Large Cardinals and Strong Logics

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This is a seminar on large cardinals and strong logics in Summer Semester 2018. in University of Bonn. Seminar begins with introductory talks, given by instructor, Dr. Peter Holy, and continues with talks given by other people, i.e. participants of the seminar. Proofs given by professor contain several exercises, as they do not contain all the required technical details. We denote with V class of all sets, i.e. universe. We denote cardinals with lower Greek letters and we denote with $P(X)$ power set of set X. All new terms are denoted in italic (like *term*) and all proofs end with $\spadesuit$. We denote structures and languages with stylized uppercase letters, like $\mathfrak{A}$, and sets on which structures stand, i.e. their universes, with uppercase letters, like A.
References: Thomas Jech, Set theory, third millenium edition, Springer, 2002., and Magidor's slides (at logic group webpage of www.math.uni-bonn.de)

Content:

1. Few introductory words

We assume familiarity with first-order logic and notions of cardinals and ordinals, as well as V_κ and rank of a set.
Here we just give a glimpse at what the seminar is even about. So, strong logics means more expressive power than the first order logic. Example:Second order logics, where we quantify not just over elements, but also over subsets.
Definition 1.1: Let γ be a cardinal and X a set. A set $\{X_\alpha : \alpha < \gamma\}$ of subsets of X (if this exists) is called γ-*sequence* in X.
Definition 1.2: Let κ be a cardinal. If there is a cardinal $\lambda < \kappa$ such that there is a λ-sequence in κ, then κ is *singular*.
For a large cardinals there is no formal definition. Those are just cardinals which can not be reached from below. Ideas are:
1. if κ is not large , then 2^κ is not large. In fact, if 2^κ was large, then that's the number of elements in the power set of κ, hence can be reached by κ, which again can be reached, as it is not large, hence 2^κ can be reached, hence not large, contradiction!
2. if κ is singular, then it is not large. Intuitively, κ can be reached by a λ-sequence hence by λ itself, put a bijection between λ-sequence and $\lambda = \{\gamma \in Ord : \gamma < \lambda\}$, where, as usual, Ord is the class of all ordinals.

2. Measurable cardinals-first part

Remark 2.1: A *regular* cardinal is the one that is not singular
Definition 2.3: A cardinal κ is a *strong limit* cardinal if for each $\lambda < \kappa$, $2^\lambda < \kappa$.
Definition 2.4: A cardinal κ is *inaccessible* if it is a regular cardinal, and a strong limit.
Remark 2.2: If κ is inaccessible, V_κ is a model of ZFC. So, $ZFC{+}\exists\, inaccessible\ cardinal \models consistency\ of\ ZFC$ (consistency of theory T we will denote by Con(T)). So, by Gdel's incompleteness theorem, $ZFC \nvdash \exists\, inaccessible\ cardinal$ and else: $(ZFC + Con(ZFC)) \nvdash ZFC + Con(ZFC + \ \exists\, inaccessible\ cardinal)$
Definition 2.5: Let κ be a cardinal. $F \subseteq P(\kappa)$ is a $filter$ on κ or on $P(\kappa)$ if:
(1) $\emptyset \notin F$, $\kappa \in F$
(2) $A, B \subset \kappa, A \subset B, A \in F \Longrightarrow B \in F$
(3) $A, B \in F \Longrightarrow A \bigcap B \in F$
Remark 2.3: So, a filter tells you which sets are big and which aren't.
Definition 2.6: A *nonprincipal* filter is the filter which can not be generated by any $X \subset \kappa$, i.e. it is not of the form $\{Y \subset \kappa : X \subset Y\}$.
Remark 2.4: The nonprincipal filter is introduced to avoid trivialities. So, a principal filter is the trivial filter, we will see what t means in more concrete situation.
Definition 2.7: A filter F on $P(\kappa)$ is $ultrafilter$ if, for each $X \subset \kappa$, either $X \in F$ or $\kappa - X \in F$.
Remark 2.5: So, ultrafilter measures a lot of sets.
Definition 2.8: For an ordinal λ, a filter F is $< \lambda$-complete if, given any $\gamma < \lambda$, and any γ-sequence $\{X_\alpha : \alpha < \gamma\}$ with $X_\alpha \in F$ for each $\alpha < \lambda$, we have $\bigcap_{\alpha<\gamma} X_\alpha \in F$.
Remark 2.6: There is another name for ω_1-completeness, and that is σ-completeness
Definition 2.9: A cardinal κ is *measurable* if there is $< \kappa$-complete nonprincipal ultrafilter on κ.
Remark 2.7: On every cardinal κ there is $< \kappa$-complete principal ultrafilter, take for any $\alpha < \kappa$. $F_\alpha = \{Y \subseteq \kappa : \alpha \in Y\}$. So that's why principal means trivial.
Definition 2.10: A filter F on κ is $uniform$ if for each $X \in F$, $|X| = \kappa$.
Theorem 2.1: If κ is measurable then it is inaccessible.
Proof:
Let F witness the measurability of κ, so, it's a nonprincipal $< \kappa$-complete ultrafilter on κ. Observe that F is uniform.
Assume first that κ is singular, i.e. $\kappa = \bigcup_{i<\gamma} \kappa_i$ where κ_i denotes an i-sequence in κ, for some $\gamma < \kappa$. Then for each i, $\kappa_i \notin F$, and so, as F is ultrafilter, $\kappa - \kappa_i \in F$ for each $i < \gamma$, and by $< \kappa$-completeness, and as $\gamma < \kappa$, $\bigcap_{i<\gamma}(\kappa - \kappa_i) = \kappa - \bigcup_{i<\gamma} \kappa_i = \kappa - \kappa = \emptyset \in F$, which is a contradiction with F being a filter.
Now assume that κ is not a strong limit, and take the witness, i.e. such a cardinal λ that $\lambda < \kappa$, and $2^\lambda \geq \kappa$. This means that we have a function $f : \kappa \to 2^\lambda$ which is injective. For every $i < \lambda$, we pick $j_i \in \{0, 1\}$, such that $X_i = \{\xi < \kappa : f(\xi)(i) = j_i\} \in F$. By the $< \kappa$-completeness of F, $\bigcap_{i<\lambda} X_i \in F$,

as $\lambda < \kappa$, in particular $|X| = \kappa$. For $\xi \in X$ then $f(\xi)(i) = j_i$, for every $i < \lambda$. But, this determines $f(\xi)$ completely, contradicting injectivity of f.♠
Definition 2.11: Given structures $\mathfrak{A}$, $\mathfrak{B}$ for the same language $\mathfrak{L}$, a map $j : A \to B$ is *elementary embedding* $j : \mathfrak{A} \to \mathfrak{B}$, if for any $\mathfrak{L}$-formula φ, and any $x_0, ..., x_n \in A$:
$\mathfrak{A} \models \varphi(x_0, ..., x_n) \iff \mathfrak{B} \models \varphi(j(x_0), ..., j(x_n))$.
In the case $j = id$, we say that $\mathfrak{A}$ is an *elementary substructure* of $\mathfrak{B}$ and denote that with $\mathfrak{A} \prec \mathfrak{B}$.
Usually, $\mathfrak{L}$ extends $\mathfrak{L}_{\in}$, the language of set theory. That language consists just of one binary relation symbol, which we call membership relation, do not define and denote with $\in$.
Definition 2.12: A structure $\mathfrak{B}$ is *transitive*, if for each $x \in y \wedge y \in B \implies x \in B$. Usually, structures are transitive, and $\mathfrak{A}, \mathfrak{B} \models ZFC$, or at least, a large part of ZFC.
Definition 2.13: A mapping $j : X \to A$ *moves* set X if $j(X) = \{j(x) : x \in X\} \neq X$.
Lemma 2.1: Assume $j : \mathfrak{A} \to \mathfrak{B}$ is elementary embedding with $\mathfrak{A}, \mathfrak{B}$ transitive models of ZFC. Then:
(1) for any ordinal α of $\mathfrak{A}$, $j(\alpha)$ is an ordinal of $\mathfrak{B}$, and $j(\alpha) \geq \alpha$
(2) Suppose $j \neq id$. Then there is an ordinal α such that $j(\alpha) > \alpha$.
Proof: (1) As elementary embedding, j preserves properties of first order. As being an ordinal is a first-order property, if α is ordinal, $j(\alpha)$ will be an ordinal. Use induction to get the second part.
(2) Exercise: If j moves no set of ordinals, then j is identity.
Let X be a set of ordinals of minimal rank such that $j(X) \neq X$. This set exists, by the exercise. Let $\alpha = rank(X)$. If $y \in x$, then $rank(y) < \alpha$, so $j(y) = y$. Therefore, $y \in x \implies j(y) = y \in j(x)$, i.e. $x \subseteq j(x)$. Pick some $z \in j(x) - x$. Assume first that $rank(j(x)) = \alpha$, as it can't be less than α because $x \subset j(x)$ and $rank(x) = \alpha$. Then $z < j(\alpha)$. So, $j(z) = z \in j(x)$. But then, by elementarity, $z \in x$. That's a contradiction. Therefore, $rank(j(x)) > \alpha$. By elementarity, $rank(j(x)) = j(rank(x))$, as rank can be expressed as first order property and apply elementarity to rank. $\implies j(\alpha) > \alpha$.♠.
Assume U is ultrafilter on κ. For any $f, g : \kappa \to V$, let $f =_U g$ iff $\{\xi < \kappa : f(\xi) = g(\xi)\} \in U$ and let $f \in_U g$ iff $\{\xi < \kappa : f(\xi) \in g(\xi)\} \in U$. $f =_U g$ is equivalence relation. Equivalence class of this relation is denoted with $[f]_U$. We define *ultrapower* by ultrafilter U by ${}^\kappa V/U = \{[f]_U : f : \kappa \to V\}$. Let $Ult(V, U) =<^\kappa V/U, \in_U>$ ultrapower of U in V. $[f]_U$ is a proper class.
Remark 2.8: So, ultrapower by ultrafilter U is a class of a classes. This has some problems, but here we are OK with that.
Remark 2.9: Instead of V we can define ultrapowers by some other class A, there we should just observe functions that are in A, everything else is the same(applied to A).
Theorem 2.2[Los's theorem] $Ult(V, U) \models \varphi([f_0]_U, ..., [f_{n-1}]_U)$ iff $\{\gamma < \kappa : \varphi(f_0(\gamma), ..., f_{n-1}(\gamma))\} \in U$
Proof: For atomic $(\in, =)$ formulas, this is by definition. Then, by induction on formula complexity.[Exercise]♠

Lemma 2.2: If U is σ-complete, then relation $\in_U$ is well-founded.
Proof: Assume that $[f_0]_U \ni_U [f_1]_U \ni_U [f_2]_U \ni_U$ ... (this is a $\ni_U$ decreasing sequence). This means that for $i < \omega$, $\{\xi < \kappa : f_{i+1}(\xi) \in f(\xi)\} \in U$. By σ-completeness, their intersection is in U, i.e. $\{\xi < \kappa : \forall i < \omega, f_{i+1}(\xi) \in f(\xi)\} \in U$. Pick any ξ in this set. Then $\in$ is ill-founded. This is a contradiction with axiom of ZFC(there is an $\in$-minimal element in any nonempty set)♠
If $< Ult(V, U), \in_U >$ is well-founded, we can identify it with its transitive collapse.
Definition 2.14: A relation E on a structure $\mathfrak{N}$ is *extensional* if for every $A, B \in N$, $A = B \iff (xEA \iff xEB)$.
Definition 2.15: If $\mathfrak{N}$ is a structure, and E is a relation on it, then a structure is $set - like$, if for each $x \in N$, $\{y \in N : yEx\}$ is a set, not a proper class.
Lemma 2.3[Mostowsky's collapsing lemma] If $< N, E >$ is such that E is well-founded, extensional and set-like, then $< N, E >$ is isomorphic to a structure $< M, \in >$ where $\in$ is here a membership relation restricted (from V) to M.
Proof: Any standard textbook, first couple pages.♠
Definition 2.16: A cardinal κ is *critical* for nontrivial elementary embedding $j : V \to M$ where M is some proper class if $j(\kappa) > \kappa$, and for each $\alpha < \kappa$, $j(\alpha) = \alpha$, i.e. κ is the least cardinal moved by j.
Theorem 2.3[Main theorem of this section] If κ is measurable, then there is an elementary embedding $j : V\ toM$, where M is a proper class in V, with κ critical point.
Proof: Assume U is a $< \kappa$-complete nonprincipal ultrafilter on $P(\kappa)$. Let $M = Ult(V, U)$. Define $j : V \to M$ by setting $j(x) = [c_x]_U$ (so class of equivalence of constant mapping that maps everything to given set x) for $x \in V$. We show that j is an elementary embedding with κ critical point. By Los's theorem, $M \models \varphi(j(x)) \iff M \models \varphi([c_x]_U) \iff \{\xi < \kappa : \varphi(x)\} \in U$. If $\alpha < \kappa$, and $[f]_U \in j(\alpha)$ then $\{\xi < \kappa : f(\xi) < \alpha\} \in U$ and by $< \kappa$-completeness, for some $\gamma < \alpha$, $\{\xi < \kappa : f(\xi) = \gamma\} \in U$. But then, $[f]_U = \gamma \iff j(\alpha) = \alpha$
It remains to show that $j(\kappa) > \kappa$. We claim that $[id_\kappa]_U \in j(\kappa)$ and $\forall \alpha < \kappa, \alpha \in [id_\kappa]_U$. By definitions:
$\alpha \in [id_\kappa]_U \iff \{\xi < \kappa : \alpha \in \xi\} \in U \iff (\alpha, \kappa) \in U$, which is the case, where (α, κ) is the interval in $\mathbb{R}$.
$[id_\kappa]_U \in j(\kappa) \iff \{\xi < \kappa : \xi \in \kappa\} \in U \iff \kappa \in U$, which is the case, by the very definitions of filters.♠
Are the measurable cardinals the only inaccessible cardinals?? A negative answer to this question gives:
Lemma 2.4: If κ is measurable, then there are κ many inaccessible cardinals below.
6. Measurable cardinals and elementary extensions [Nikola Kovačević]
There is a very difficult task what to write or speak about measurable cardinals. You could actually write a lot of pages about measurable cardinals, even arise to a full book. As I have no time, I will just refer reader to the measurable cardinals history, whose beginnings is written in notes on the end of a page in a book K.Kuratowski, A. Mostowski, Set theory with an introduction to descriptive set theory, Studies of Logics and foundations of mathematics, volume 86,

second edition, North-Holland Publishing Company and Polish Scientific publishers, 1976., which has a full chapter on measurable cardinals, which is very good to read as it is explained, among other things, why are they called "measurable", and the rest of the history of measurable cardinals is in introduction by Akihiro Kanamori in Matthew Foreman, Akihiro Kanamori, Handbook of Set Theory, Volume 1, Springer, 2010. Also, what was introduced in X.1. in Kuratowski book is here written in pretty advanced fashion, by Thomas Jech himself. Also, for a more advanced look at the measure cardinals you might have a look at Drake "Set theory", as he also devotes the full chapter to measure cardinals. Full research-level strength is shown in one of the last chapters of Akihiro Kanamori, Matthew Foreman, third volume. All of this is advanced in a way that you need firm knowledge of forcing to even read. Our concern here is not about any forcing, but about measurability and elementary extensions. A lot of definitions were introduced in 2.
So much for trivia about measurable cardinals. We will here concentrate on three theorems. First theorem is explaining why are they well-defined and will be important argument towards the end of the proof of main theorem. In theorem 2.3 there is one thing proved, but that's an equivalence, out of which I will here prove other part. Third, and most important theorem, gives out a relationship between measure cardinals and elementary extensions, which is why the title of the section is like that.
To begin, recall Lemma 2.2 which says that for a σ-complete ultrafilter, the relation $\in_U$ is well-defined. Also, we will need all the range of ultrapowers given in 2. Let U be an ultrafilter on S and let $a \in S$ be arbitrary element. If $a \in U$ then we could define a constant mapping $c_a(x) = a$ for each $x \in S$.
Lemma 6.1 The canonical embedding $j : U \to Ult(U, V)$ is an elementary embedding.
Proof: Choose a formula φ, and then by Los's theorem, $U \models \varphi(j(a))$ iff $U \models \varphi([c_a])$ iff $U \models \varphi([a])$ for almost all x iff $U \models \varphi([a])$. ♠
We also need to clarify that the class model of ZFC is just a class of all sets which satisfy some given formula in a first-order language. Also, we need the following:
Definition 6.2 Let U be a model and take a natural number n and denote the presentation of n in U as $[n]$. So, U is *correct about* ω, or U is an ω-*model* if there are not any more natural numbers in U but the interpretation of natural numbers. Otherwise told: $U \models$ "$a \in \omega$" $\implies (\exists n \in \mathbb{N} a = [n]$.
Let's start proving theorems:
Theorem 1: Let κ be the least cardinal such that on κ there is a σ-complete nonprincipal ultrafilter U. Then, U is κ-complete.
Proof: Suppose U is not κ-complete. Then, there is a $\gamma < \kappa$ with $\{X_\alpha : \alpha < \gamma\}$ not in U for all $\alpha < \gamma$. So, now form a mapping $f : \kappa \to \gamma$ via: $f(x) = \alpha$ iff $x \in X_\alpha$. This is easily seen to be a well-defined mapping. Also, define $D \subset P(\gamma)$ as $Z \in D \iff f_{-1}(Z) \in U$.

$\emptyset \in D$ means $f_{-1}(\emptyset) \in U$. As f is a well-defined map, $f_{-1}(\emptyset) = \emptyset$.