Variable-Length Pendulum

Eric Wang, Nikolas Hall, Kevin Kim

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1 Introduction

The system consists of a single pendulum with a point mass m and massless string with varying length $r = \ell_0 + \alpha \sin \omega t$, where the length oscillates between $\ell_0 - \alpha$ and $\ell_0 + \alpha$ with drive frequency ω, as illustrated in Figure 1. We will use θ as the generalized coordinate. First, we consider velocity

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$
$$\vec{v}^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

where

$$r = \ell_0 + \alpha \sin \omega t \tag{1}$$
$$\dot{r} = \alpha\omega \cos \omega t. \tag{2}$$

From this we may compute the potential and kinetic energies,

$$U = mgy = -mg\left(\ell_0 + \alpha \sin(\omega t)\right)\cos\theta \tag{3}$$
$$T = \frac{m}{2}\vec{v}^2 = \frac{m}{2}\left[\alpha^2\omega^2\cos^2\omega t + \left(\ell_0^2 + 2\ell_0\alpha\sin\omega t + \alpha^2\sin^2\omega t\right)\dot{\theta}^2\right] \tag{4}$$
$$\tag{5}$$

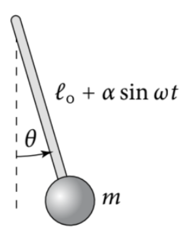

Figure 1: Diagram of the 2-D system. The pendulum's motion can be described by the length of the string and angle displaced from the vertical dashed line.

The Lagrangian is $L = T - U$, such that

$$L = \frac{m}{2}\left[\alpha^2\omega^2\cos^2\omega t + \left(\ell_0^2 + 2\ell_0\alpha\sin\omega t + \alpha^2\sin^2\omega t\right)\dot{\theta}^2\right]$$
$$+ mg\left(\ell_0 + \alpha\sin\omega t\right)\cos\theta$$

It has explicit time dependence on the varying pendulum's length, so the Hamiltonian is not conserved. Furthermore, the equations of transformation, Eq. (2), also have explicit time dependence, so the Hamiltonian is not equal to the energy of the system.

Lagrange's equation for this system is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = \frac{d}{dt}\left[m\left(\ell_0^2 + 2\ell_0\alpha\sin\omega t + \alpha^2\sin^2\omega t\right)\dot{\theta}\right] = \frac{d}{dt}\left[mr^2\dot{\theta}\right]$$
$$= mr^2\ddot{\theta} + m\dot{\theta}\left(2\ell_0\alpha\omega\cos\omega t + 2\alpha^2\omega\sin\omega t\cos\omega t\right)$$
$$\frac{\partial L}{\partial\theta} = -mg\ell_0\sin\theta - mg\alpha\sin\omega t\cdot\sin\theta.$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0 = mr^2\ddot{\theta} + m\dot{\theta}\left(2\ell_0\alpha\omega\cos\omega t + 2\alpha^2\omega\sin\omega t\cos\omega t\right)$$
$$+ mg\ell_0\sin\theta + mg\alpha\sin\omega t\cdot\sin\theta.$$

Dividing both sides by the varying length mr gives us

$$(\ell_0 + \alpha\sin\omega t)\ddot{\theta} + 2\omega\alpha\cos(\omega t)\dot{\theta} = -g\sin\theta$$
$$\left(1 + \frac{\alpha}{\ell_0}\sin(\omega t)\right)\ddot{\theta} + 2\omega\frac{\alpha}{\ell_0}\cos(\omega t)\dot{\theta} = -\frac{g}{\ell_0}\sin\theta$$

θ is time-dependent, so we want to replace time with a dimensionless variable $\tau = \omega_0 t$, where $\omega_0 = \sqrt{g/\ell_0}$ is the pendulum's natural frequency (where the pendulum length isn't varying). This can provide us a natural time scale for the motion. Let $f = \omega/\omega_0$ be the unitless ratio of the drive frequency to the pendulum frequency. Then the equation of motion in dimensionless variables becomes

$$\boxed{\frac{d^2\theta}{d\tau^2} = \frac{1}{1 + \frac{\alpha}{\ell_0}\sin(f\tau)}\left[-2\frac{\alpha}{\ell_0}f\cos(f\tau)\frac{d\theta}{d\tau} - \sin\theta\right]} \tag{6}$$

We now vary the parameters in this equation and observe the effects on the pendulum's motion.

2 Experiments With System Parameters

To explore the physics of our system, we vary the following parameters: g (acceleration due to gravity), l_0 (the base length of the pendulum), ω (the frequency of oscillations in pendulum length), α (the amplitude of those oscillations), θ_0 (the pendulum's initial angle from the vertical), and $\frac{d\theta}{d\tau}_0$ (its the initial angular speed). Unless otherwise noted, in each set of experiments, we try to vary the parameter in question while holding all others constant. The results of our experiments are presented below.

2.1 First Exploration: A Simple Pendulum

As a first test of our code, we set $\omega = 0$. Doing so should turn our variable-length pendulum into a simple pendulum. As a result, we expect to observe simple harmonic oscillations at small angles. Plotting $\theta(\tau)$ in Figure 2, we see that this is indeed the case.

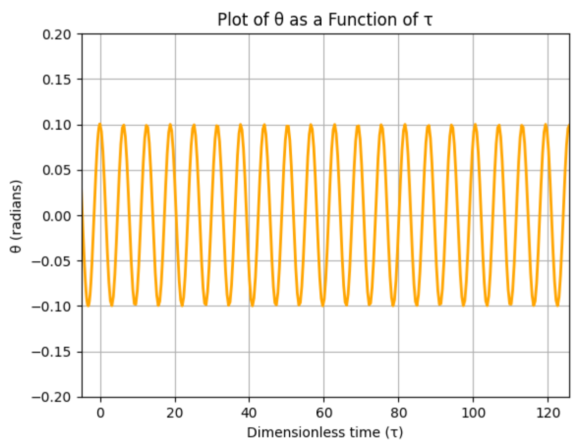

Figure 2: When we set $\omega = 0$ with $\theta_0 = 0.1$ rad, we observe sinusoidal oscillations consistent with simple harmonic motion.

Our state space plot of $\frac{d\theta}{d\tau}$ against θ resembles an ellipse (see Figure 3). This too is expected for a simple pendulum: $\frac{d\theta}{d\tau}$ is maximized where θ is minimized (and vice versa) as energy changes from kinetic to potential form. Because the pendulum oscillates symmetrically about the vertical, our plot is symmetric about $\theta = 0$ in state space.

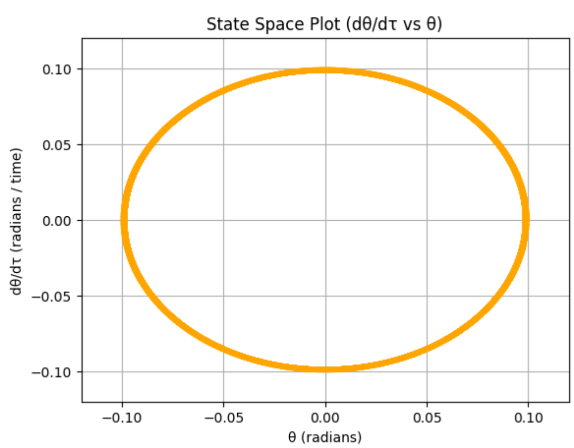

Figure 3: Our state space plot for $\omega = 0$ with small θ_0 resembles that of a simple harmonic oscillator.

2.2 Varying g

Simulating the system for relatively large g ($g = 2000$) without altering the initial conditions produces these plots:

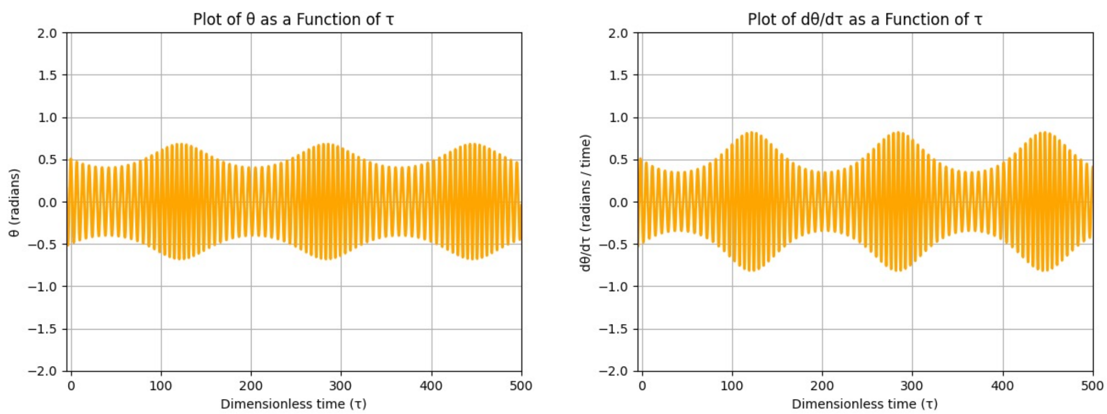

Figure 4: Graphs of θ (left) and $\frac{d\theta}{d\tau}$ (right) against τ for $g = 2000$.

For small values of g, we see that:

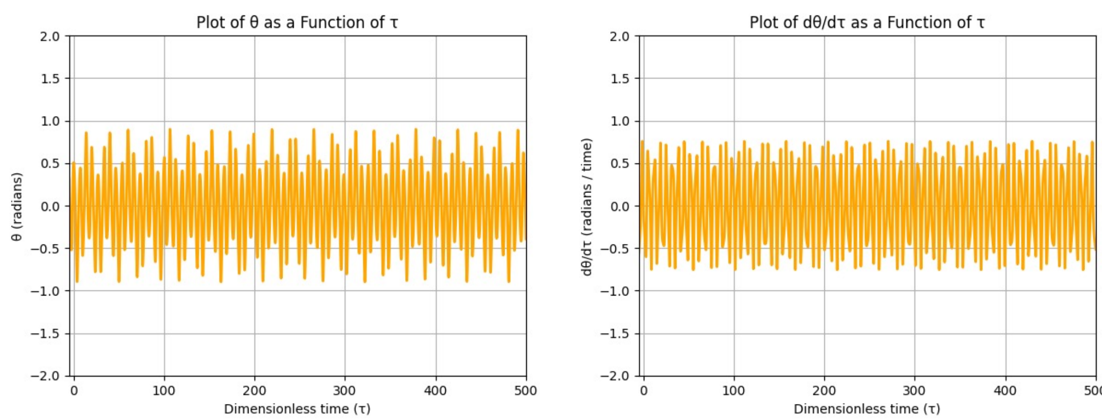

Figure 5: Graphs of θ (left) and $\frac{d\theta}{d\tau}$ (right) against τ for $g = 0.002$.

The key takeaway is that for large values of g we observe a "beat" effect in which the wavepacket seems to be amplitude-modulated by a second sinusoid. Changing g to large or small values does not seem to cause chaotic behavior.

2.3 Varying l_0

Simulating the system for relatively large and small values of l_0 also yields different end behaviors. For small values of l_0, we see that:

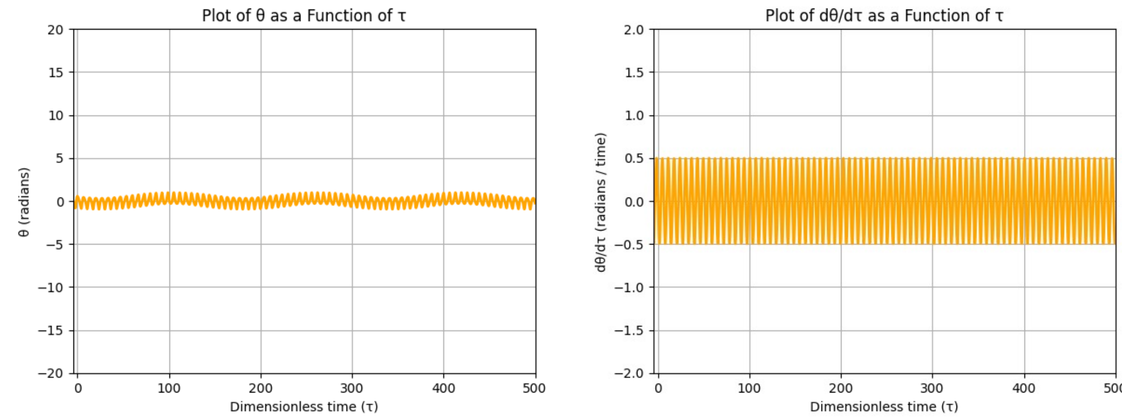

Figure 6: Graphs of θ (left) and $\frac{d\theta}{d\tau}$ (right) against τ for $l_0 = 2$. Note that the scale of the left-hand graph is larger than that of the others by a factor of 10.

Note that for smaller values of $l_0 \leq \alpha$ the result is not physical (the pendulum's length must always be positive). For large values of l_0,

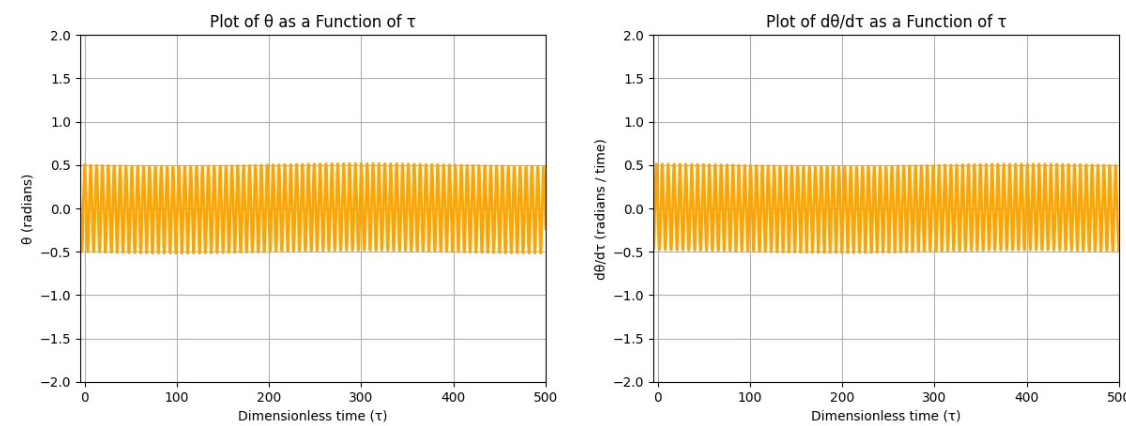

Figure 7: Graphs of θ (left) and $\frac{d\theta}{d\tau}$ (right) against τ for $l_0 = 30$.

The key idea is that while increasing l_0 does decrease ω_0, it does not seem to have a strong effect on the frequency or amplitude of the pendulum's oscillation. It also does not seem to be linked to chaotic behavior.

2.4 Varying ω

We study the effect of changing ω, the frequency with which the pendulum length oscillates. Plots of $\theta(\tau)$ and state space plots for three values of ω are shown in Figures 8 and 9.

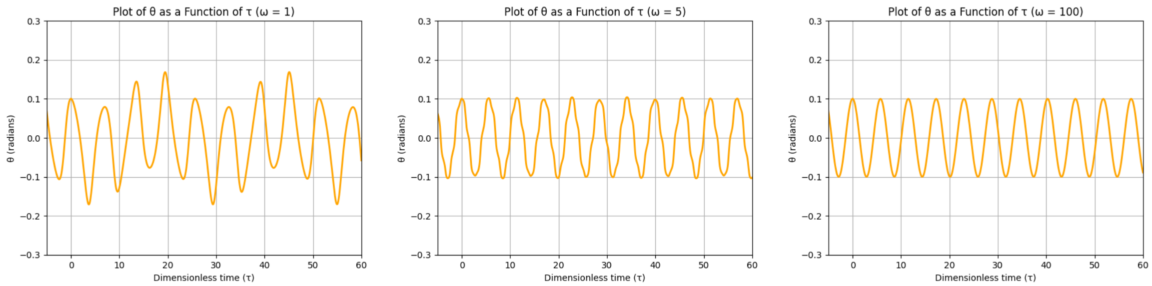

Figure 8: Plots of $\theta(\tau)$ for $\omega = 1, 5, 100$. In all three experiments: $g = 2$, $l_0 = 3$, $\alpha = 1$, $\theta_0 = 0.1$, and $\frac{d\theta}{d\tau}_0 = 0$.

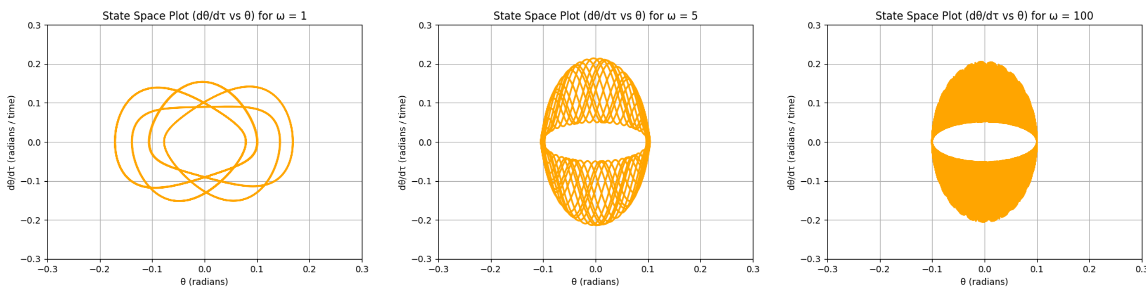

Figure 9: State space plots for $\omega = 1, 5, 100$.

From Figure 8, we see that in all three cases, θ oscillates with a period of about 2π, suggesting that the pendulum's frequency of oscillation is independent of ω. We also see that when $\omega = 1$, some cycles have a larger amplitude than others. However, these variations in amplitude seem to follow a repeating pattern with a period of about 25. As ω increases, the variations in amplitude become less and less pronounced, and when $\omega = 100$, $\theta(\tau)$ resembles a sinusoid. A possible explanation is as follows: with length changing on time scales much shorter than the back-and-forth swing of the pendulum, the pendulum can be treated as having a constant, average length (and, therefore, an average energy). This causes both the amplitude and the frequency of oscillation to resemble those of a simple pendulum.

From Figure 9, we can also see that as ω increases, the system explores more points in state space. This suggests that, although the pendulum's overall oscillations become more regular as ω increases, its instantaneous angular speed varies widely. This too may be explained by rapid changes in pendulum length: to conserve angular momentum, $\frac{d\theta}{d\tau}$ must increase and decrease as the pendulum shrinks and grows. This happens over small ranges of θ values, so we see a greater per-θ density of $\frac{d\theta}{d\tau}$ points in our state space plot.

As a final note, we notice large, chaotic oscillations in theta when ω lies in the approximate range of 1.3 to 2.1 (see Figure 10).

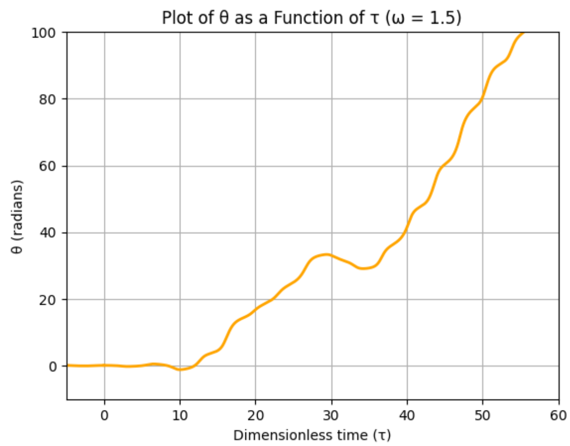

Figure 10: When $\omega = 1.5$, we see large spikes in θ beginning at about $\tau = 12$. θ's rise beyond 2π indicates that the pendulum undergoes a complete, 360-degree loop.

The spike in θ over a relatively narrow range of ω-values brings to mind our class's discussion of resonance. We continue to observe large spikes in θ when we change θ_0, suggesting that when our system is arranged with the parameters listed under Figure 8, there is something fundamental about the 1.3–2.1 frequency range. If we were "driving" the oscillations in pendulum length with ω near a resonant frequency, it might explain the spike in θ.

2.5 Varying α

Here, we experiment with α, the amplitude of oscillations in pendulum length. When $\alpha = 0$, we again have a simple pendulum, yielding our familiar elliptical plot in phase space.

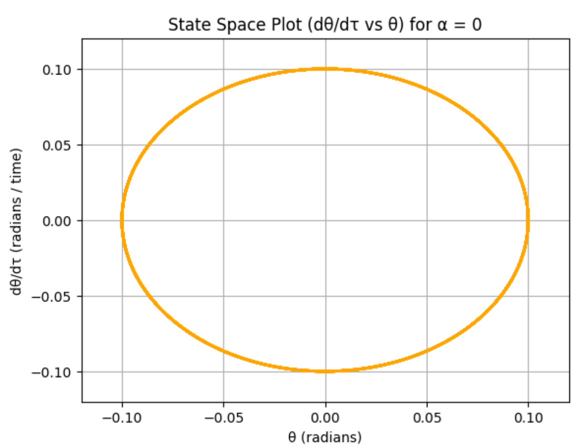

Figure 11: When $\alpha = 0$ and θ is small, our pendulum resembles a simple harmonic oscillator.

As we increase alpha from 0, we still see oscillations, but they are more erratic, with the amplitude of θ varying by more. We also see a less regular, discernible period (see Figure 12).

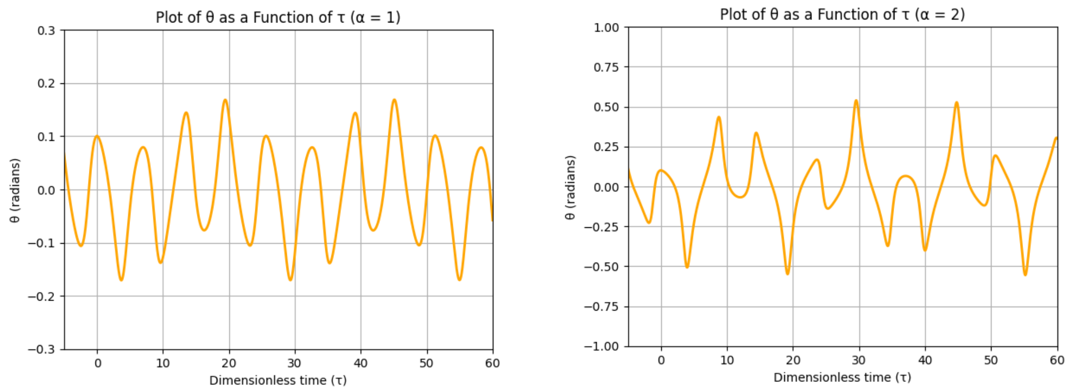

Figure 12: Plots of $\theta(\tau)$ for $\alpha = 1, 2$. In all experiments in this section: $\omega = 1$, $g = 2$, $l_0 = 3$, $\theta_0 = 0.1$, and $\frac{d\theta}{d\tau}_0 = 0$.

As α approaches the rest length of the pendulum (in this case, $l_0 = 3$), the oscillations in θ become very large (see Figure 13).

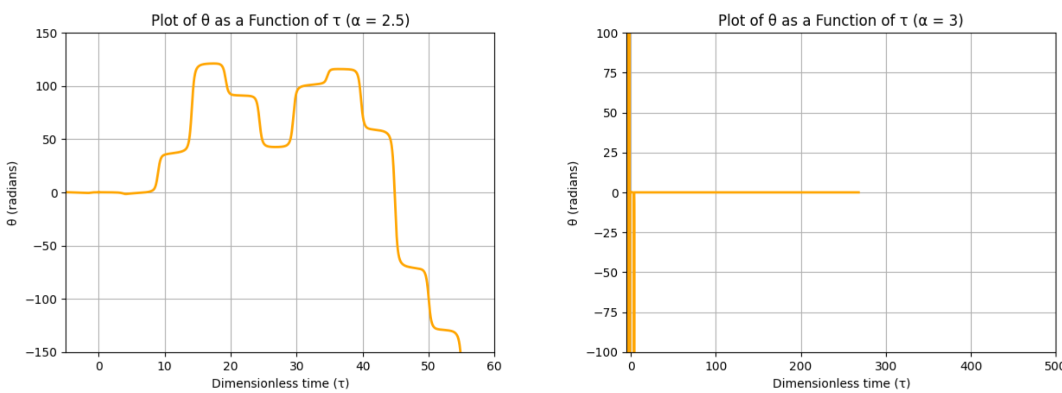

Figure 13: Plots of $\theta(\tau)$ for $\alpha = 2.5, 3$.

This result might be explained by conservation of angular momentum, L. If we choose the pendulum's pivot point as our origin, then the external agent changing the pendulum's length (r) can only exert a force in the radial direction. As a result, the pendulum mass experiences no torque, and its angular momentum must be conserved. It follows that, when the pendulum becomes shorter, $\frac{d\theta}{d\tau}$ must increase. The increase in $\frac{d\theta}{d\tau}$ translates to an increase in energy. Another way of thinking about this is to note that kinetic energy is $T = L^2/2mr^2$ — since L is constant, decreasing r increases T. Furthermore, as the pendulum becomes shorter, its potential energy increases as well (see Equation 3 on Page 1). The increased energy allows the pendulum to quickly whip around the pivot many times, leading to large spikes in θ well above 2π.

As shown in Figure 13, at $\alpha = 3$ (the pendulum's rest length), our simulation breaks down. Physically, this is because r going to zero implies infinite energy, which is not possible.

2.6 Varying Initial Conditions

To consider the effects of our initial conditions, we run several simulations with $\omega = 1$, $g = 2$, $l_0 = 3$, and $\alpha = 1$.

We begin by varying $\theta(0) = \theta_0$. For these tests, we set $\frac{d\theta}{d\tau}_0 = 0$. Up to about $\theta_0 = 0.8$ rad, θ experiences varying-amplitude oscillations that repeat on long time scales. However, as we push θ_0 beyond 0.8, the pendulum's behavior becomes chaotic, and oscillations follow no regular pattern (see Figure 14). While we are not sure why this happens, we notice that the θ_0 above which we observe chaotic behavior grows as g increases and as l_0 decreases. This threshold θ_0 also grows as α decreases, suggesting that changes in g, l, and α can all make the system more sensitive to a wider range of θ_0's.

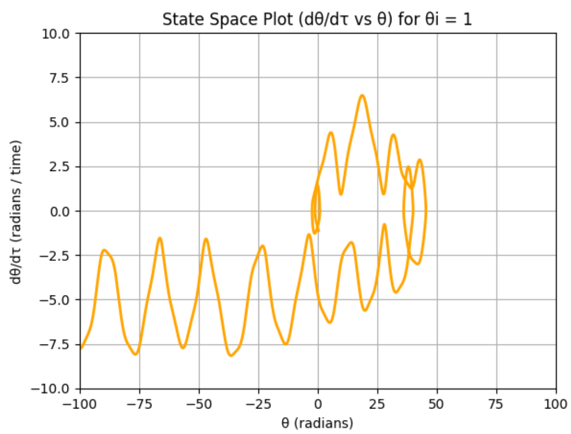

Figure 14: State space plot for $\theta_0 = 1$, evaluated from $\tau = -5$ to $\tau = 500$. For $\theta_0 > 0.8$ rad, θ undergoes large, irregular oscillations, and we discern no pattern in our state space plots.

If we instead increase $\frac{d\theta}{d\tau}_0$ from 0, we at first see θ and $\frac{d\theta}{d\tau}$ oscillate with a pattern of amplitudes that repeat over several periods, as before. The only meaningful change that we see is an increase in the amplitudes of both functions. This makes sense: starting the pendulum with nonzero speed should give it more energy at the start of its motion, which intuitively should result in a larger maximum θ.

With $\theta_0 = 0.1$ rad, when we set $\frac{d\theta}{d\tau}_0 > 0.8$ rad / time, we again observe chaotic behavior (see Figure 15). Once again, increasing g, decreasing l_0, and decreasing α all raise the $\frac{d\theta}{d\tau}_0$ value that serves as the threshold for chaos. Our θ_0 and $\frac{d\theta}{d\tau}_0$ threshold values are both very close to $\omega_0 = \sqrt{g/l_0} = 0.816$, hinting at a possible connection to ω_0. However, the similarity starts to break down as we increase g to 3 (which yields $\omega_0 = 1$ and rough threshold values of $\theta_0 = 1.4$ and $\frac{d\theta}{d\tau}_0 = 1.2$). This suggests that the threshold values are separately influenced by g and l_0, not by their ratio.

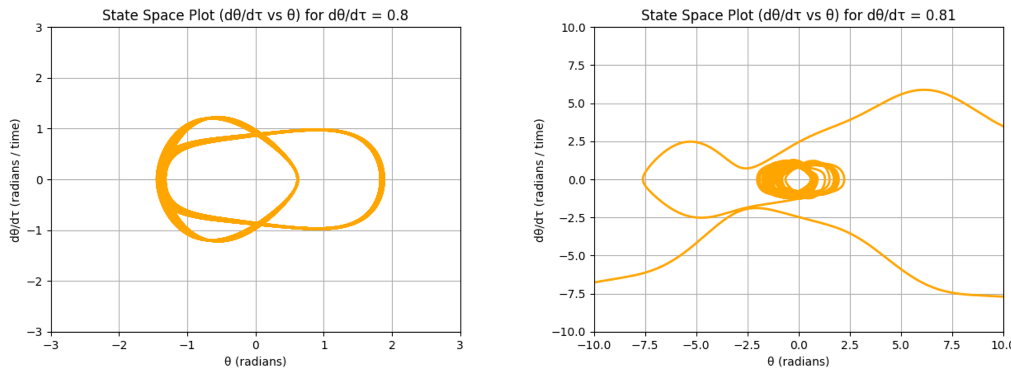

Figure 15: State space plots for $\frac{d\theta}{d\tau}_0 = 0.80, 0.81$. Notice how, when $\frac{d\theta}{d\tau}_0 = 0.81$, the system begins with a bounded oscillation. A spike in $\frac{d\theta}{d\tau}$ eventually leads to unpredictable motion.

3 Conclusion

We have studied the motion of a pendulum whose length changes sinusoidally. To do so, we varied four parameters — the strength of gravity (g), the pendulum's base length (l_0), the frequency of oscillations in length (ω), and the amplitude of those oscillations (α) — as well as initial conditions, and numerically solved the pendulum's equation of motion. We verified that, when the pendulum's length is fixed, it behaves as a simple harmonic oscillator, a reassuring check of our code's functionality. We found that, with the exception of a relatively narrow range from 1.3 to 2.1, changing ω does not seem to induce chaotic behavior. Increasing α from 0 likewise does not — to a point. As α approaches l_0, however, the system undergoes large, rapid swings in θ (the pendulum angle), an effect that may be explained by conservation of angular momentum. Increasing l_0 seems to have a relatively small effect on the pendulum's motion, whereas increasing g overlays a beat-like envelope on its oscillations. Beyond threshold values in both θ_0 and $\frac{d\theta}{d\tau}_0$, the system also becomes highly sensitive to initial conditions, exhibiting irregular, chaotic behavior. Future work might investigate the physical origins of these threshold values and their dependence on g, l_0, α, and ω. The resonance-like amplification of θ over a narrow range of ω-values could also be a topic for further study.