hline
-w + z - 2 & = & 0 & (4) = (2) - (1) \\
-w + y - 2 & = & 0 & (5) = (3) - (1) \\
y - z & = & 0 & (6) = (5) - (4)
\end{array}
$$

- Equation (6) implies that $y = z$. It depends on (1), (2), and (3).

Precise Lemmas: simple approach

- Example: assume equations (1), (2) and (3) were asserted into the logical context.

$$
\begin{array}{rcll}
x + w + 3 & = & 0 & (1) \\
x + z + 1 & = & 0 & (2) \\
x + y + 1 & = & 0 & (3) \\
\hline
-w + z - 2 & = & 0 & (4) = (2) - (1) \\
-w + y - 2 & = & 0 & (5) = (3) - (1) \\
y - z & = & 0 & (6) = (5) - (4)
\end{array}
$$

- Equation (6) implies that $y = z$. It depends on (1), (2), and (3).
- Equation (1) is not necessary to derive $y = z$.

Precise Lemmas: auxiliary variables

- Use auxiliary/zero variables to "name" linear polynomials.

$$
\begin{array}{rcl}
x + w + 3 & = & s_1 \\
x + z + 1 & = & s_2 \\
x + y + 1 & = & s_3 \\
\hline
\end{array}
$$

Precise Lemmas: auxiliary variables

- Use auxiliary/zero variables to “name” linear polynomials.

$$
\begin{array}{rcl}
x + w + 3 & = & s_1 \\
x + z + 1 & = & s_2 \\
x + y + 1 & = & s_3 \\
\hline
-w + z - 2 & = & s_2 - s_1
\end{array}
$$

Precise Lemmas: auxiliary variables

- Use auxiliary/zero variables to “name” linear polynomials.

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x + w + 3 & = & s_1 \\
x + z + 1 & = & s_2 \\
x + y + 1 & = & s_3 \\
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-w + z - 2 & = & s_2 - s_1 \\
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y - z & = & s_3 - s_1 - s_2 + s_1
\end{array}
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- The last equation implies $y = z$ when s_2 and s_3 are equal to 0.
- This is the approach used in the Simplex based solver.
- A similar approach is used to implement incremental SAT solvers.

Efficient Backtracking

- One of the most important improvements in SAT was efficient backtracking.
- Until recently, backtracking was ignored in the design of theory solvers.
- Extreme (inefficient) approach: restart from scratch on every conflict.
- Other approaches:
  - Functional data-structures.
  - Backtrackable data-structures
  - Trail-stack.
- Restore to a logically equivalent state.

Reduction Functions

- A reduction function reduces the satisfiability problem for a theory $\mathcal{T}_1$ to the satisfiability problem of a simpler theory $\mathcal{T}_2$.
- Reduction functions simplify the implementation.
- Potential disadvantages:
  - "Information loss".
  - Eager addition of irrelevant information.
- Theory of commutative functions.
  - Deductive closure of: $\forall x, y. f(x, y) = f(y, x)$
  - Reduction to $\mathcal{T}_{\mathcal{E}}$.
  - For every $f(a, b)$ in ϕ, add the equality $f(a, b) = f(b, a)$.

Reduction Functions: Ackermann's reduction

- Ackermann's reduction is used to remove uninterpreted functions.
  - For each application $f(\vec{a})$ in ϕ create a fresh variable $f_{\vec{a}}$.
  - For each pair of applications $f(\vec{a})$, $f(\vec{c})$ in ϕ add the clause $\vec{a} \neq \vec{c} \vee f_{\vec{a}} = f_{\vec{c}}$.
  - Replace $f(\vec{a})$ with $f_{\vec{a}}$ in ϕ.
- It is used in some SMT solvers to reduce $\mathcal{T}_{\mathcal{LA}} \cup \mathcal{T}_{\mathcal{E}}$ to $\mathcal{T}_{\mathcal{LA}}$.
- Main problem: quadratic number of new clauses.
- It is also problematic to use this approach in the context of several theories and when combining SMT solvers with quantifier instantiation.

Reduction Functions: Ackermann's reduction

- Congruence closure based algorithms miss the following inference rule

$$f(\overline{n}) \neq f(\overline{m}) \implies \bigvee n_i \neq m_i$$

- Following simple formula takes $\mathcal{O}(2^N)$ time to be solved using SAT + Congruence closure.

$$\bigwedge_{i=1}^{N} (p_i \vee x_i = v_0),\ (\neg p_i \vee x_i = v_1),\ (p_i \vee y_i = v_0),\ (\neg p_i \vee y_i = v_1),$$
$$f(x_N, \ldots, f(x_2, x_1) \ldots) \neq f(y_N, \ldots, f(y_2, y_1) \ldots)$$

- It can be solved in polynomial time with Ackermann's reduction.
- A similar behavior is also observed in several pipeline verification problems.

Dynamic Ackermann's reduction

- This performance problem reflects a limitation in the current congruence closure algorithms used in SMT solvers.
- It is not related with the theory combination problem.
- Dynamic Ackermannization: clauses corresponding to Ackermann's reduction are added when a congruence rule participates in a conflict.

	CC		Ack		Dyn Ack	
	conflicts	time (s)	conflicts	time (s)	conflicts	time (s)
c10bi	217232	143.87	6880	6.09	5885	1.75
f10id	$>$ 8752181	$>$ 1800	22038	16.20	21220	7.20

Delaying inference rules

- A commonly used approach: delay the application of “expensive” inference rules.
- Examples:
  - Inference rules that produce new case-splits.
  - Non-linear arithmetic.
- Potential problem: solver may waste time searching an infeasible part of the search space.

Quantifiers

- Since first-order logic is undecidable, satisfiability is not solvable for arbitrary quantified formulas.
- Some theories, e.g., datatypes, linear arithmetic over integers, arithmetic over reals, support quantifier elimination.
- Existential quantifiers can be skolemized, but the problem of instantiating universal quantifiers for detecting unsatisfiability remains.

Heuristic Quantifier Instantiation

- Semantically, $\forall x_1, \ldots, x_n.F$ is equivalent to the infinite conjunction $\bigwedge_\beta \beta(F)$.
- Solvers use heuristics to select from this infinite conjunction those instances that are "relevant".
- The key idea is to treat an instance $\beta(F)$ as relevant whenever it contains enough terms that are represented in the solver state.
- Non ground terms p from F are selected as patterns.
- E-matching (matching modulo equalities) is used to find instances of the patterns.
- Example: $f(a, b)$ matches the pattern $f(g(x), x)$ if $a = g(b)$.

E-matching

- E-matching is NP-hard.
- The number of matches can be exponential.
- It is not refutationally complete.
- In practice:
  - Indexing techniques for fast retrieval.
  - Incremental E-matching.

E-matching: example

- $\forall x. f(g(x)) = x$
- Pattern: $f(g(x))$
- Atoms: $a = g(b), b = c, f(a) \neq c$
- $\longrightarrow$ *instantiate* $f(g(b)) = b$

Quantifiers in Z3

- Z3 uses a E-matching abstract machine.
  - Patterns $\rightsquigarrow$ code sequence.
  - Abstract machine executes the code.
- Z3 uses new algorithms that identify matches on E-graphs incrementally and efficiently.
  - E-matching code trees.
  - Inverted path index.
- Z3 garbage collects clauses, together with their atoms and terms, that were useless in closing branches.

E-matching code trees

- In practice, there are several similar patterns.
- Idea: combine several code sequences in a code tree.
- Factor out redundant work.
- Match several patterns simultaneously.
- Saturation based theorem provers use a different kind of code tree to implement:
  - Forward subsumption.
  - Forward demodulation.

Incremental E-matching

- Z3 uses a backtracking search.
- New terms are created during the search.
  - A code tree for each function symbol f.

    Patterns that start with a f-application.
  - Execute code-tree for each new term.
- New equalities are asserted during the search.
  - New equalities $\rightsquigarrow$ new E-matching instances.
  - Example:

    $f(a, b)$ matches $f(g(x), x)$ after

    $a = g(b)$ is asserted.

E-matching limitations

- E-matching needs ground (seed) terms.
  - It fails to prove simple properties when ground (seed) terms are not available.
  - Example:

$$(\forall x. f(x) \leq 0) \wedge (\forall x. f(x) > 0)$$

- Matching loops

$$(\forall x. f(x) = g(f(x))) \wedge (\forall x. g(x) = f(g(x)))$$

  - Inefficiency and/or non-termination.
  - Some solvers have support for detecting matching loops based on instantiation chain length.

Quantifiers: future work

- Model checking.
- Superposition calculus + SMT.
- Decidable fragments.

Conclusion

- SMT is the next generation of verification engines.
- More automation: it is push-button technology.
- SMT solvers are used in different applications.
- The breakthrough in SAT solving influenced the new generation of SMT solvers:
  - Precise lemmas.
  - Theory Propagation.
  - Incrementality.
  - Efficient Backtracking.
- Z3 website:

  `http://research.microsoft.com/projects/z3`