On the Brownian motion on Lie groups*

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Abstract

Brownian motion relies on the metric structure of the smooth manifold. The extra symmetry provided by Lie groups facilitates implementation via the associated Lie algebra.

1 Preliminaries

A smooth manifold M of dimension d is a Hausdorff space locally isomorphic to $\mathbb{R}^d$, along with a smooth structure. A Riemannian manifold is a smooth manifold together with a Riemannian metric. Given an orientation, Riemannian metrics induce the linear measurement of volumes. A Lie group is a smooth manifold which is also a group. The group operations are smooth and provide extra isomorphisms with the Lie algebra$-$the tangent space at the identity.

1.1 The gradient

On a smooth manifold M, $\dim(M) = n$, consider a smooth, symmetric, positive definite covariant 2-tensor field $g : C^\infty(TM) \times C^\infty(TM) \to C^\infty(M)$, with value at a point $p \in U$

$$g(X, Y)(p) = \sum_{i,j} g_{i,j}(p)\ e^i_p(X_p) \otimes e^j_p(Y_p), \tag{1}$$

where $\{e^i\}_{i=1}^n$ is the dual to the frame $\{\mathrm{E}_i\}_{i=1}^n$; the latter naturally defined by the vector space isomorphism $\varphi_*^{-1} : T_{\phi(p)}(M) \to T_p(M)$, $\varphi_*^{-1}(\frac{\partial}{\partial x_i} \mid_{x=\varphi(p)}) = \mathrm{E}_{i,p}$. Here (U, φ) denotes a smooth chart of the (smooth) covering of M.

Provided smooth section $X : M \to TM$, with $X_p = \sum_{j=1} X_{i,p}\mathrm{E}_i(p)$ in $U \subseteq M$, let $X^* : M \to T^*M$ be the smooth co-vector field determined by X and the Riemannian metric g:

$$X^*(p) := g(X)(Y)(p) = \sum_j X^{*j}_p e^j_p(Y), \tag{2}$$

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where

$$X_p^{*,j} = \sum_i g_{i,j}(p) X_{i,p}, \tag{3}$$

are the components of X^* w.r.t. the co-frame[1] $\{e^i(\cdot)\}_i$. Conversely, given a smooth co-vector field $X^* : M \to T^*M$ with value at $p \in U$ as in (2), (4) along with the fact that g^{-1} exists, determine a smooth vector field $X : M \to TM$ that in U can be expressed w.r.t. the frame $\{E_i\}$ as above. The components of this vector field in U are

$$X_{i,p} = \sum_j g^{i,j}(p) X_p^{*,j}, \tag{4}$$

where $g^{i,j}$ is the representation of g^{-1} in U. That being said, we isomorphically define the gradient $\mathrm{grad}f : M \to TM$ of a $0-$differential form $f \in C^\infty(M)$ to be the vector field such that

$$g(\mathrm{grad}f, Y)(p) = \mathrm{d}f(Y)(p), \ \forall p \in M, \tag{5}$$

where $\mathrm{d}f : \Omega^0 \to \Omega^1$ is the exterior derivative of the $0-$form f

$$\mathrm{d}f(X) := Xf, \tag{6}$$

where of course $Xf(p) := X_p f$. Inside (U, φ),

$$\mathrm{d}f(Y_p)(p) = \sum_i c^i e^i(Y_p),$$

which for $Y_p = E_{k,p}$ gives $c^k = E_{k,p} f$. This implies

$$\mathrm{grad}f(p)_i = \sum_j g^{i,j}(p) \mathrm{E}_{i,p} f,$$

or

$$\mathrm{grad}(f)(p) = \sum_i \Big(\sum_j g^{i,j}(p)(E_{j,p} f) \Big) \mathrm{E}_{i,p}, \ p \in U.$$

Take $h \in C^\infty(M)$. Then

$$\begin{aligned}
\mathrm{grad}f_p(h) &= \sum_i \Big(\sum_j g^{i,j}(p) \mathrm{E}_{j,p} f \Big) \mathrm{E}_{i,p} h \\
&= \sum_i \Big(\sum_j g^{i,j} \circ \varphi^{-1}(x) \frac{\partial}{\partial x_j}\Big|_{x=\varphi(p)} \varphi^{*-1} f \Big) \frac{\partial}{\partial x_j}\Big|_{x=\varphi(p)} \varphi^{*-1} h
\end{aligned}$$

[1] The topology of the tangent bundle is given by the pre-image of the projection map. Thus, open sets of the tangent bundle are of the form $W = \pi^{-1}(U) := U \times E^n$. This implies that frames and co-frames are considered always w.r.t. some open chart. Besides, they are both constant assignments only in some open $U \subseteq M$.

$$= \sum_i \Big(\sum_j g^{i,j} \circ \varphi^{-1}(x) \frac{\partial}{\partial x_j}\Big|_{x=\varphi(p)} \widehat{f} \Big) \frac{\partial}{\partial x_j}\Big|_{x=\varphi(p)} \widehat{h}.$$

which holds for all $\widehat{h} \in C^\infty(\mathbb{R}^{\dim(M)=n})$, and thus for $\widehat{h} = x_l$ too. As a result, the representation (ith component) of the gradient in local coordinates is

$$\mathrm{grad} f_x = \sum_j g^{i,j} \circ \varphi^{-1}(x) \frac{\partial \widehat{f}}{\partial x_j}\Big|_x, x \in \bar{U} = \varphi(U) \subseteq \mathbb{R}^n,$$

and we write $\mathrm{grad} f_x = g^{-1} \nabla \widehat{f}$.

1.1.1 Gradient on Lie groups

We may extend the previous result on Lie groups. It is

$$\begin{aligned} \mathrm{d}f(Y_g)(g) &= \mathrm{d}f(L_{g*} \circ L_{g*}^{-1} Y_g)(g) \\ &= (L_g^* \mathrm{d}f)(L_{g*}^{-1} Y_g)(e) \end{aligned} \tag{7}$$

At this point consider a symmetric, positive definite covariant $2-$tensor on the tangent space at the identity of the group, $\tau : T_eG \times T_eG \to \mathbb{R}$, and define the Riemannian metric $\bar{g} : C^\infty(TG) \times C^\infty(TG) \to C^\infty(G)$ such that its value at $g \in G$ satisfies

$$\bar{g}(X_g, Y_g)(g) := \tau(L_{g*}^{-1} X_g, L_{g*}^{-1} Y_g),\ g \in G, X_g, Y_g \in T_gG. \tag{8}$$

It is easy to show that $\bar{g}$ is a left-invariant Riemannian metric on G. With respect to that metric, the gradient of a function $f \in C^\infty(M)$ satisfies

$$\bar{g}(\mathrm{grad} f_g, Y_g)(g) = \mathrm{d}f(Y_g)(g),\ \forall g \in G.$$

Thus, by combining (7), and (8)

$$\tau(L_{g*}^{-1} \mathrm{grad} f_g, L_{g*}^{-1} Y_g) = (L_g^* \mathrm{d}f)(L_{g*}^{-1} Y_g)(e),\ \forall Y_g \in T_gG$$

The latter expression shows that the tangent vector $L_{g*}^{-1} \mathrm{grad} f_g \in T_eG$ uniquely determines a covector $\omega(Y_e) := \tau(L_{g*}^{-1} \mathrm{grad} f(g), Y_e) \in T_e^*G$. So there is a vector space isomorphism $Q : T_eG \to T_e^*G$ such that

$$\Big(Q \circ L_{g*}^{-1} \mathrm{grad} f_g\Big)(Y_e)(e) = (L_g^* \mathrm{d}f)(Y_e)(e)$$

for all $Y_e \in T_eG$. Point-wise definition of equality of co-vectors yields:

$$Q \circ L_{g*}^{-1} \mathrm{grad} f_g = L_g^* \mathrm{d}f,$$

or

$$\mathrm{grad} f_g = L_{g*} \circ Q^{-1} \circ L_g^*\ \mathrm{d}f.$$

1.2 The volume form

A $n-$differential form $\omega_g : M \to \Lambda^n(T^*M)$ has value at a point $p \in (U, \varphi)$

$$\omega_g(X_1, \dots, X_n)(p) = \alpha(p)\ e_p^1 \wedge \dots \wedge e_p^n(X_1, \dots, X_n), \tag{9}$$

where $\alpha \in C^\infty(M)$, and $\{e^i\}_i$ is the dual to the coordinate frame $\{\mathrm{E}_i\}_i$ in (U, φ). In addition, given a Riemannian metric as in (1), let $\{\widehat{\mathrm{E}}_i\}_i$ be the orthonormal frame w.r.t. to which we assign positive orientation in (U, φ), and $\{\hat{e}_i\}_i$ the corresponding dual. With respect to this frame, the Riemannian volume form reads

$$\omega_g(X_1, \dots, X_n)(p) = \widehat{e}_p^1 \wedge \dots \wedge \widehat{e}_p^n(X_1, \dots, X_n), \tag{10}$$

Further, since the orthonormal at any point spans the tangent space at that point, the coordinate frame can be expressed w.r.t. the orthonormal frame as

$$E_{i,p} = \sum_j c_i^j \widehat{E}_{j,p} \tag{11}$$

Further,

$$\widehat{e}_p^i = \lambda_1 e_p^1 + \lambda_2 e_p^2 + \dots + \lambda_n e_p^n$$

from which we obtain $\widehat{e}_p^i\left(\mathrm{E}_{k,p}\right) = \lambda_k$. In addition, $\hat{e}_p^i\left(E_{k,1}\right) = c_k^i$, and therefore

$$\widehat{e}_i^i = \sum_j c_j^i e_p^j$$

By equating (9), and (10) on the coordinate frame components, we obtain:

$$\begin{aligned}
\omega\left(\mathrm{E}_{1,p}, \dots, \mathrm{E}_{n,p}\right) &= \widehat{e}_p^1 \wedge \dots \wedge \widehat{e}_p^n\left(\mathrm{E}_{1,p}, \dots, \mathrm{E}_{n,p}\right) \\
&= \mathrm{Alt}\left(\widehat{e}_p^1\left(\mathrm{E}_{1,p}\right) \otimes \dots \otimes \widehat{e}_p^n\left(E_{n,p}\right)\right) \\
&= \frac{1}{n!}\sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\widehat{e}_p^1\left(\mathrm{E}_{\sigma(1),p}\right) \otimes \dots \otimes \widehat{e}_p^n\left(\mathrm{E}_{\sigma(n),p}\right)
\end{aligned}$$

Further,

$$\begin{aligned}
\widehat{e}_p^i\left(\mathrm{E}_{\sigma(i)p}\right) &= \widehat{e}_p^i\left(\varphi_*^{-1}\left(\partial x_{\sigma(i)}\Big|_{x=\varphi(p)}\right)\right) = \left(\varphi^{*-1}\widehat{e}_1^i\right)\left(\partial x_{\sigma(i)}\Big|_{x=\varphi(p)}\right) \\
&= \left(\varphi^{*-1}\sum_j c_j^i e_p^j\right)\left(\partial x_{\sigma(i)}\Big|_{x=\varphi(p)}\right) = \left(\sum_j c_j^i \varphi^{*-1} e_p^j\right)\left(\partial x_{\sigma(i)}\Big|_{x=\varphi(p)}\right) \\
&= \sum_j c_j^i dx^j\left(\partial x_{\sigma(i)}|_{x=\varphi(p)}\right) = c_{\sigma(i)}^i(\varphi(p)) = c_{\sigma(i)}^i(x).
\end{aligned}$$

Therefore,

$$\alpha(p) = \omega\left(\mathrm{E}_{1,p}, \ldots, \mathrm{E}_{n,p}\right) = \frac{1}{n!}\sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) c^1_{\sigma(1)} \cdots c^n_{\sigma(n)}$$
$$= \det\left(C^\top(\varphi(p))\right).$$

On top of that, since $\{\widehat{\mathrm{E}}_{i,p}\}_i$ are orthonormal w.r.t. the Riemannian metric g,

$$g_{i,k}(p) = g\Big(\mathrm{E}_{i,p}, \mathrm{E}_{k,p}\Big)(p) = g\Bigg(\sum_j c^j_i \widehat{\mathrm{E}}_{j,p}, \sum_{\dot\rho} c^\rho_k \widehat{\mathrm{E}}_{\rho,p}\Bigg)(p)$$
$$= \sum_{j,\rho} c^j_i c^p_k \delta^j_\rho = \left[CC^\top\right]_{i,k}.$$

As a result,

$$\alpha(p) = \sqrt{\det(g(p))}.$$

1.3 The Laplacian

At this point we assume that given $U \subseteq M$, we can find an oriented orthonormal frame $\{\mathrm{E}_i\}^n_i$, with co-frame $\{e_i\}^n_i$. Given an orientation in M, the Laplacian of a smooth function $f \in C^\infty(M)$ is defined as:

$$\Delta f(p) := \star^{-1}\mathrm{d}\Big(\ (\mathrm{d}f)^\sharp \lrcorner\ \omega_g\ \Big),$$

where $(\ )^\sharp : T^*M \to TM$ denotes the dual pairing induced by the Riemannian metric, and $\omega_g : M \to \Lambda^n(T^*M)$ is the Riemannian volume form; the latter with value at $p \in U$

$$\omega_g(X_1, \ldots, X_n)(p) = \sqrt{\det(g)(p)}\ e^1_p \wedge \cdots \wedge e^n_p(X_1, \ldots, X_n).$$

The hodge star operator $\star : \Omega^n(M) \to C^\infty(M)$, and the interior product $\lrcorner : \Omega^n(M) \to \Omega^{n-1}(M)$.

To begin with, it is

$$((\mathrm{d}f)^\sharp \lrcorner\ \omega_g)(p) = (\mathrm{grad} f \lrcorner\ \omega_g)_p := \mathrm{grad} f_p \lrcorner\ \omega_{gp},$$

and from the product rule for the interior product, and given that it is zero for a real-valued function,

$$\begin{aligned}
\mathrm{grad} f_p \lrcorner\ \omega_{gp} &= \sqrt{\det(g)(p)}\ \mathrm{grad} f_p \lrcorner\ (e^1_p \wedge \cdots \wedge e^n_p) \\
&= \sqrt{\det(g)(p)}\ \sum_{i=1}^n (-1)^{i-1} e^i(\mathrm{grad} f_p) e^1_p \wedge \cdots \wedge \widehat{e^i_p} \wedge \cdots \wedge e^n_p \\
&= \sum_{i=1}^n \left[\sqrt{\det(g)(p)}(-1)^{i-1}\Big(\sum_j g^{i,j}(p)\,(\mathrm{E}_{j,p} f)\Big)\right] e^1_p \wedge \cdots \wedge \widehat{e^i_p} \wedge \cdots \wedge e^n_p \in \Lambda^{n-1}(T^*M).
\end{aligned} \tag{12}$$

The linearity and product rule of the exterior derivative gives for (12):

$$\mathrm{d}(\mathrm{grad} f_p \lrcorner\, \omega_{gp}) = \sum_{i=1}^{n} \mathrm{d}\Big[\sqrt{\det(g)(p)}(-1)^{i-1}\Big(\sum_j g^{i,j}(p)\,(\mathrm{E}_{j,p}f)\Big)\Big] \wedge e_p^1 \wedge \cdots \wedge \widehat{e_p^i} \wedge \cdots \wedge e_p^n$$
$$+ \sum_{i=1}^{n}(-1)^0\sqrt{\det(g)(p)}(-1)^{i-1}\Big(\sum_j g^{i,j}(p)\,(\mathrm{E}_{j,p}f)\Big)\mathrm{d}\Big(e_p^1 \wedge \cdots \wedge \widehat{e_p^i} \wedge \cdots \wedge e_p^n\Big) \tag{13}$$

The second term in (13) is zero due to repeated indices. Further,

$$\mathrm{d}\Big[\sqrt{\det(g)(p)}(-1)^{i-1}\Big(\sum_j g^{i,j}(p)\,(\mathrm{E}_{j,p}f)\Big)\Big] = \sum_{k=1}^{n} E_{k,p}\Big(\sqrt{\det(g)(p)}(-1)^{i-1}\Big(\sum_j g^{i,j}(p)\,(\mathrm{E}_{j,p}f)\Big)\Big)e_p^k. \tag{14}$$

After plugging (14) into (13) the only non-zero component corresponds to the index i. Also, for $e_p^{k=i}$ to go from the 'zero position' to the $\widehat{e_p^i}$ position, i 'swaps' are needed. This means that (14) is multiplied by $(-1)^i$. Thus,

$$\begin{aligned}\mathrm{d}(\mathrm{grad} f_p \lrcorner\, \omega_{gp}) &= -\Big[\sum_{i=1}^{n} \mathrm{E}_{i,p}\Big(\sqrt{\det(g)(p)}\Big(\sum_j g^{i,j}(p)\,(\mathrm{E}_{j,p}f)\Big)\Big)\Big] e_p^1 \wedge \cdots \wedge e_p^n \\ &= -\frac{1}{\sqrt{\det(g)(p)}}\Big[\sum_{i=1}^{n} \mathrm{E}_{i,p}\Big(\sqrt{\det(g)(p)}\Big(\sum_j g^{i,j}(p)\,(\mathrm{E}_{j,p}f)\Big)\Big)\Big]\omega_{gp},\end{aligned}$$

and subsequently,

$$\star^{-1}\mathrm{d}(\mathrm{grad} f_p \lrcorner\, \omega_{gp}) = -\frac{1}{\sqrt{\det(g)(p)}}\Big[\sum_{i=1}^{n} \mathrm{E}_{i,p}\Big(\sqrt{\det(g)(p)}\Big(\sum_j g^{i,j}(p)\,(\mathrm{E}_{j,p}f)\Big)\Big)\Big]$$

As a result the Laplacian reads,

$$\Delta_p f = -\frac{1}{\sqrt{\det(g)(p)}}\Bigg[\sum_{i=1}^{n} \mathrm{E}_{i,p}\Bigg(\sqrt{\det(g)(p)}\Big(\sum_j g^{i,j}(p)\,(\mathrm{E}_{j,p}f)\Big)\Bigg)\Bigg]\ ,\ p \in U. \tag{15}$$

We can further expand (15) as follows:

$$\begin{aligned}\Delta_p f &= -\frac{1}{\sqrt{\det(g)}}\Bigg[\sum_{i=1}^{n} \mathrm{E}_i\Bigg(\sqrt{\det(g)}\Big(\sum_j g^{i,j}(x)\,(\mathrm{E}_j f)\Big)\Bigg)\Bigg] \\ &= -\sum_{i,j}\frac{\mathrm{E}_i\Big(\sqrt{\det(g)}g^{i,j}\Big)}{\sqrt{\det(g)}}\mathrm{E}_j f - \sum_{i,j} g^{i,j}\mathrm{E}_i\mathrm{E}_j f\end{aligned} \tag{16}$$

the latter can be written as

$$\Delta_p f = \sum_j \left(\sum_i \frac{\mathrm{E}_{ip}\Big(\sqrt{\det(g)}g^{i,j}\Big)}{\sqrt{\det(g)}} \right)\mathrm{E}_{jp} f + \sum_{i,j} g^{i,j}\mathrm{E}_{ip}\mathrm{E}_{jp} f$$
$$= \mathbf{d}(x)^\top \nabla f + g^{-1}(x) : \mathrm{Hess} f, \tag{17}$$

where $\Big[\mathbf{d}(x)\Big]_i = \mathrm{div}\Big(g^{-1,i}(x_t)\Big)$, where the divergence takes the ith column of the matrix g^{-1}. More generally we have the following

Lemma 1 (Riemannian divergence)**.** *On a smooth manifold M, the value of the divergence of a smooth vector field $X : M \to TM$, inside a smooth chart (U, φ) is expressed as*

$$\mathrm{div}(X)(p) = \frac{1}{\sqrt{\det(g(p))}} \sum_i \mathrm{E}_{i,p}\Big(\sqrt{\det(g(p))}X_i(p)\Big), \ p \in U \tag{18}$$

where $\{E_i\}_i$ is the coordinate frame naturally defined in (U, φ).

Proof. By definition

$$\mathrm{div}(X) := \star^{-1}\mathrm{d}\Big(X \lrcorner\, \omega_g\Big). \tag{19}$$

Consider now a smooth chart (U, φ) with the associated coordinate frame $\{\mathrm{E}_i\}_i$, and the corresponding dual frame $\{\mathrm{e}_i\}_i$. Inside this chart, we do have an explicit expression for the volume form and thus, by utilizing the product rule for the interior product, we do have an explicit expression for the interior product too. The interior product of the vector field X and the Riemannian volume form is a one order less alternating tensor with value at $p \in U$

$$(X \lrcorner\, \omega_g)(p) = X \lrcorner \left(\sqrt{\det(g(p))}e_p^1 \wedge \cdots \wedge e_p^n\right) \tag{20}$$

At this point recall that all $0-$tensors are alternating tensors [Lee, 2012, p. 350], and therefore, the wedge product of a $0-$tensor and any alternating tensor is well-defined. On top of that, the interior product of a $0-$ differential form is zero and thus, by the product rule of interior product

$$(X \lrcorner\, \omega_g)(p) = (-1)^0\sqrt{\det(g(p))}\Big(X \lrcorner(e_p^1 \wedge \cdots \wedge e_p^n)\Big)(p)$$
$$= \sqrt{\det(g(p))}\sum_{i=1}^n(-1)^{i-1}e_p^i(X) \wedge \dots \widehat{e}_p^i \cdots \wedge e_p^n \tag{21}$$

Now, inside U, we have the following expression for the value of the vector field X at the point p $X_p = \sum_k X_k(p)\mathrm{E}_{kp}$. Since the co-vector $e_p^i(\ ) : T_pM \to \mathbb{R}$ is by definition a linear map, evaluated at X cancels-out all the components different than i. Therefore the last row of (21) reads:

$$(X \lrcorner\, \omega_g)(p) = \sqrt{\det(g(p))}\sum_{i=1}^n(-1)^{i-1}X_i(p)e_p^1 \wedge \dots \widehat{e}_p^i \cdots \wedge e_p^n \tag{22}$$

Now we take the exterior derivative of (22) and again we use product rule for exterior derivatives [Lee, 2012, p. 365]. Note that $(X \lrcorner\, \omega_g)$ is a one-order lower differential form than ω_g, and therefore, its exterior derivative (by definition) will be a $n-$differential form. The exterior derivative of $(X \lrcorner\, \omega_g)$ is a $n-$differential form, with value at $p \in U$

$$\begin{aligned} \mathrm{d}(X \lrcorner\, \omega_g)(p) &= \mathrm{d}\left(\sum_i \sqrt{\det(g(p))}(-1)^{i-1} X_i(p) e_p^1 \wedge \ldots \widehat{e}_p^i \cdots \wedge e_p^n \right) \\ &\overset{\text{linearity of d and product rule of d}}{=} \sum_i \mathrm{d}\Big(\sqrt{\det(g(p))}(-1)^{i-1} X_i(p)\Big) \wedge e_p^1 \wedge \ldots \widehat{e}_p^i \cdots \wedge e_p^n \\ &+ \sum_i \sqrt{\det(g(p))}(-1)^{i-1} X_i(p) \mathrm{d}\Big(e_p^1 \wedge \ldots \widehat{e}_p^i \cdots \wedge e_p^n\Big) \end{aligned} \tag{23}$$

The second term is zero (why?). For the first term, the differential $\mathrm{d}\Big(\sqrt{\det(g(p))}(-1)^{i-1} X_i(p)\Big)$ is an element of $\Omega^1(M)$ (i.e. a co-vector field) with value at p being a co-vector and therefore it can be written as

$$\mathrm{d}\Big(\sqrt{\det(g(p))}(-1)^{i-1} X_i(p)\Big)(Y)(p) = \sum_l \beta_l(p) e_p^l(Y), \;\; p \in U, \;\; Y \in T_pM \tag{24}$$

For $Y_p = \mathrm{E}_{\eta p}$ in (24) we obtain

$$\beta_\eta(p) = \mathrm{d}\Big(\sqrt{\det(g(p))}(-1)^{i-1} X_i(p)\Big)(\mathrm{E}_{\eta p})(p) \tag{25}$$

On top of that for the exterior derivative of a $0-$form we have the following definition

$$\mathrm{d}f(Y)(p) := Y_p f. \tag{26}$$

Therefore, from (24), and (25) we obtain

$$\beta_\eta(p) = \mathrm{E}_{\eta p} \sqrt{\det(g(p))}(-1)^{i-1} X_i(p). \tag{27}$$

Thus, the exterior derivative reads:

$$\begin{aligned} \mathrm{d}\Big(\sqrt{\det(g(p))}(-1)^{i-1} X_i(p)\Big)(Y)(p) &= \sum_l \mathrm{E}_{lp}\left(\sqrt{\det(g(p))}(-1)^{i-1} X_i(p)\right) e_p^l \\ &= (-1)^{i-1} \sum_l \mathrm{E}_{lp}\left(\sqrt{\det(g(p))} X_i(p)\right) e_p^l \end{aligned} \tag{28}$$

By plugging (28) into (23), we can easily observe that the only term that remains from the sum in (28) is the ith term. Thus,

$$\begin{aligned} \mathrm{d}(X \lrcorner\, \omega_g)(p) &= \sum_i \left((-1)^{i-1} \sum_l \mathrm{E}_{lp}\left(\sqrt{\det(g(p))} X_i(p)\right) e_p^l \right) \wedge e_p^1 \wedge \ldots \widehat{e}_p^i \cdots \wedge e_p^n \\ &= \sum_i \mathrm{E}_{ip}\left(\sqrt{\det(g(p))} X_i(p)\right) e_p^i (-1)^{i-1} \wedge \Big(e_p^1 \wedge \ldots \widehat{e}_p^i \cdots \wedge e_p^n\Big) \end{aligned} \tag{29}$$

In the above expression, for the e_p^i co-vector to go to the ith position , $i-1$ swaps are needed, and thus

$$\begin{aligned}\mathrm{d}(X \lrcorner\, \omega_g)(p) &= \sum_i \mathrm{E}_{ip}\Big(\sqrt{\det(g(p))}X_i(p)\Big) e_p^1 \wedge \cdots \wedge e_p^n \\ &= \frac{\sum_i \mathrm{E}_{ip}\Big(\sqrt{\det(g(p))}X_i(p)\Big)}{\sqrt{\det(g(p))}}\omega_g(p) \end{aligned} \tag{30}$$

As a result, the Riemannian divergence of the vector field X at a point $p \in U$ can be expressed by

$$\mathrm{div}_M X(p) = \frac{\sum_i \mathrm{E}_{ip}\Big(\sqrt{\det(g(p))}X_i(p)\Big)}{\sqrt{\det(g(p))}},\ p \in U \tag{31}$$

$\square$

For $X_i(p) = g^{-1,\cdot,i}(p)$ (i.e. the ith column of the inverse metric matrix) in Lemma (1) we obtain $\Big[\mathbf{d}(x)\Big]_i = \mathrm{div}\Big(g^{-1,i}(x_t)\Big)$. So each component of the vector $\mathbf{d}$ is the divergence of the ith column of the matrix $g^{-1}(p)$. In other words, we may consider a vector field with components in U the entries of the ith column of $g^{-1}(p)$. The generator of the Riemannian Brownian motion w.r.t. the metric g reads:

$$\Delta_p f = \mathrm{div}_M g^{-1}(p)^\top \nabla_p f + \frac{1}{2} g^{-1}(p) : 2\mathrm{Hess}_p f \tag{32}$$

From [Pavliotis, 2016, p. 66] we can identify the Riemannian Brownian motion as the following SDE

$$\mathrm{d}x_t = \mathrm{div}_M\Big(g^{-1}(x_t)\Big)\mathrm{d}t + \sqrt{2g^{-1}(x_t)}\mathrm{d}\omega_t, \tag{33}$$

where the divergence is w.r.t. the Riemannian metric and is applied to every column of the inverse metric matrix representation. Note that the Riemannian Brownian motion, as opposed to the standard Brownian motion, incorporates a drift term. We can do more by writing the term inside the first sum in (16) as

$$\frac{\mathrm{E}_i\Big(\sqrt{\det(g)}g^{i,j}\Big)}{\sqrt{\det(g)}} = \frac{\mathrm{E}_i(\sqrt{\det(g)})g^{i,j}}{\sqrt{\det(g)}} + \mathrm{E}_i g^{i,j} \tag{34}$$

At this point we can further express the first term in (34) by using the fact that the Riemannian metric automatically determines uniquely a Riemannian torsion-free connection on M [Boothby, 1986][2]. A connection is represented by the Christoffel symbols, and

[2]In general, a connetion indicates how motion in the total space, in that case in the tangent bundle, induces motion along the fibre (tangent space). It is a vector-valued 1−form.

in this case the latter are determined by the Riemannian metric. We have the following usefull identities:

$$\mathrm{E}_k g_{ij} = \Gamma_{ikj} + \Gamma_{jki} = \sum_l g_{lj}\Gamma^l_{ik} + g_{li}\Gamma^l_{jk}, \tag{35}$$

$$\mathrm{E}_k g^{ij} = \sum_l -g^{lj}\Gamma^i_{lk} - g^{li}\Gamma^j_{lk}, \tag{36}$$

where (36) follows from differentiating the identity $g_{ij}g^{jk} = \delta^k_i$ and inserting (35). From the expression of the derivative of the metric determinant we also find

$$\begin{aligned}
\mathrm{E}_k \det(g) &= \sum_{i,j} \det(g) g^{ij} \mathrm{E}_k g_{ij} \\
&= \sum_{i,j} \det(g) g^{ij} \left(\Gamma_{ikj} + \Gamma_{jki}\right) \\
&= \sum_{i,j} \det(g)(\Gamma^i_{ik} + \Gamma^j_{jk}) = 2\sum_i \det(g)\Gamma^i_{ik}
\end{aligned}$$

From this we obtain

$$\sum_i \Gamma^i_{ik} = \frac{1}{2\det(g)} E_k \det(g) = \frac{1}{\sqrt{\det(g)}} E_k \sqrt{\det(g)}$$

Thus, (34) reads

$$\frac{\mathrm{E}_i\left(\sqrt{\det(g)}g^{i,j}\right)}{\sqrt{\det(g)}} = \sum_l \Gamma^l_{l,i} g^{i,j} + \sum_l -g^{l,j}\Gamma^i_{l,i} - \sum_l g^{l,i}\Gamma^j_{l,i},$$

and subsequently the first term in (16)

$$\sum_{i,j} \frac{\mathrm{E}_i\left(\sqrt{\det(g)}g^{i,j}\right)}{\sqrt{\det(g)}} \mathrm{E}_j f = \sum_{i,j,l} \left(\Gamma^l_{l,i} g^{i,j} - g^{l,j}\Gamma^i_{l,i} - g^{l,i}\Gamma^j_{l,i}\right) \mathrm{E}_j f$$

At this point we can use the symmetry of lower indexes of the Christofell symbols, isolate the second term, and swap the indexes l, and i, to obtain

$$\sum_{i,j} \frac{\mathrm{E}_i\left(\sqrt{\det(g)}g^{i,j}\right)}{\sqrt{\det(g)}} \mathrm{E}_j f = -\sum_{i,j,l} g^{l,i}\Gamma^j_{l,i} \mathrm{E}_j f$$

As a result, the Laplace operator reads:

$$\Delta_p f = \sum_{i,j,l} g^{l,i}(p)\Gamma^j_{l,i}(p)\mathrm{E}_j f(p) - \sum_{i,j} g^{i,j}(p)\mathrm{E}_i \mathrm{E}_j f(p), \;\; p \in U \subseteq M. \tag{37}$$

Again, we can identify the Riemannian Brownian motion as

$$\mathrm{d}x_t = \left[\sum_{i,l} g^{l,i}(x_t)\Gamma^j_{l,i}(x_t)\right]_{j=1:n} \mathrm{d}t + \sqrt{2g^{-1}(x_t)}\mathrm{d}\omega_t \tag{38}$$

Note that the drift term of the Riemannian Brownian motion depends on the Riemannian connection. Both stochastic differential equations (33), and (38) are understood in the Ito sense.

1.4 The Left-invariant Laplacian

The result in (37) relies on the smooth structure of M, and thus, it can be transferred unaltered into the Lie group setting by plugging-in the algebraic structure provided by the group.

Our main goal is to express the coordinate frame induced by the smooth chart (U, φ) w.r.t. the left-invariant vector-fields made by a basis in the tangent space at the identity of the group, and in such a way extend the Laplacian to the entire manifold. In other words, we want to perform a transformation at every tangent space T_gG at $g \in U$.

To do so, first take U in (37) to be an open set that contains the identity of the group, and subsequently choose a basis $\{\epsilon_i\}_{i=1}^n$ on T_eG.

The exponential map $\exp : T_eG \to G$ is a diffeomorphism from an open neighborhood of T_eG around zero to an open neighborhood $V \subseteq U$ [Helgason, 1979, p.104, Proposition 1.6], and therefore, with $g = \exp(x_i\epsilon_i)$, we can define $\varphi(g) = (x_1(g), \dots, x_n(g))$ such that $x_i(e) = 0$, and $x_i(g) = x_i \in \mathbb{R}$ and close to zero.

Let now $\{\mathrm{E}_i\}_i$ be the coordinate frame inside this small neighborhood V, i.e. $\mathrm{E}_i x_k = \delta_{ik}$, or equivalently $\mathrm{E}_{ig} = \varphi_*^{-1}\Big(\partial_i\Big|_{x=\varphi(g)}\Big)$. Note that $\{\mathrm{E}_{ie}\}_i = \{\epsilon_i\}_i$. Consider (37) w.r.t. this frame

$$\Delta_g f = \sum_{i,j,l} g^{l,i}(g)\Gamma^j_{l,i}(g)\mathrm{E}_{jg}f - \sum_{i,j} g^{i,j}(g)\mathrm{E}_{ig}\mathrm{E}_{jg}f, \;\; g \in V. \tag{39}$$

At this point, take the group homomorphism of the line (with shift symmetry) $\gamma : t \mapsto G$, which we know it reads $\gamma = \exp(tX), \;\; X \in T_eG$, and let it act on G from the right

$$\theta_\gamma(t) := R_{\gamma(t)}(g). \tag{40}$$

The infinitesimal generator $\bar{X} : G \to TG \equiv G \times T_eG$ of $\theta : \gamma \times G \to G$ is a left-invariant vector field and by definition (it is the velocity vector of the orbit of $g \in G$ under the action θ_γ), its value at $g \in G$ is given by

$$\bar{X}_g f := \mathrm{d}_t f(R_{\exp(tX)}(g))\Big|_{t=0}, \tag{41}$$

for all $f \in C^\infty(V \subseteq G)$. Further, $\bar{X}_g$ is a tangent vector at T_gG, and thus, it can be written w.r.t. the coordinate frame-basis as

$$\bar{X}_g = \sum_j \alpha_j(g)\mathrm{E}_{jg}. \tag{42}$$

Now, we know that every tangent vector at the identity completely determines a left-invariant vector field through the tangent map of the left-shift $L_g : G \to G$. Essentially, this is because through the additional algebraic structure we can map diffeomorphically any neighborhood of the identity to every neighborhood that contains a point of the group.

By applying this to the basis $\{\epsilon_i\}_i \in T_eG$, we have:

$$\bar{\epsilon}_{ig} = \sum_j \alpha_{ij}(g)\mathrm{E}_{jg}. \tag{43}$$

Essentially, we want to find the inverse of the transformation $\{\alpha_{ij}(g)\}_{ij=1}^{\dim(G)}$. The first step to do so is to plug-in (43) into (41), and subsequently to view $g \in V$ as the member of the one-parameter sub-group $g = \exp(tY),\ Y \in T_eG,$ for $t = 1$, thus obtaining

$$\sum_j \alpha_{ij}(\exp(Y))\mathrm{E}_{j\exp(\mathrm{Y})}f = \mathrm{d}_t f\Big(\exp(Y)\exp(t\epsilon_i)\Big)\Big|_{t=0}. \tag{44}$$

Now, we use the formula for the product of the exponential map [Helgason, 1979, page 106]

$$\exp(Y)\exp(tX) = \exp\Big(Y + t\epsilon_i + \frac{t}{2}\mathrm{ad}(Y)(\epsilon_i) + \mathrm{O}(t^2)\Big),$$

which after plugging it into (44) yields:

$$\sum_j \alpha_{ij}(\exp(Y))\mathrm{E}_{j\exp(\mathrm{Y})}f = \mathrm{d}_t f\Big(\exp\Big(Y + t\epsilon_i + \frac{t}{2}\mathrm{ad}(Y)(\epsilon_i) + \mathrm{O}(t^2)\Big)\Big)\Big|_{t=0}. \tag{45}$$

The term inside the exponential map in the right-hand side of (45) is an element $v \in (T_eG, [\ ,\ ]) \equiv \mathfrak{g}$. Observe now that since the exponential map is a diffeomorphism, $f = x_k$ in would give us the kth component of v. That is,

$$\alpha_{ik}(\exp(Y)) = \mathrm{d}_t x_k(Y + t\epsilon_i + \frac{t}{2}\mathrm{ad}(Y)(\epsilon_i) + \mathrm{O}(t^2))\Big|_{t=0},$$

or

$$\alpha_{ik}(g) = \mathrm{d}_t\Big(Y_k + t(\epsilon_i)_k + \frac{t}{2}\mathrm{ad}(Y)(\epsilon_i)_k + \mathrm{O}(t^2)\Big)\Big|_{t=0}$$

or

$$\alpha_{ik}(g) = (\epsilon_i)_k + \frac{1}{2}\mathrm{ad}(Y)(\epsilon_i)_k,\ g \in V. \tag{46}$$

Note that the dependency on g in the right hand-side of (46) 'is hidden' in Y and arises again from the fact that exp is a diffeomorphism. With $Y = \sum_i y_i\epsilon_i \in \mathfrak{g}$, and close to zero,

$$\exp(Y) = g,$$

with $\varphi(g) = \{y_i\}_i$. Therefore, $Y = \sum_i y_i(g)\epsilon_i$, and

$$\alpha_{ik}(g) = (\epsilon_i)_k + \frac{1}{2}\mathrm{ad}\Bigg(\sum_\eta y_\eta(g)\epsilon_\eta\Bigg)(\epsilon_i)_k. \tag{47}$$

Due to linearity of the bracket

$$\alpha_{ik}(g) = (\epsilon_i)_k + \frac{1}{2}\sum_\eta y_\eta(g)\mathrm{ad}(\epsilon_\eta)(\epsilon_i)_k,\ g \in V,$$

where the terms $\mathrm{ad}(\epsilon_\eta)(\epsilon_i) \in \mathfrak{g}$ can be expresses w.r.t. the basis $\{\epsilon_i\}_i$ and the structure coefficients of the group.

The tangent map of the left-shift of the ith basis vector of the tangent space at the identity at $g \in V$ reads

$$\bar{\epsilon}_{ig} = \sum_k \Big((\epsilon_i)_k + \frac{1}{2} \sum_\eta x_\eta(g) \mathrm{ad}(\epsilon_\eta)(\epsilon_i)_k \Big) \mathrm{E}_{kg},$$

where x_η are the coordinates of $g \in V$. To find the inverse transformation, we start by the fact that the tangent map of the exponential map $\exp(X)_* : T_e G \to T_{\exp(X)} G$ reads [Tuynman, 1995]

$$\exp(X)_* = L_{\exp(X)*} \circ \Big(\sum_{k=0}^{+\infty} \frac{(-1)^k}{(k+1)!} \big(\mathrm{ad}(X)\big)^k \Big),$$

where

$$\big(\mathrm{ad}(X)\big)^{k+1}(Y) = \Big[X, \big(\mathrm{ad}(X)\big)^k (Y)\Big], \quad \mathrm{ad}(X)(Y) = [X, Y]. \tag{48}$$

The map

$$\sum_{k=0}^{+\infty} \frac{(-1)^k}{(k+1)!} \big(\mathrm{ad}(X)\big)^k = L^{-1}_{\exp(X)*} \circ \exp(X)_* : \mathfrak{g} \to \mathfrak{g}$$

is invertible with

$$\Big(\sum_{k=0}^{+\infty} \frac{(-1)^k}{(k+1)!} \big(\mathrm{ad}(X)\big)^k \Big)^{-1} = \mathrm{id}_{\mathfrak{g}} + \frac{1}{2} \mathrm{ad}(X) + \sum_{k=2}^{\infty} \beta_k \, \mathrm{ad}(X)^k,$$

where the third term vanishes for X close to zero. As a result

$$\mathrm{E}_{ig} = \sum_k \Big((\epsilon_i)_k - \frac{1}{2} \sum_\eta x_\eta(g) \big[\mathrm{ad}(\epsilon_\eta)(\epsilon_i)\big]_k \Big) \bar{\epsilon}_{kg}, \tag{49}$$

or, since $\{\epsilon_i\}$ are basis vectors,

$$\mathrm{E}_{ig} = \ \bar{\epsilon}_{ig} - \frac{1}{2} \sum_{k,\eta} x_\eta(g) \gamma^k_{\eta,i} \bar{\epsilon}_{kg}, \tag{50}$$

where the constants $\gamma^k_{\eta,i} := \big[\mathrm{ad}(\epsilon_\eta)(\epsilon_i)\big]_k$ are the structural coefficients of G. Thus,

$$\begin{aligned} \mathrm{E}_{jg}\mathrm{E}_{kg} f &= \Big(\bar{\epsilon}_{jg} - \frac{1}{2} \sum_{\xi,\eta} x_\eta(g) \gamma^\xi_{\eta,i} \bar{\epsilon}_{\xi g} \Big) \Big(\bar{\epsilon}_{kg} f - \frac{1}{2} \sum_{\rho,\lambda} x_\lambda(g) \gamma^\rho_{\lambda,k} \bar{\epsilon}_{\rho g} f \Big) \\ &= \bar{\epsilon}_{jg}\bar{\epsilon}_{kg} f - \frac{1}{2} \bar{\epsilon}_{jg} \Big(\sum_{\rho,\lambda} x_\lambda(g) \gamma^\rho_{\lambda,k} \bar{\epsilon}_{\rho g} f \Big) - \frac{1}{2} \sum_{\xi,\eta} x_\eta(g) \gamma^k_{\eta,i} \bar{\epsilon}_{\xi g} (\bar{\epsilon}_{kg} f) \\ &+ \frac{1}{4} \Big(\sum_{\xi,\eta} x_\eta(g) \gamma^k_{\eta,i} \bar{\epsilon}_{\xi g} \Big) \Big(\sum_{\rho,\lambda} x_\lambda(g) \gamma^\rho_{\lambda,k} \bar{\epsilon}_{\rho g} f \Big). \end{aligned} \tag{51}$$

Let us now consider a symmetric, positive definite covariant $2-$tensor on the tangent space at the identity, $\tau : T_eG \times T_eG \to \mathbb{R}$, and we impose an extra condition to the basis $\{\epsilon_i\}_i$. That is, we assume that it is orthonormal w.r.t. τ, the latter being defined as

$$\tau(X,Y) := \sum_{i,j} q_{i,j} e_i(X) \otimes e_j(Y), \ \forall X, Y \in T_eG, \tag{52}$$

where $\{e_i\}_i \in T_e^*G$ is the dual basis to $\{\epsilon_i\}$ In that case orthonormal means $\tau(\epsilon_{ie}, \epsilon_{ke}) = \sum_{j,\rho} q_{ij} q_{pk} \delta_{j\rho} = [QQ^\top]_{ik}$, where $Q : T_eG \to T_e^*G$ is the vector space isomorphism induced by the tensor (inner product). By choosing $Q = I$, $\tau(\epsilon_{ie}, \epsilon_{ke}) = \delta_{ik}$.

Subsequently, by utilizing τ, we can impose an extra condition on the Riemannian metric, that is, to be left-invariant. On top of that left-invariant Riemannian metrics $\bar{g} : C^\infty(TG) \times C^\infty(TG) \to C^\infty(G)$, are uniquely determined via the symmetric, positive definite covariant $2-$tensor on the tangent space at the identity, according to the prescription

$$\bar{g}(X_g, Y_g)(g) := \tau(L_{g*}^{-1} X_g, L_{g*}^{-1} Y_g), \ g \in G, X_g, Y_g \in T_gG. \tag{53}$$

From (50), (53) we can express the metric components w.r.t. the left-invariant vector field as

$$\begin{aligned}
\bar{g}_{ij}(g) &= \bar{g}_{ij} \left(\mathrm{E}_{ig}, \mathrm{E}_{jg}\right)(g) \\
&= \bar{g}\left(\bar{\epsilon}_{ig} - \frac{1}{2}\sum_{k,\eta} x_\eta(g)\gamma^k_{\eta i}\bar{\epsilon}_{kg}\ ,\ \bar{\epsilon}_{jg} - \frac{1}{2}\sum_{k,\eta} x_\eta(g)\gamma^k_{\eta j}, \bar{\epsilon}_{kg}\right) \\
&= \delta_{ij} - \frac{1}{2}\sum_{\eta} x_\eta(g)\left(\sum_k \gamma^k_{\eta i}\delta_{kj}\right) - \frac{1}{2}\sum_{\eta} x_\eta(g)\left(\sum_k \gamma^k_{\eta j}\delta_{ki}\right) \\
&\qquad + \frac{1}{4}\sum_{k,\sigma,l,\eta} x_\eta(g)\gamma^k_{\eta i} x_\sigma(g)\gamma^l_{\sigma_j}\delta_{kl} \\
&= \delta_{ij} - \frac{1}{2}\sum_{\eta} x_\eta(g)\gamma^j_{\eta i} - \frac{1}{2}\sum_{\eta} x_\eta(g)\gamma^i_{\eta j} + \frac{1}{4}\sum_{k,\eta,\sigma} x_\eta(g)x_\sigma(g)\gamma^k_{\eta,i}\gamma^k_{\sigma l}
\end{aligned} \tag{54}$$

In addition, the connection coefficients in (39) read:

$$\Gamma^j_{l,i}(g) = \frac{1}{2}\sum_s \bar{g}^{js}(g)\Big(\mathrm{E}_{ig}\bar{g}_{sl} - \mathrm{E}_{sg}\bar{g}_{li} + \mathrm{E}_{lg}\bar{g}_{is}\Big), \tag{55}$$

where

$$\begin{aligned}
E_{ig}\bar{g}_{sl} &= -\frac{1}{2}\gamma^l_{is} - \frac{1}{2}\gamma^s_{il} + O\left(x\right) \\
E_{sg}\bar{g}_{li} &= -\frac{1}{2}\gamma^i_{sl} - \frac{1}{2}\gamma^l_{si} + O\left(x\right) \\
E_{lg}\bar{g}_{is} &= -\frac{1}{2}\gamma^s_{li} - \frac{1}{2}\gamma^i_{ls} + O\left(x\right)
\end{aligned} \tag{56}$$

Unfortunately(39) needs the coefficients of the inverse Riemannian metric, and obtaining them from the elements of the metric is nothing but apparent. Therefore we are going to follow another argument.

Definition 1. *On a smooth manifold M, consider the differential operator $D : C^\infty(M) \to C^\infty(M)$. The operator D is called invariant under the diffeomorphism $\psi : M \to M$ iff $D_{\psi(p)} f = D_p f \circ \psi$.*

Next, we have the following very usefull proposition:

Proposition 1. *[Gallier, 2013, Proposition 16.2] If $\varphi \in C^\infty(M)$ is a local isometry, then: $\varphi \circ \mathbf{exp}_p = \mathbf{exp}_{\varphi(p)} \circ \varphi_*$*

and further an alternative definition of the Laplacian

Definition 2 ([Helgason, 1979])**.** *On a Riemannian manifold M the Laplace operator is defined as*

$$\Delta_p f := \sum_i \frac{\mathrm{d}^2}{\mathrm{d}t^2} f(\mathbf{exp}_p(tY_i))\Big|_{t=0}, \tag{57}$$

where $\{Y_i\} \in T_pM$ is any orthonormal basis of T_pM.

The Laplace operator 'measures' the local curvature of the real-valued map $f : M \to \mathbb{R}$ at the point $p \in M$ w.r.t. M. It does so by evaluating the function f along geodesics that initiate from the point $p \in M$ and have directions Y_i, thus obtaining functions $h(t) := f(\exp_p(tY_i))$ and compute their second derivative w.r.t. $t \in \mathbb{R}$. Note that by definition, the Laplacian is invariant w.r.t. to the orthonormal bases. In addition, Definition 2 in conjuction to Proposition 1 reveals directly the invariance of the Laplacian w.r.t. local isometries:

$$\begin{aligned}\Delta_{\varphi(p)} f &= \sum_i \frac{\mathrm{d}^2}{\mathrm{d}t^2} f(\mathbf{exp}_{\varphi(p)}(t\varphi_* Y_i))\Big|_{t=0} \\ &= \sum_i \frac{\mathrm{d}^2}{\mathrm{d}t^2} f(\varphi \circ \mathbf{exp}_p(tY_i))\Big|_{t=0} \\ &= \Delta_p(f \circ \varphi) \end{aligned} \tag{58}$$

On top of that, it is easy to show that for a Riemannian Lie group, with a left-invariang Riemannian metric, $\varphi = L_g$ is an isometry, and

$$\Delta_g f = \Delta_e(f \circ L_g).$$

That is, to derive the expression for the left-invariant Laplacian at a point $g \in G$, all we need is its value at the identity of the group. For $g = e$ in (51), (54), and (56) we obtain

$$\mathrm{E}_{je}\mathrm{E}_{ke} f = \bar{\epsilon}_{je}\bar{\epsilon}_{ke} f + 0,$$

$$\bar{g}^{i,j}(e) = \delta_{ij},$$

$$
\begin{aligned}
\Gamma^j_{l_1 i}(e) &= \frac{1}{2}\sum_s \delta_{js}\left(-\frac{1}{2}\gamma^l_{is} - \frac{1}{2}\gamma^s_{ie} + \frac{1}{2}\gamma^i_{si} + \frac{1}{2}\gamma^l_{si} - \frac{1}{2}\gamma^s_{li} - \frac{1}{2}\gamma^i_{ls}\right)\\
&= \frac{1}{2}\left(-\frac{1}{2}\gamma^l_{ij} - \frac{1}{2}\gamma^j_{il} + \frac{1}{2}\gamma^i_{jl} + \frac{1}{2}\gamma^l_{ji} - \frac{1}{2}\gamma^j_{li} - \frac{1}{2}\gamma^i_{li}\right)\\
&= \frac{1}{2}\left(\gamma^l_{ji} + \gamma^i_{jl}\right),
\end{aligned}
$$

respectively. Therefore,

$$
\Delta_e = -\sum_i \bar{\epsilon}_{ie}\bar{\epsilon}_{ie} + \sum_{i,j}\gamma^j_{ij}\bar{\epsilon}_{ie}.
$$

As a result, due to left-invariance, the left-invariant Laplacian on G reads:

$$
\Delta_g = -\sum_i \bar{\epsilon}_{ig}\bar{\epsilon}_{ig} + \sum_{i,j}[\mathrm{ad}(\epsilon_i)(\epsilon_j)]_j\bar{\epsilon}_{ig}. \tag{59}
$$

Note that (39) corresponds to the Riemannian Brownian motion on G which is given in local coordinates by either (33), or (38). On the contrary, (59) corresponds to the *left-invariant* Riemannian Brownian motion on G

$$
\mathrm{d}g_t = L_{g_t *}\Big(-\frac{1}{2}\sum_{i,j}\gamma^j_{ij}\epsilon_i \mathrm{d}t + \sum_i \epsilon_i(\hookleftarrow)\mathrm{d}\omega_{ti}\Big), \tag{60}
$$

where $\epsilon_i(\hookleftarrow)\mathrm{d}\omega_{ti}$ denotes the injection of the individual one-dimensional standard Brownian motion $\mathrm{d}\omega_{ti}$, into the corresponding basis vector ϵ_i of $\mathfrak{g}$. Note that (60) is based on the differentiable, Riemannian, *and algebraic* structure of the manifold, while (38) only on the differentiable and Riemannian structure.

1.5 On the special orthogonal group

The set

$$
M = \Big\{X \in \mathbb{R}^{n\times n} \ : \ \det(X) \neq 0\Big\}
$$

is open as the complement of the closed set $\Big\{X \in \mathbb{R}^{n\times n} \ : \ \det(X) = 0\Big\}$. On top of that, the continuity of det induces the standard topology from $\mathbb{R}$ to M.

Together with the open set M, and the empty set, the collection of the previously constructed open sets form a valid topology T, and thus a valid topological space (M, T). In addition, the map $e^{(\cdot)} : \mathbb{R}^{n\times n} \to M$, is onto M, and smooth. Therefore, the pre-image under $e^{(\cdot)}$ of any open set $U \in M$ is open in $\mathbb{R}^{n\times n}$ which makes $e^{(\cdot)}$ a local homomorphism, and therefore $(M, T, e^{(\cdot)})$ a topological manifold. The smooth covering of $(M, T, e^{(\cdot)})$ can be then established by smooth coordinate changes from $\mathbb{R}^{n^2}$ to itself.

On top of that, the pair $(M, \cdot)$, with the second entry referring to the standard matrix multiplication, is a group since for $X_1, X_2 \in M$, $\det(X_1 \cdot X_2) = \det(X_1)\det(X_2) \neq 0$ (the identity matrix serves as the identity of the group, and every element has a well-defined inverse).

In fact, the map $\mathrm{L}_{\bar{X}} : M \to M$ defined as $\mathrm{L}_{\bar{X}} \equiv \bar{X} \cdot X$ is smooth (since the entries of the value are rational functions of the entries), and the inverse of the matrix is also smooth since it can be written as $X^{-1} = \frac{1}{\det(X)}\mathrm{adj}(X)$. Thus, the object $(M, T, e^{(\ )}, \cdot)$ is a Lie group called *the general linear group*, and denoted as $\mathbb{GL}(\mathbb{R}, n)$

Moving forward, the set $F = \Big\{\mathrm{R} \in \mathbb{R}^{n\times n} : \mathrm{R}^\top \mathrm{R} = I,\ \det(\mathrm{R}) = +1\Big\}$ is a subset of M, and the pair $(F, \cdot)$ is a group (again $\cdot$ refers to the standard matrix multiplication). Therefore, the object $(F, T, e^{(\ )}, \cdot)$ is a (sub) Lie group (of $\mathbb{GL}(\mathbb{R}, n)$) called *the special orthogonal group*, denoted as $\mathbb{SO}(\mathbb{R}, n)$.

1.5.1 Tangent vectors

The tangent space T_eG of every Lie group G at the identity $e \in G$ is a vector space and it can always be equipped with an additional binary operator $\star$ so that the triple $\Big(T_eG, +, \cdot, \star\Big)$ is a Lie algebra. Clearly the tangent space at the identity of $\mathbb{GL}(\mathbb{R}, n)$ is $\mathbb{R}^{n\times n}$, and thus $\dim(\mathbb{GL}(\mathbb{R}, n)) = n^2$. To find the tangent space of the special orthogonal group, consider a smooth group homomorphism $\gamma : (\mathbb{R}, +) \to \mathbb{SO}(\mathbb{R}, n)$. We know that the isomorphism $\gamma_* : T_0\mathbb{R} \to T_I\mathbb{SO}(\mathbb{R}, n)$, and therefore, $\omega \equiv \dot{\gamma}(0) \in T_I\mathbb{SO}(\mathbb{R}, n)$. Clearly, $\gamma(t)\gamma(t)^\top = I$ for all $t \in \mathbb{R}$, and thus

$$\dot{\gamma}(0)\gamma(0)^\top + \gamma(0)\dot{\gamma}(0)^\top = 0,$$

or

$$\omega + \omega^\top = 0.$$

That is, $T_I\mathbb{SO}(\mathbb{R}, n) = \Big\{\omega \in \mathbb{R}^{n\times n}\ :\ \omega + \omega^\top = 0\Big\}$, and as a result $\dim(\mathbb{SO}(\mathbb{R}, n)) = \frac{n(n-1)}{2}$. Surprisingly, $\dim(\mathbb{SO}(\mathbb{R}, 3)) = 3$. It can be shown that $(T_I\mathbb{SO}(\mathbb{R}, n), +, \cdot, [\ ,\ ])$ is the Lie algebra denoted by $\mathfrak{so}(\mathbb{R}, n)$, where $[\ ,\ ] : T_I\mathbb{SO}(\mathbb{R}, n) \times T_I\mathbb{SO}(\mathbb{R}, n) \to T_I\mathbb{SO}(\mathbb{R}, n)$. In particular, for $n = 3$, the commutator is the known cross product between vectors (a lot of intuition can be gained from here).

1.5.2 Gradient flow

The tangent space at the identity, inherits the inner product from $\mathbb{R}^{n\times n}$.

$$\tau(X_e, Y_e) \equiv \mathrm{tr}(X_e Y_e^\top), \tag{61}$$

where $e = I$. Left-invariant Riemannian metrics are in 1-1 correspondence with the inner products in the Lie algebra. Define

$$\bar{g}(X_\mathrm{R}, Y_\mathrm{R})(\mathrm{R}) \equiv \tau(\mathrm{R}^\top X_\mathrm{R}, \mathrm{R}^\top Y_\mathrm{R}). \tag{62}$$

Then the gradient $\mathrm{grad}_\mathrm{R} f : \mathbb{SO}(\mathbb{R}, n) \to T\mathbb{SO}(\mathbb{R}, n)$ w.r.t. that left-invariant metric is the vector field for which

$$\bar{g}(\mathrm{grad}_\mathrm{R} f, Y_\mathrm{R})(\mathrm{R}) = \mathrm{d}f(Y_\mathrm{R})(\mathrm{R}). \tag{63}$$

In the following example, we compute the gradient of a real-valued map defined on the special orthogonal group

Example 1 (Brockett [1989], Tao and Ohsawa [2020]). *Consider the real-valued map* $f : \mathbb{SO}(\mathbb{R}, n) \to \mathbb{R}$*, defined as* $f(x) = \operatorname{tr}\Big(x^\top A x N\Big)$*, where* A *is symmetric, and* N *diagonal. It is*

$$
\begin{aligned}
\mathrm{d}f(Y_{\mathrm{R}}) &\equiv Y_{\mathrm{R}} f \\
&= Y_I (f \circ L_{\mathrm{R}}) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left[f(\mathrm{R}\cdot x)\Big|_{x=\Theta(t,I)} \right]\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big[f(\mathrm{R}\cdot I \cdot \gamma(t)) \Big]\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t} f(\mathrm{R}\cdot \gamma(t))\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t} f(\mathrm{R}\cdot \exp(tY_I))\Big|_{t=0},
\end{aligned} \tag{64}
$$

where $\exp(\cdot) = e^{(\cdot)}$*, and*

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} f(\mathrm{R}\cdot \exp(tY_e))\Big|_{t=0} &= \operatorname{tr}\Big(- Y_I \mathrm{R}^\top A \mathrm{R} N \Big) + \operatorname{tr}\Big(Y_I N \mathrm{R}^\top A \mathrm{R} \Big) \\
&= \operatorname{tr}\Big(Y_I (\mathrm{R}^\top A^\top \mathrm{R} N^\top - N \mathrm{R}^\top A \mathrm{R}) \Big) \\
&= \operatorname{tr}\Big(\mathrm{R}^\top Y_{\mathrm{R}} (\mathrm{R}^\top A^\top \mathrm{R} N^\top - N \mathrm{R}^\top A \mathrm{R}) \Big) \\
&= \operatorname{tr}\Big(Y_{\mathrm{R}} [\mathrm{R}(\mathrm{R}^\top A^\top \mathrm{R} N^\top - N \mathrm{R}^\top A \mathrm{R})]^\top \Big) \\
&= \operatorname{tr}\Big(\mathrm{R}^\top Y_{\mathrm{R}} [\mathrm{R}^\top \mathrm{R}(\mathrm{R}^\top A^\top \mathrm{R} N^\top - N \mathrm{R}^\top A \mathrm{R})]^\top \Big) \\
&= \tau(\mathrm{R}^\top \mathrm{R}(\mathrm{R}^\top A \mathrm{R} N - N \mathrm{R}^\top A \mathrm{R}), \mathrm{R}^\top Y_{\mathrm{R}}).
\end{aligned} \tag{65}
$$

Due to uniqueness of the gradient

$$
\operatorname{grad}_{\mathrm{R}} f = \mathrm{R}\Big(\mathrm{R}^\top A \mathrm{R} N - N \mathrm{R}^\top A \mathrm{R} \Big). \tag{66}
$$

Further note that $\omega_*^\top = \Big(\mathrm{R}^\top A \mathrm{R} N - N \mathrm{R}^\top A \mathrm{R}\Big)^\top = -\Big(\mathrm{R}^\top A \mathrm{R} N - N \mathrm{R}^\top A \mathrm{R}\Big) = -\omega_*$

The gradient flow reads

$$
\dot{\mathrm{R}} = -\mathrm{R}\Big(\mathrm{R}^\top A \mathrm{R} N - N \mathrm{R}^\top A \mathrm{R} \Big) \tag{67}
$$

The gradient is the left-translated version of a tangent vector at the identity.

1.5.3 Gradient Descent

As we already mentioned, for each $t \in \mathbb{R}$, $-\omega_*(t) \in \mathfrak{so}(3)$, and thus the unique one-parameter sub-group $\gamma(\tau) \equiv e^{-\tau\omega_*(t)}$, $\tau \in \mathbb{R}$, such that $\dot{\gamma}(0) = -\omega_*(t)$. Thus, for $\tau = t + h$, we obtain $\gamma(t+h) = e^{-(t+h)\omega_*(t)} = \gamma(t)e^{-h\omega_*(t)}$. Thus, the corresponding to (67) gradient descent reads:

$$\gamma(t+h) = \gamma(t)e^{-h\omega_*(t)}, \tag{68}$$

or

$$\mathrm{R}(t+h) = \mathrm{R}(t)e^{-h\omega_*(t)}, \tag{69}$$

where $\omega_*(t) = \mathrm{R}^\top(t)A\mathrm{R}(t)N - N\mathrm{R}^\top(t)A\mathrm{R}(t)$.

2 Conclusion

In these notes, we studied Brownian motion on Lie groups. Brownian motion is a metric property and together with the algebraic structure of the Lie group, they facilitate implementation through its associative Lie algebra.

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