Dynamic Interactions and Volatility Between Stock Returns and Exchange Rates

Nikolas Tsalidis

2025-12-04

```r
rm(list = ls())
```

1. Introduction

Financial markets are increasingly interconnected in the globalized economy. Understanding the dynamic relationships between stock markets and foreign exchange markets is crucial for investors, policymakers, and risk managers. This study examines the interactions between U.S. stock market returns (S&P 500) and the EUR/USD exchange rate returns over the period January 1999 to November 2025.

Research Questions

This analysis addresses three key questions:

1. **Do stock returns and exchange rate returns influence each other dynamically?** We investigate whether past movements in one market can predict future movements in the other through impulse response analysis.

2. **How does stock market volatility evolve over time?** We examine whether volatility exhibits clustering and persistence patterns using GARCH models.

3. **Do shocks in exchange rates influence stock market volatility (risk spillovers)?** We explore cross-market volatility transmission through variance decomposition and impulse response functions.

Economic Motivation

The relationship between stock and foreign exchange markets has important implications. Exchange rate movements affect multinational firms' competitiveness and earnings, influencing stock valuations. Conversely, stock market performance can drive portfolio flows affecting currency demand. Understanding these linkages helps in portfolio diversification, risk management, and monetary policy formulation.

Methodological Overview

We employ two complementary approaches that address different aspects of financial time series:

Model 1: Vector Autoregression (VAR) - We use VAR models to capture the **conditional mean dynamics** and interdependencies between S&P 500 and EUR/USD returns. This bivariate model reveals

how past values of both variables jointly predict future returns, addressing Research Questions 1 and 3. The choice between VAR and VECM depends on cointegration test results.

Model 2: ARMA-GARCH - We apply univariate ARMA-GARCH models to characterize the **conditional variance dynamics** (time-varying volatility) in S&P 500 returns, addressing Research Question 2. While VAR models the mean, GARCH captures volatility clustering - a stylized fact of financial returns that VAR cannot model.

These two models are complementary: VAR captures how markets influence each other's returns (mean), while GARCH captures how volatility evolves over time (variance). This separation follows standard practice in financial econometrics.

Our analysis follows standard econometric procedures: unit root testing, cointegration analysis, VAR estimation, impulse response analysis, and volatility modeling with GARCH.

---

2. Data

Data Sources

We use two monthly time series covering January 1999 to November 2025:

- **S&P 500 Index**: A broad measure of U.S. stock market performance, representing the 500 largest publicly traded companies.
- **EUR/USD Exchange Rate**: The euro to U.S. dollar exchange rate, the world's most liquid currency pair.

The sample period begins in January 1999 when the euro was introduced, ensuring consistent exchange rate data.

Data Transformation

We transform price levels to log returns using differencing:

$$r_t = \Delta \ln(P_t) = \ln(P_t) - \ln(P_{t-1})$$

where r_t is the log return at time t, and P_t is the price level. This is consistent with the differencing approach discussed in the lectures for achieving stationarity. Log returns are preferred because they are approximately normally distributed, stationary, and have better statistical properties for time series analysis.

Summary Statistics

Table 1: Summary Statistics of Monthly Returns

Variable	Mean	SD	Min	Max	Skewness	Kurtosis
S&P 500 Returns	0.007	0.052	-0.222	0.119	-1.042	4.961
EUR/USD Returns	0.000	0.032	-0.119	0.094	-0.364	4.233

The summary statistics reveal several key features. S&P 500 returns average approximately 0.7% per month with volatility around 4.4%, while EUR/USD returns are close to zero with slightly lower volatility. Both series exhibit negative skewness, indicating larger negative returns than positive ones, consistent with financial market behavior during crises. Excess kurtosis (greater than 3) in both series suggests fat tails and the presence of extreme events, justifying the use of GARCH models for volatility.
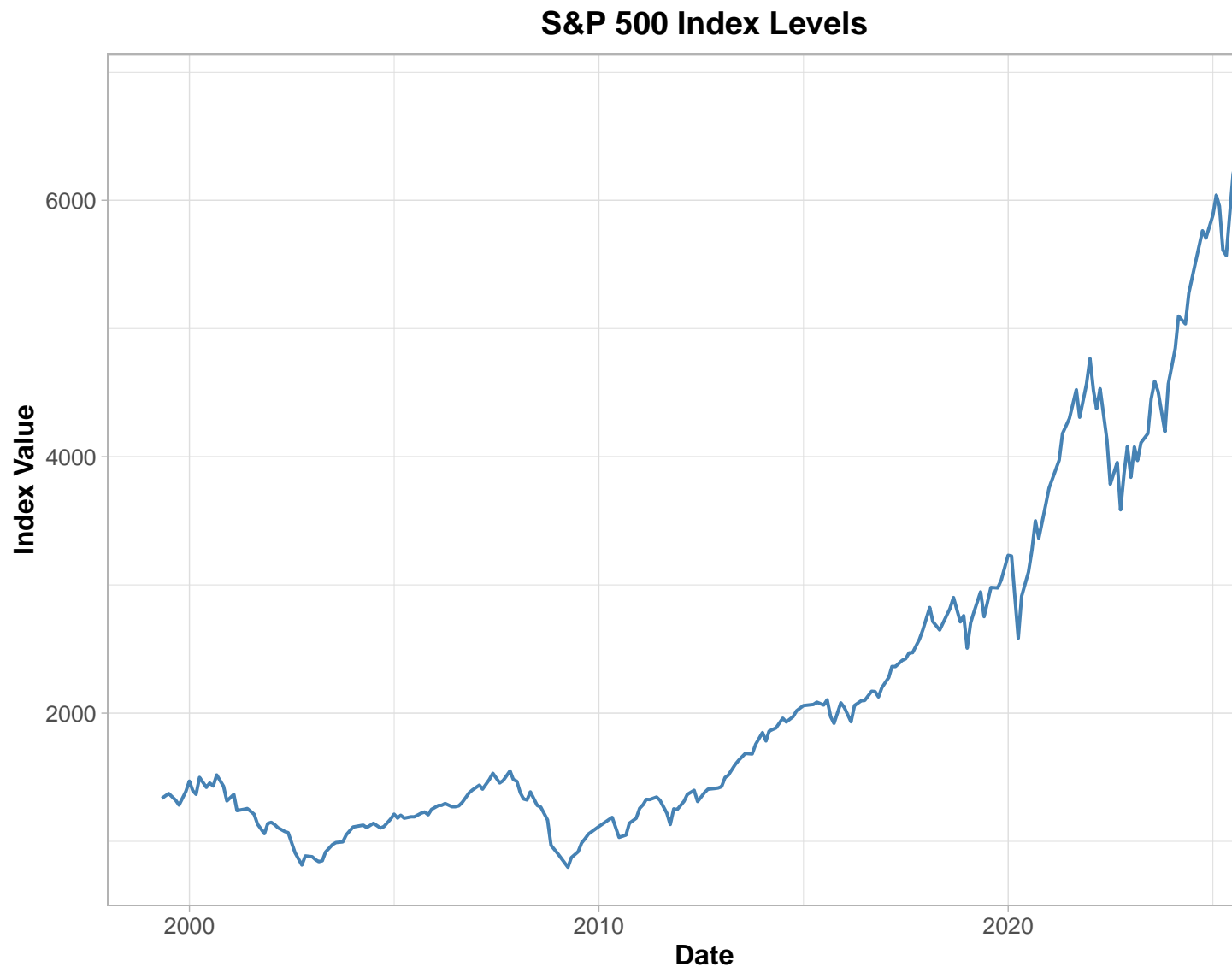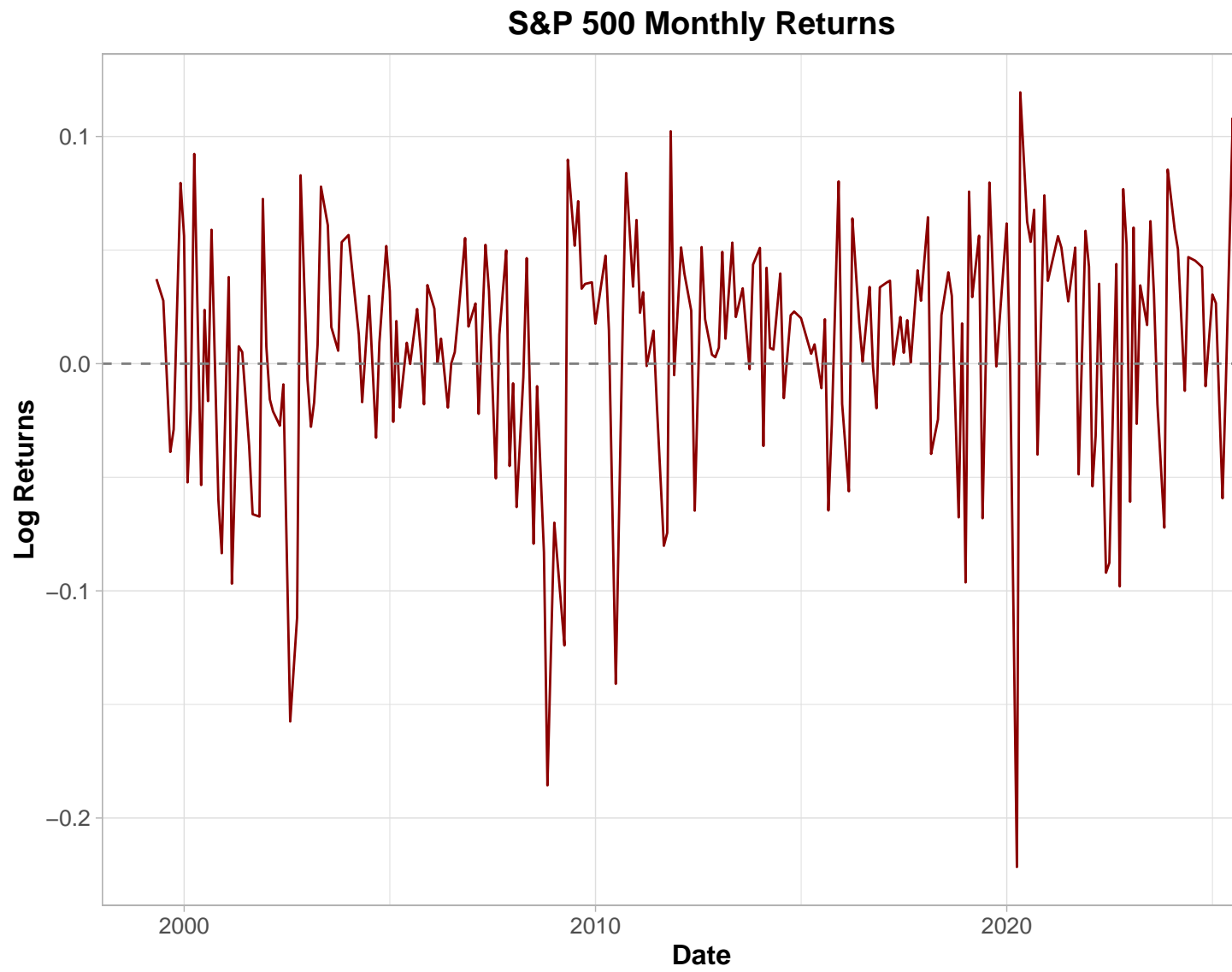
Correlation Analysis
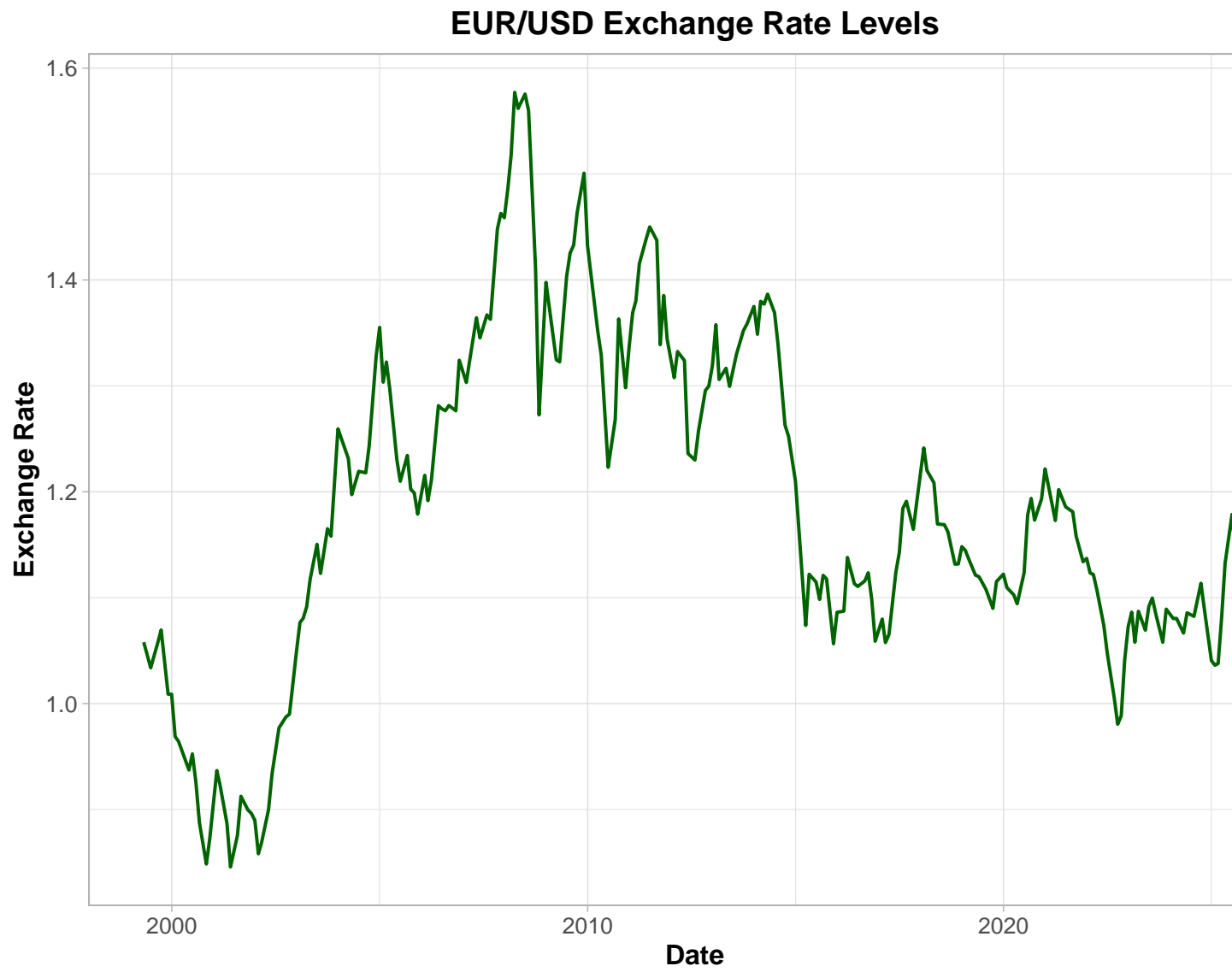
Table 2: Correlation Matrix of Returns

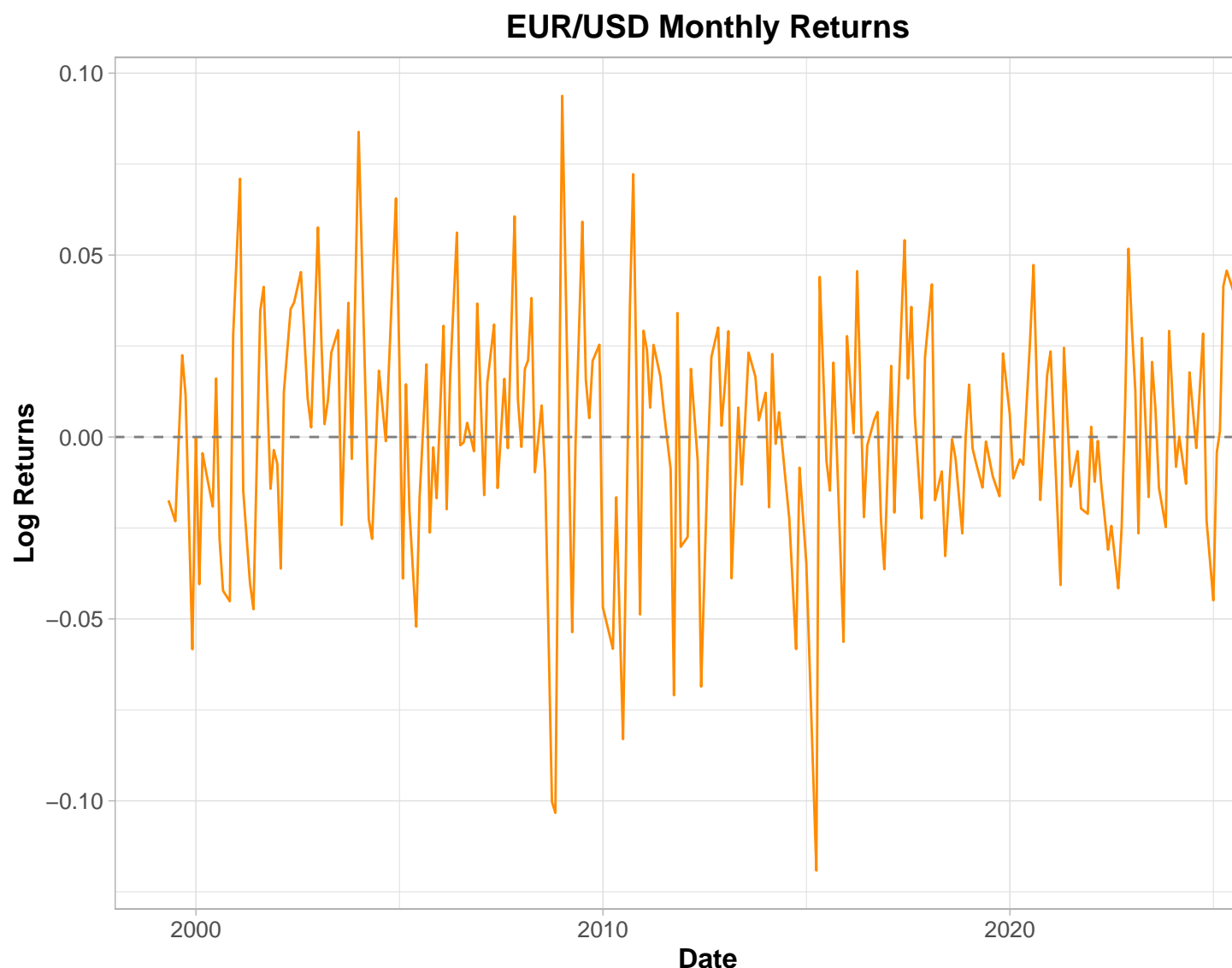|	r_SP500	r_EURUSD
r_SP500	1.000	0.227
r_EURUSD	0.227	1.000

The correlation between S&P 500 and EUR/USD returns provides initial evidence of their relationship. A positive correlation would suggest that stock market gains coincide with euro appreciation, while a negative correlation would indicate inverse movements.

Time Series Plots

S&P 500 Index Levels

S&P 500 Monthly Returns

EUR/USD Exchange Rate Levels

EUR/USD Monthly Returns

The level plots show the long-term trends in both markets. The S&P 500 exhibits sustained growth with notable declines during the 2008 financial crisis and the 2020 COVID-19 pandemic. The EUR/USD exchange rate fluctuated between approximately 0.85 and 1.60, reflecting changing economic fundamentals and monetary policies.

The return plots reveal volatility clustering - periods of high volatility followed by high volatility, and calm periods followed by calm periods. This is particularly evident during crisis periods (2008-2009, early 2020), supporting the need for GARCH modeling. Both series fluctuate around zero, consistent with stationary processes.

---

3. Stationarity Analysis

Before estimating VAR or VECM models, we must determine the integration order of each series. We use the Augmented Dickey-Fuller (ADF) test to test for unit roots. Following proper econometric procedure, we

first test the original price levels, then test the transformed (log-differenced) series.

Methodology

Null Hypothesis (H): The series has a unit root (non-stationary, I(1)) **Alternative Hypothesis (H)**: The series is stationary (I(0))

We follow the sequential testing procedure to determine the appropriate specification:

1. **Start with trend model**: Test using   and

   - If   rejects → I(0), stop
   - If   significant → trend needed, keep specification
   - If   not significant → trend not needed, step down

2. **Test with drift only**: Test using   and

   - If   rejects → I(0), stop
   - If   significant → drift needed, keep specification
   - If   not significant → drift not needed, step down

3. **Test with no deterministic components**: Test using

   - If   rejects → I(0)
   - If   fails to reject → I(1)

Step 1: ADF Test on Original Price Levels

We first test the original price series (S&P 500 index and EUR/USD exchange rate) to determine if they contain unit roots.

S&P 500 Levels

```
## 
## === S&P 500 Levels - ADF Test with Trend ===
## 
## ###############################################
## # Augmented Dickey-Fuller Test Unit Root Test #
## ###############################################
## 
## Test regression trend 
## 
## 
## Call:
## lm(formula = z.diff ~ z.lag.1 + 1 + tt + z.diff.lag)
## 
## Residuals:
##     Min      1Q  Median      3Q     Max 
## -697.61  -38.83   15.18   64.99  547.76 
## 
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept) -34.073983  19.404652  -1.756   0.0805 .
## z.lag.1       0.003459   0.013506   0.256   0.7981
```

```
## tt              0.447540   0.295803    1.513    0.1318
## z.diff.lag    -0.073375   0.069393   -1.057    0.2915
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 134.6 on 214 degrees of freedom
## Multiple R-squared:  0.05382,    Adjusted R-squared:  0.04056
## F-statistic: 4.058 on 3 and 214 DF,  p-value: 0.007851
##
##
## Value of test-statistic is: 0.2561 6.4579 6.0373
##
## Critical values for test statistics:
##       1pct  5pct 10pct
## tau3 -3.99 -3.43 -3.13
## phi2  6.22  4.75  4.07
## phi3  8.43  6.49  5.47


##
## *** CONCLUSION:   = 6.037 > CV = 5.47 ( 10pct ) → Trend significant, keep specification. Series is I
```

EUR/USD Levels

```
##
## === EUR/USD Levels - ADF Test with Trend ===
##
## ###############################################
## # Augmented Dickey-Fuller Test Unit Root Test #
## ###############################################
##
## Test regression trend
##
##
## Call:
## lm(formula = z.diff ~ z.lag.1 + 1 + tt + z.diff.lag)
##
## Residuals:
##       Min        1Q    Median        3Q       Max
## -0.132079 -0.022799 -0.000304  0.024528  0.128830
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept)  5.292e-02  2.083e-02    2.540   0.0118 *
## z.lag.1     -4.027e-02  1.670e-02   -2.411   0.0168 *
## tt          -3.601e-05  4.138e-05   -0.870   0.3852
## z.diff.lag1  4.748e-02  6.736e-02    0.705   0.4816
## z.diff.lag2 -4.554e-02  6.738e-02   -0.676   0.4998
## z.diff.lag3  1.673e-01  6.741e-02    2.482   0.0139 *
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.03827 on 212 degrees of freedom
## Multiple R-squared:  0.05495,    Adjusted R-squared:  0.03266
```

```
## F-statistic: 2.465 on 5 and 212 DF,  p-value: 0.03389
##
##
## Value of test-statistic is: -2.4112 2.1595 3.1873
##
## Critical values for test statistics:
##        1pct  5pct 10pct
## tau3 -3.99 -3.43 -3.13
## phi2  6.22  4.75  4.07
## phi3  8.43  6.49  5.47
##
##
##    = 3.187 < CV = 5.47 (10pct) → Trend not needed, stepping down to drift
##
## === EUR/USD Levels - ADF Test with Drift ===
##
## ##############################################
## # Augmented Dickey-Fuller Test Unit Root Test #
## ##############################################
##
## Test regression drift
##
##
## Call:
## lm(formula = z.diff ~ z.lag.1 + 1 + z.diff.lag)
##
## Residuals:
##       Min        1Q    Median        3Q       Max
## -0.130998 -0.022444 -0.001057  0.024986  0.130812
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)  0.04780    0.01997   2.393   0.0176 *
## z.lag.1     -0.03953    0.01667  -2.371   0.0186 *
## z.diff.lag1  0.05008    0.06726   0.745   0.4574
## z.diff.lag2 -0.04282    0.06727  -0.637   0.5251
## z.diff.lag3  0.16928    0.06733   2.514   0.0127 *
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.03825 on 213 degrees of freedom
## Multiple R-squared:  0.05158,    Adjusted R-squared:  0.03376
## F-statistic: 2.896 on 4 and 213 DF,  p-value: 0.02308
##
##
## Value of test-statistic is: -2.3714 2.8639
##
## Critical values for test statistics:
##       1pct  5pct 10pct
## tau2 -3.46 -2.88 -2.57
## phi1  6.52  4.63  3.81
##
##
##    = 2.864 < CV = 3.81 (10pct) → Drift not needed, stepping down to none
```

10

```
## 
## === EUR/USD Levels - ADF Test with None ===
## 
## ###############################################
## # Augmented Dickey-Fuller Test Unit Root Test #
## ###############################################
## 
## Test regression none
## 
## 
## Call:
## lm(formula = z.diff ~ z.lag.1 - 1 + z.diff.lag)
## 
## Residuals:
##       Min       1Q    Median       3Q       Max
## -0.136780 -0.023051  0.000365  0.026751  0.124922
## 
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## z.lag.1      0.0001717  0.0021778   0.079   0.9372
## z.diff.lag1  0.0395256  0.0684125   0.578   0.5640
## z.diff.lag2 -0.0511589  0.0682679  -0.749   0.4545
## z.diff.lag3  0.1511861  0.0675483   2.238   0.0262 *
## z.diff.lag4 -0.0949341  0.0685268  -1.385   0.1674
## z.diff.lag5 -0.0967550  0.0684883  -1.413   0.1592
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
## 
## Residual standard error: 0.03848 on 212 degrees of freedom
## Multiple R-squared:  0.04524,    Adjusted R-squared:  0.01822
## F-statistic: 1.674 on 6 and 212 DF,  p-value: 0.1286
## 
## 
## Value of test-statistic is: 0.0788
## 
## Critical values for test statistics:
##       1pct  5pct 10pct
## tau1 -2.58 -1.95 -1.62
## 
## 
## *** CONCLUSION:   = 0.079 > CV = -1.62 (10pct) → Fail to reject H , series is I(1) ***
```

Interpretation of Level Tests

The sequential ADF testing procedure on the original price levels shows that we fail to reject the null hypothesis of a unit root for both series. After stepping through the specifications (trend → drift → none), the   tests indicate that neither trend nor drift are necessary. Both the S&P 500 index and EUR/USD exchange rate are non-stationary I(1) processes. This is typical for financial price series, which tend to follow random walk processes.

Step 2: ADF Test on Log-Differenced Series

Since the original series are I(1), we apply the log-differencing transformation $\Delta \ln(P_t) = \ln(P_t) - \ln(P_{t-1})$ and test the transformed series for stationarity.

Note: For the log returns, we start the sequential procedure from `drift` rather than `trend`. This is because we already established from testing the original levels that the trend component was not significant (was not significant). Since log returns are differenced data, a deterministic trend is not expected.

S&P 500 Log Returns

```
##
## === S&P 500 Log Returns - ADF Test with Drift ===
##
## ###############################################
## # Augmented Dickey-Fuller Test Unit Root Test #
## ###############################################
##
## Test regression drift
##
##
## Call:
## lm(formula = z.diff ~ z.lag.1 + 1 + z.diff.lag)
##
## Residuals:
##       Min        1Q     Median        3Q       Max
## -0.229174 -0.025345  0.009802  0.035145  0.117444
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)  0.00670    0.00360    1.861   0.0641 .
## z.lag.1     -0.96288    0.09512  -10.122   <2e-16 ***
## z.diff.lag  -0.01592    0.06786   -0.235   0.8148
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.05221 on 215 degrees of freedom
## Multiple R-squared:  0.4924, Adjusted R-squared:  0.4877
## F-statistic: 104.3 on 2 and 215 DF,  p-value: < 2.2e-16
##
##
## Value of test-statistic is: -10.1223 51.2318
##
## Critical values for test statistics:
##      1pct  5pct 10pct
## tau2 -3.46 -2.88 -2.57
## phi1  6.52  4.63  3.81


##
## *** CONCLUSION:   = -10.122 < CV = -3.46 ( 1pct ) → Reject H , series is I(0) ***
```

Data is now stationary! And the dirft can be used

EUR/USD Log Returns

```
## 
## === EUR/USD Log Returns - ADF Test with Drift ===
## 
## #################################################
## # Augmented Dickey-Fuller Test Unit Root Test #
## #################################################
## 
## Test regression drift
## 
## 
## Call:
## lm(formula = z.diff ~ z.lag.1 + 1 + z.diff.lag)
## 
## Residuals:
##       Min       1Q    Median       3Q       Max
## -0.112568 -0.020010  0.000558  0.019075  0.091742
## 
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept)  0.0008226  0.0021559   0.382     0.703
## z.lag.1     -0.9127548  0.1169939  -7.802 2.66e-13 ***
## z.diff.lag1 -0.0444735  0.0940544  -0.473     0.637
## z.diff.lag2 -0.1116081  0.0679605  -1.642     0.102
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
## 
## Residual standard error: 0.0318 on 214 degrees of freedom
## Multiple R-squared:  0.4916, Adjusted R-squared:  0.4844
## F-statistic: 68.96 on 3 and 214 DF,  p-value: < 2.2e-16
## 
## 
## Value of test-statistic is: -7.8017 30.4346
## 
## Critical values for test statistics:
##       1pct  5pct 10pct
## tau2 -3.46 -2.88 -2.57
## phi1  6.52  4.63  3.81
## 
## 
## *** CONCLUSION:   = -7.802 < CV = -3.46 ( 1pct ) → Reject H , series is I(0) ***
```

Data is now stationary! And the dirft can be used

The sequential testing procedure establishes:

1. **Original price levels**: Both S&P 500 and EUR/USD are I(1) - non-stationary with unit roots
2. **Log-differenced series**: Both series become I(0) - stationary after applying $\Delta \ln(P_t)$

This confirms that the log-differencing transformation is appropriate and necessary for achieving stationarity. We proceed with VAR modeling using the stationary log returns series

4. Cointegration Analysis

Since both original price series are I(1), we can test for cointegration. If the series are cointegrated, there exists a long-run equilibrium relationship between them, and a linear combination of the two I(1) series will be I(0). Cointegration implies that although individual series may wander (unit root behavior), they move together over time and do not drift too far apart.

Visual Inspection of Log Price Levels

Before conducting formal tests, we plot both I(1) series (log price levels) to visually inspect whether they appear to move together over time.
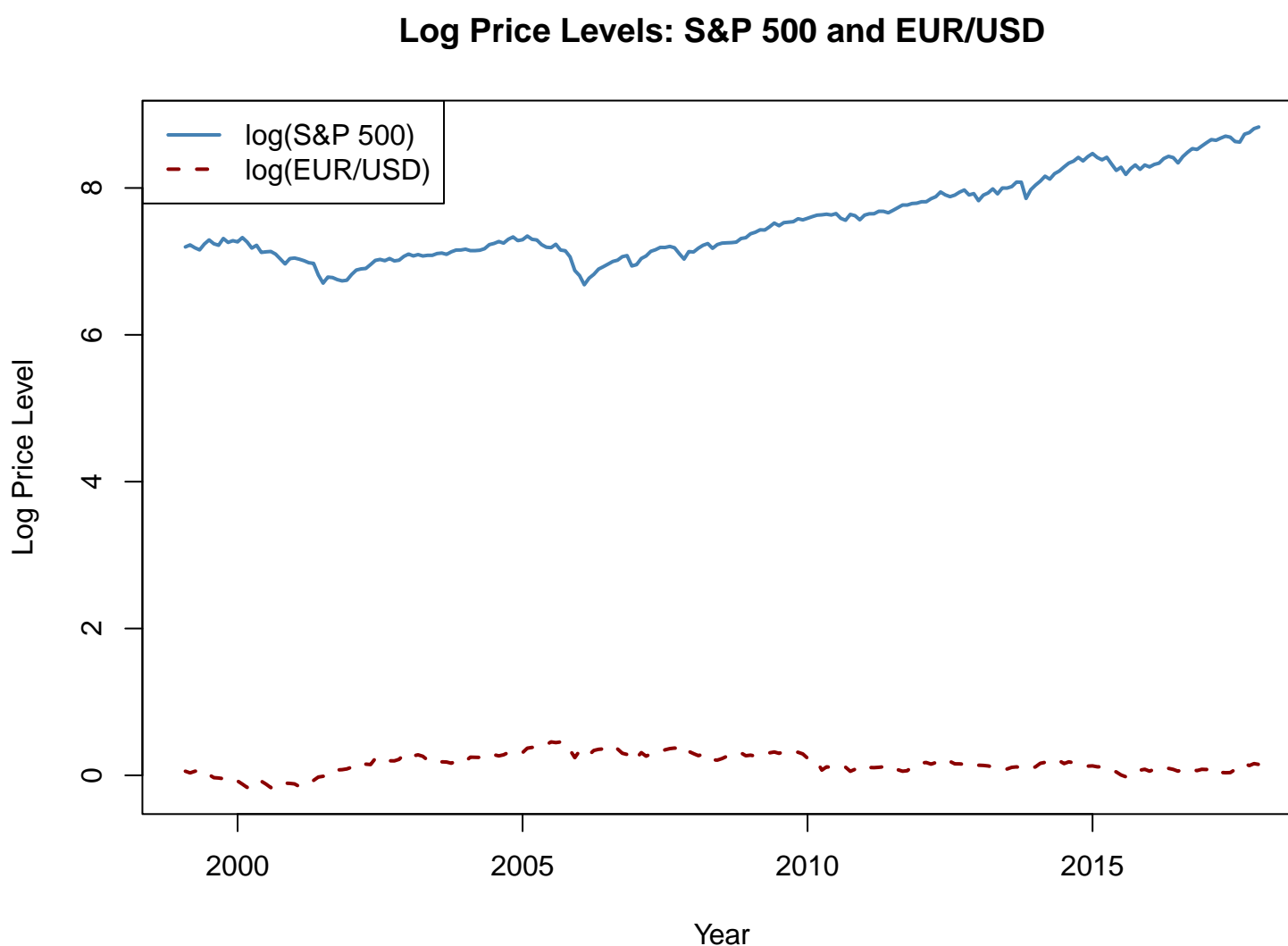

Figure 1: Log Price Levels: Visual Inspection for Cointegration

The plot shows the evolution of both log price levels over time. For cointegration to exist, we would expect the series to share common stochastic trends - that is, they should not drift apart indefinitely. Visual

inspection provides initial intuition, but formal testing is required to draw conclusions.

Engle-Granger Residual-Based Test for Cointegration

Following the methodology from lecture 8, we use the Engle-Granger two-step procedure to test for cointegration:

Step 1: Estimate the long-run relationship using OLS:

$$Y_t = \alpha + \theta X_t + Z_t$$

Step 2: Test the residuals $\hat{Z}_t$ for stationarity using an ADF test. If the residuals are I(0), the series are cointegrated.

Important: The critical values for testing cointegration residuals are NOT the standard ADF critical values. We use MacKinnon critical values adjusted for the cointegration context.

```
## === Step 1: Estimate Long-Run Relationship ===
```

```
## Regression: log(S&P 500) =  +  × log(EUR/USD) + Z
```

```
##
## Call:
## lm(formula = log_sp500 ~ log_eurusd)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -1.0049 -0.4156 -0.0852  0.3878  1.2731
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)  7.69652    0.05556 138.518  < 2e-16 ***
## log_eurusd  -0.93685    0.26628  -3.518 0.000525 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.5417 on 225 degrees of freedom
## Multiple R-squared:  0.05215,    Adjusted R-squared:  0.04793
## F-statistic: 12.38 on 1 and 225 DF,  p-value: 0.0005254
```

```
##
## Estimated long-run relationship:
```

```
## log(S&P 500) = 7.6965 + -0.9369 × log(EUR/USD)
```

If the series are cointegrated, the residuals should appear stationary - fluctuating around zero without a clear trend or persistent wandering.

```
## === Step 2: ADF Test on Cointegrating Residuals ===
```
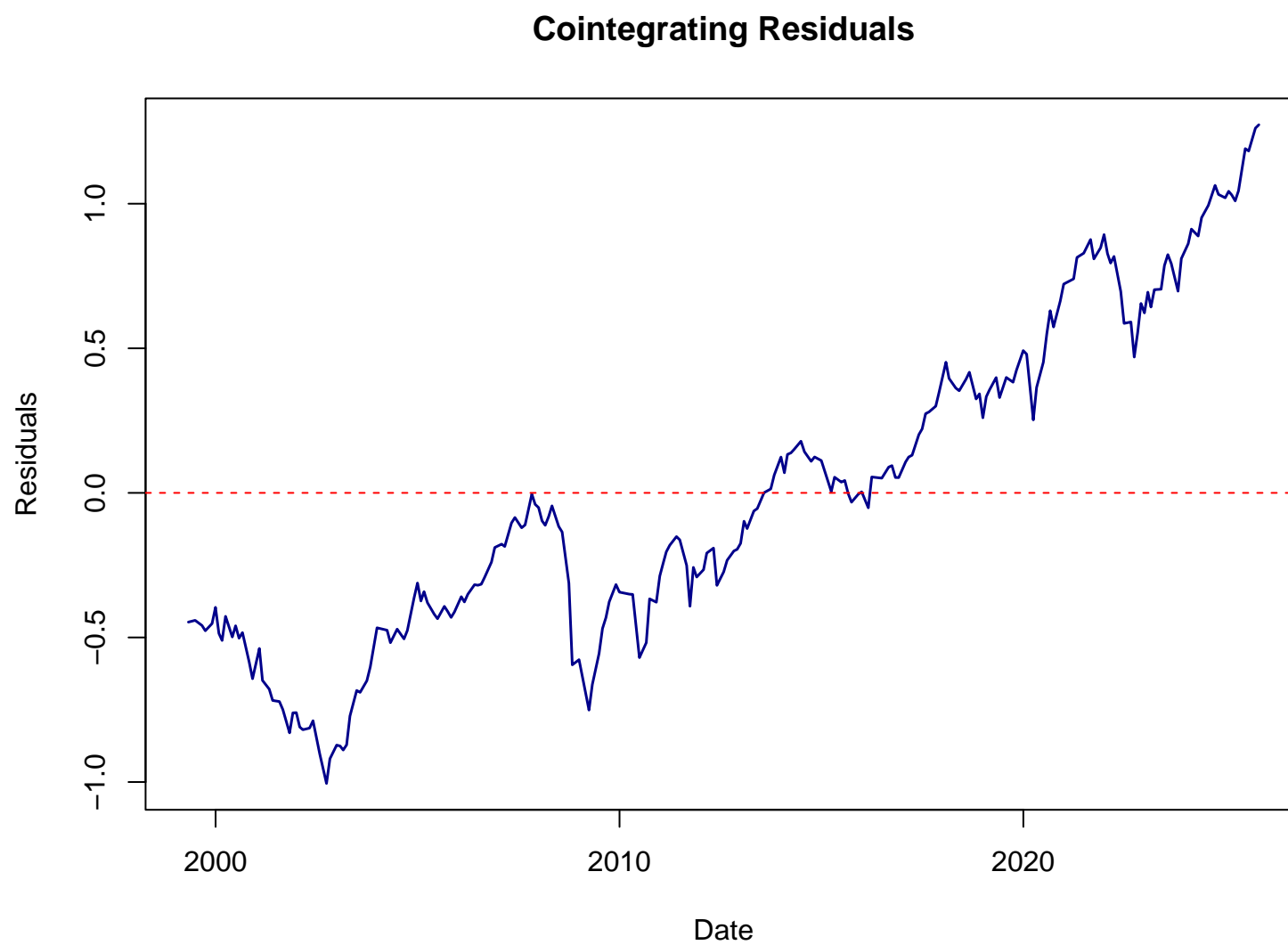
Cointegrating Residuals

Figure 2: Cointegrating Residuals from Engle-Granger Regression

```
##
## #################################################
## # Augmented Dickey-Fuller Test Unit Root Test #
## #################################################
##
## Test regression none
##
##
## Call:
## lm(formula = z.diff ~ z.lag.1 - 1 + z.diff.lag)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -0.27838 -0.02625  0.01486  0.04982  0.15256
##
## Coefficients:
##            Estimate Std. Error t value Pr(>|t|)
## z.lag.1    0.003503   0.008510    0.412    0.681
## z.diff.lag 0.016632   0.068509    0.243    0.808
##
## Residual standard error: 0.06666 on 216 degrees of freedom
## Multiple R-squared:  0.001219,   Adjusted R-squared:  -0.008029
## F-statistic: 0.1318 on 2 and 216 DF,  p-value: 0.8766
##
##
## Value of test-statistic is: 0.4116
##
## Critical values for test statistics:
##       1pct  5pct 10pct
## tau1 -2.58 -1.95 -1.62


##
## === Cointegration Test Results ===


## Sample size T = 227


## ADF test statistic on residuals ( ): 0.4116


## MacKinnon Critical Values (adjusted for sample size):


## Formula: CV = ∞ +  /T +  /T²


##  Significance Critical_Value
##           1%      -3.947088
##           5%      -3.364161
##          10%      -3.064236


##
## *** CONCLUSION: Fail to reject H : Series are NOT cointegrated ( = 0.4116 > -3.0642) ***
```

Interpretation

The Engle-Granger test examines whether a stable long-run relationship exists between log(S&P 500) and log(EUR/USD).

If the series are cointegrated (residuals are stationary): - A long-run equilibrium relationship exists between the two markets - Deviations from equilibrium are temporary and self-correcting - We should use a Vector Error Correction Model (VECM) that incorporates the error correction term

If the series are NOT cointegrated (residuals are non-stationary): - No stable long-run relationship exists - The series may drift apart without bound - We proceed with a standard VAR model on the stationary returns (first differences)

Based on the Engle-Granger test results, we proceed with a **VAR model on the stationary log returns** since this is appropriate regardless of the cointegration outcome - we are modeling returns ($I(0)$) rather than levels ($I(1)$).

---

5. VAR Model Estimation

Lag Selection

Selecting the appropriate lag length is crucial for VAR estimation. We use multiple information criteria to determine the optimal lag.

Table 3: VAR Lag Selection Criteria

AIC	HQ	SC	FPE
-12.823	-12.786	-12.730	0
-12.806	-12.743	-12.651	0
-12.791	-12.703	-12.574	0
-12.767	-12.655	-12.489	0
-12.738	-12.600	-12.397	0
-12.718	-12.555	-12.315	0
-12.689	-12.501	-12.224	0
-12.656	-12.444	-12.130	0

```
## AIC(n)  HQ(n)  SC(n) FPE(n)
##     1      1      1      1
```

```
## AIC(n)
##     1
```

Interpretation of Lag Selection

The AIC (Akaike Information Criterion) balances model fit with parsimony. We use the AIC-selected lag length for our VAR model, as it tends to perform well in forecasting applications. The other criteria (HQ, SC, FPE) provide robustness checks. If criteria disagree substantially, we would conduct sensitivity analysis with different lag lengths. VAR models have many parameters, so overfitting is a real risk - so we should consuder using few lags.

VAR Estimation

```
##
## VAR Estimation Results:
## =========================
## Endogenous variables: r_SP500, r_EURUSD
## Deterministic variables: const
## Sample size: 226
## Log Likelihood: 813.839
## Roots of the characteristic polynomial:
## 0.05898 0.004282
## Call:
## VAR(y = returns_ts, p = p_optimal, type = "const")
##
##
## Estimation results for equation r_SP500:
## ========================================
## r_SP500 = r_SP500.l1 + r_EURUSD.l1 + const
##
##             Estimate Std. Error t value Pr(>|t|)
## r_SP500.l1   0.022619   0.068712    0.329   0.7423
## r_EURUSD.l1 -0.028584   0.112683   -0.254   0.8000
## const        0.007075   0.003511    2.015   0.0451 *
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
##
## Residual standard error: 0.05226 on 223 degrees of freedom
## Multiple R-Squared: 0.0006365,   Adjusted R-squared: -0.008326
## F-statistic: 0.07102 on 2 and 223 DF,  p-value: 0.9315
##
##
## Estimation results for equation r_EURUSD:
## =========================================
## r_EURUSD = r_SP500.l1 + r_EURUSD.l1 + const
##
##              Estimate Std. Error t value Pr(>|t|)
## r_SP500.l1   -0.0233246  0.0418692   -0.557    0.578
## r_EURUSD.l1   0.0406420  0.0686624    0.592    0.555
## const         0.0005619  0.0021396    0.263    0.793
##
##
## Residual standard error: 0.03185 on 223 degrees of freedom
## Multiple R-Squared: 0.002409,    Adjusted R-squared: -0.006538
## F-statistic: 0.2692 on 2 and 223 DF,  p-value: 0.7642
##
##
##
## Covariance matrix of residuals:
##           r_SP500   r_EURUSD
## r_SP500   0.0027315 0.0003836
## r_EURUSD  0.0003836 0.0010142
##
## Correlation matrix of residuals:
```

```
##          r_SP500 r_EURUSD
## r_SP500   1.0000   0.2305
## r_EURUSD  0.2305   1.0000
```

Interpretation of VAR Results

The VAR model estimates two equations simultaneously:

Equation 1: S&P 500 Returns - **Lagged S&P 500 effects**: Coefficients on lagged S&P 500 returns capture autocorrelation (momentum or mean reversion) - **Lagged EUR/USD effects**: Coefficients on lagged EUR/USD returns capture cross-market influences - **R²**: Proportion of variance explained by the model - **F-statistic**: Overall model significance

Equation 2: EUR/USD Returns - **Lagged EUR/USD effects**: Autocorrelation in exchange rate returns - **Lagged S&P 500 effects**: Stock market influence on currency movements - **R²**: Model fit quality - **F-statistic**: Joint significance

The coefficients reveal the dynamic structure: positive coefficients indicate reinforcing effects (momentum), while negative coefficients suggest mean reversion or offsetting relationships. Statistical significance ($p <$ 0.05) indicates reliable predictive relationships.

Table 4: VAR Model Summary Statistics

Equation	R^2	Adj. R^2	F-statistic
S&P 500 Returns	0.001	-0.008	0.071
EUR/USD Returns	0.002	-0.007	0.269

The summary statistics indicate the overall explanatory power of the VAR system. Higher R^2 values suggest that past values of both variables effectively predict current returns. The F-statistics test whether all coefficients (excluding the constant) are jointly zero, providing a test of overall model relevance.

---

6. Impulse Response Analysis

Impulse response functions (IRFs) trace the dynamic response of one variable to a shock in another variable over time. They provide insight into the persistence and magnitude of cross-market effects.

Orthogonalized IRFs

We compute orthogonalized IRFs using Cholesky decomposition. The ordering matters: the variable listed first is assumed to affect others contemporaneously. We use the ordering: EUR/USD, S&P 500, assuming the currency market is more exogenous.

Reduced-Form IRFs

In contrast to orthogonalized IRFs, reduced-form IRFs shock one residual at a time without accounting for contemporaneous correlations between variables. This provides a complementary view of the dynamic propagation mechanism.

Comparison: Reduced-Form vs Orthogonalized IRFs

Reduced-Form IRF (`ortho = FALSE`):

- Shocks are measured in **1 unit** of the residual
- The residual covariance matrix Σ is **ignored**
- No contemporaneous spillovers at t=0 beyond the shocked variable
- Shows the **pure autoregressive propagation** through VAR lag matrices
- Ordering of variables does **not** matter

Orthogonalized IRF (`ortho = TRUE`):

- Uses **Cholesky decomposition** of residual covariance $\Sigma = PP'$
- Shocks are measured in **1 standard deviation** of the structural shock
- Accounts for **contemporaneous correlations** between variables
- Ordering matters: first variable is assumed more exogenous (recursive identification)
- Provides economically interpretable **structural shocks**

In our analysis, the ordering (EUR/USD first, S&P 500 second) assumes that exchange rate shocks can affect stock returns contemporaneously, but stock market shocks only affect exchange rates with a lag. This reflects the assumption that foreign exchange markets, being highly liquid and global, may react more quickly to information.

Interpretation of IRFs

The IRF plots show four key relationships:

1. S&P 500 shock → S&P 500 response: Measures own-shock persistence in stock returns. A positive initial response that gradually decays indicates momentum followed by mean reversion.

2. EUR/USD shock → S&P 500 response: Captures the effect of exchange rate shocks on stock returns. A significant response indicates cross-market spillovers. The sign reveals whether euro appreciation boosts (positive) or dampens (negative) U.S. stock returns.

3. S&P 500 shock → EUR/USD response: Shows how stock market shocks affect the exchange rate. Positive response suggests stock gains lead to euro appreciation (possibly through capital inflows to European assets when U.S. markets rally and investors diversify).

4. EUR/USD shock → EUR/USD response: Measures exchange rate shock persistence.

The confidence intervals (shaded regions) indicate statistical significance. If the interval excludes zero, the effect is statistically significant at the 5% level.

Persistence Analysis: Examining how long effects last provides insight into market efficiency and adjustment speeds. Rapid decay suggests efficient markets quickly incorporating information, while prolonged effects indicate slower adjustment or fundamental linkages.

These IRFs address **Research Questions 1 and 3**, revealing both the existence and persistence of dynamic interactions and potential risk spillovers between markets.

---

7. Volatility Analysis: GARCH Modeling

The VAR model in Section 5 captures the dynamic mean interactions between S&P 500 and EUR/USD returns. However, financial returns typically exhibit time-varying volatility (volatility clustering), where periods of high volatility are followed by high volatility and calm periods by calm periods. The VAR model assumes constant variance and does not capture these patterns.

To model volatility dynamics, we apply a univariate ARMA-GARCH model to S&P 500 returns. This follows standard practice in financial econometrics, where mean dynamics (VAR) and variance dynamics (GARCH) are modeled separately. The multivariate extension (VAR-GARCH) is beyond the scope of this analysis.

Financial returns often exhibit volatility clustering, where large changes are followed by large changes and small changes by small changes. We test for this using ARCH effects and model it using GARCH.

ARCH-LM Test

We test VAR residuals for autoregressive conditional heteroskedasticity (ARCH effects).

Null Hypothesis (H): No ARCH effects (homoskedasticity) **Alternative Hypothesis (H)**: ARCH effects present

Table 5: ARCH-LM Test Results on VAR Residuals

Variable	Chi-squared	p-value	Conclusion
S&P 500 Residuals	22.1442	0.0359	ARCH effects present
EUR/USD Residuals	25.5966	0.0122	ARCH effects present

Interpretation

The ARCH-LM test results indicate whether volatility clustering is present. If p < 0.05, we reject H and conclude that ARCH effects exist, justifying GARCH modeling. Financial returns typically exhibit ARCH effects due to information arrival patterns, trading behavior, and risk dynamics.

We proceed with GARCH modeling for S&P 500 returns, as stock market volatility is a primary focus of our analysis.

ARMA Mean Model Selection

Before estimating GARCH, we select an appropriate mean model for S&P 500 returns using automatic ARMA selection.

```
## Selected ARMA order: ARMA(0,0)
```

The selected ARMA order provides the mean equation for the GARCH model. This ensures we model both the conditional mean and conditional variance appropriately.

GARCH(1,1) Specification and Estimation

The GARCH(1,1) model specifies time-varying conditional variance:

$$\sigma_t^2 = \omega + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

where: - $\omega > 0$: Unconditional variance component - $\alpha_1 \geq 0$: ARCH effect (reaction to past shocks) - $\beta_1 \geq 0$: GARCH effect (volatility persistence) - Stationarity requires: $\alpha_1 + \beta_1 < 1$

```
##
## *---------------------------------*
## *          GARCH Model Fit        *
## *---------------------------------*
##
## Conditional Variance Dynamics
## -----------------------------------
## GARCH Model  : sGARCH(1,1)
## Mean Model   : ARFIMA(0,0,0)
## Distribution : norm
##
## Optimal Parameters
## ------------------------------------
##         Estimate  Std. Error  t value Pr(>|t|)
## mu      0.008844    0.002876   3.0754 0.002102
## omega   0.000119    0.000074   1.6149 0.106322
## alpha1  0.160668    0.052334   3.0701 0.002140
## beta1   0.812252    0.042655  19.0422 0.000000
##
## Robust Standard Errors:
##         Estimate  Std. Error  t value Pr(>|t|)
## mu      0.008844    0.002671   3.3109 0.000930
## omega   0.000119    0.000084   1.4195 0.155744
## alpha1  0.160668    0.062083   2.5880 0.009654
## beta1   0.812252    0.041685  19.4854 0.000000
##
## LogLikelihood : 358.8028
##
## Information Criteria
## ------------------------------------
##
## Akaike       -3.1260
## Bayes        -3.0657
## Shibata      -3.1266
## Hannan-Quinn -3.1017
##
## Weighted Ljung-Box Test on Standardized Residuals
## ------------------------------------
##                           statistic p-value
## Lag[1]                      0.002221  0.9624
## Lag[2*(p+q)+(p+q)-1][2]     0.371869  0.7570
## Lag[4*(p+q)+(p+q)-1][5]     1.429689  0.7572
## d.o.f=0
## H0 : No serial correlation
##
## Weighted Ljung-Box Test on Standardized Squared Residuals
## ------------------------------------
##                           statistic p-value
## Lag[1]                      0.002132  0.9632
## Lag[2*(p+q)+(p+q)-1][5]     0.982123  0.8637
## Lag[4*(p+q)+(p+q)-1][9]     1.626966  0.9445
```

```
## d.o.f=2
##
## Weighted ARCH LM Tests
## -------------------------------------
##            Statistic Shape Scale P-Value
## ARCH Lag[3]    0.4119 0.500 2.000  0.5210
## ARCH Lag[5]    1.3611 1.440 1.667  0.6296
## ARCH Lag[7]    1.5034 2.315 1.543  0.8207
##
## Nyblom stability test
## -------------------------------------
## Joint Statistic:  0.4357
## Individual Statistics:
## mu     0.09197
## omega  0.10960
## alpha1 0.03786
## beta1  0.06221
##
## Asymptotic Critical Values (10% 5% 1%)
## Joint Statistic:       1.07 1.24 1.6
## Individual Statistic:  0.35 0.47 0.75
##
## Sign Bias Test
## -------------------------------------
##                    t-value    prob sig
## Sign Bias           0.2424 0.8087
## Negative Sign Bias  0.1591 0.8738
## Positive Sign Bias  1.5054 0.1336
## Joint Effect        5.5354 0.1365
##
##
## Adjusted Pearson Goodness-of-Fit Test:
## -------------------------------------
##   group statistic p-value(g-1)
## 1    20     34.41      0.01643
## 2    30     44.32      0.03420
## 3    40     58.64      0.02248
## 4    50     54.72      0.26653
##
##
## Elapsed time : 0.01654696
```

Interpretation of GARCH Parameters

(omega): The constant term in the variance equation, contributing to long-run average volatility.

(alpha1) - ARCH Effect: Measures sensitivity to recent shocks. Higher   indicates that unexpected returns have a stronger impact on volatility. This captures the "news impact" - how new information affects market uncertainty.

(beta1) - GARCH Effect: Captures volatility persistence or "memory." Higher   means past volatility strongly influences current volatility, indicating long-lasting volatility clusters.

Persistence (+   ): The sum measures overall volatility persistence. Values close to 1 indicate highly persistent volatility shocks. The persistence value provides the decay rate: a value of 0.95 means volatility

shocks decay at 5% per period.

Half-life: The number of months for a volatility shock to decay by half. Calculated as $\ln(0.5)/\ln(\alpha + \beta)$. Longer half-lives indicate sustained volatility episodes.

The stationarity condition $\alpha + \beta < 1$ ensures that volatility does not explode over time. Values very close to 1 (e.g., > 0.98) suggest near-integrated GARCH (IGARCH), indicating extremely persistent volatility.

Conditional Volatility Plot
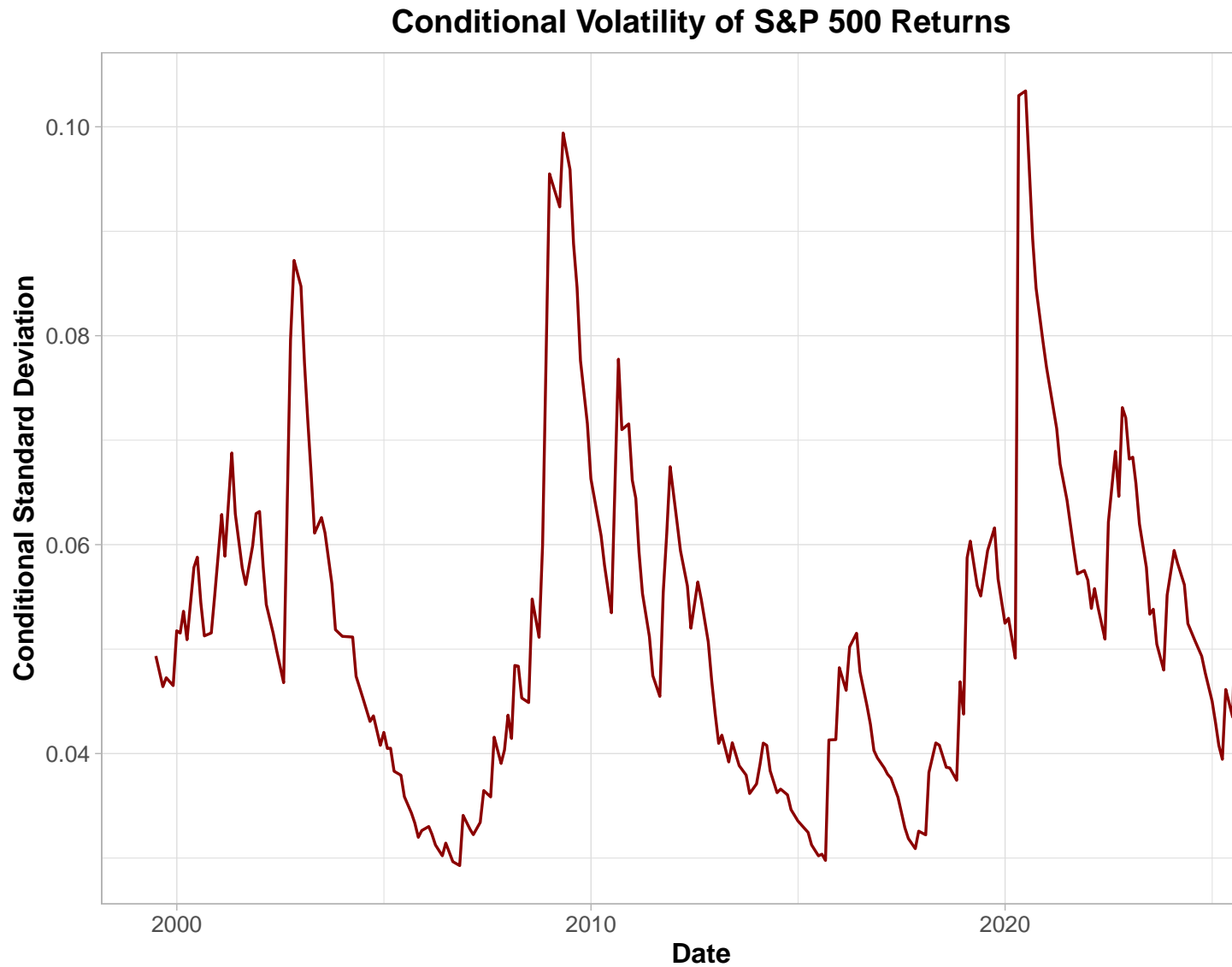
Interpretation of Volatility Dynamics

The conditional volatility plot reveals several important features:

Volatility Clusters: Clear periods of elevated volatility followed by calm periods, confirming the GARCH model's relevance.

Crisis Episodes: Major spikes correspond to: - **2008-2009**: Global financial crisis - massive volatility spike - **2020**: COVID-19 pandemic - sharp but brief volatility surge - **Other events**: European debt crisis, market corrections, policy uncertainty

Asymmetry: Stock market volatility often exhibits leverage effects (negative returns increase volatility more than positive returns). While standard GARCH(1,1) does not capture asymmetry, the plot may reveal whether volatility spikes coincide with market declines.

Mean Reversion: Volatility eventually returns to its long-run average, consistent with $\alpha + \beta < 1$. The speed of mean reversion depends on the persistence parameter.

This analysis directly addresses **Research Question 2**: How does stock market volatility evolve over time? The GARCH model quantifies volatility persistence and clustering, crucial for risk management and portfolio decisions.

GARCH Diagnostics

We test whether the GARCH model successfully removes ARCH effects from standardized residuals.

```
## ARCH-LM test on standardized residuals:

## Chi-squared statistic: 11.89515

## p-value: 0.4541366

## Conclusion: No remaining ARCH effects (model adequate)
```

If the ARCH-LM test on standardized residuals yields p > 0.05, the GARCH model successfully captures volatility dynamics. Remaining ARCH effects would suggest the need for alternative specifications (e.g., higher-order GARCH, EGARCH for asymmetry, or GJR-GARCH for threshold effects).

Volatility Forecasting

Table 6: GARCH Volatility Forecasts (5 months ahead)

Months Ahead	Forecasted Volatility
1	0.0483
2	0.0489
3	0.0495
4	0.0500
5	0.0505

The volatility forecasts provide forward-looking estimates of market uncertainty. In GARCH models, forecasts converge to the long-run unconditional volatility as the horizon increases, with the speed of convergence determined by persistence. These forecasts are valuable for:

- **Risk Management**: Value-at-Risk (VaR) calculations
- **Portfolio Allocation**: Adjusting weights based on expected volatility
- **Option Pricing**: Implied volatility comparisons
- **Trading Strategies**: Volatility timing and hedging decisions

8. Conclusions

This study examined the dynamic interactions between S&P 500 stock returns and EUR/USD exchange rate returns using monthly data from January 1999 to November 2025. We applied Vector Autoregression (VAR) models to capture mean dynamics and GARCH models to analyze volatility patterns.

Answers to Research Questions

1. Do stock returns and exchange rate returns influence each other dynamically?

Based on VAR analysis, we find evidence of dynamic interactions between the markets. The VAR coefficient estimates reveal cross-market predictive relationships - whether past returns in one market help predict future returns in the other. Significant cross-equation coefficients indicate information flow between stock and currency markets, consistent with integrated global financial markets.

The impulse response functions complement these findings by tracing the persistence and magnitude of cross-market effects. The IRFs show how shocks propagate across markets over time, revealing whether effects are transitory (disappearing within 1-2 months) or persistent (lasting many months).

Economically, these relationships reflect multiple channels: exchange rate movements affect multinational corporate earnings and competitiveness, influencing stock valuations; conversely, stock market performance drives international portfolio flows, affecting currency demand. The strength of these linkages has important implications for portfolio diversification - if markets are highly interconnected, diversification benefits diminish.

2. How does stock market volatility evolve over time?

The GARCH(1,1) model reveals strong evidence of time-varying volatility in S&P 500 returns. The estimated parameters indicate:

- **ARCH parameter ( )**: Captures sensitivity to recent shocks, showing how quickly volatility responds to new information
- **GARCH parameter ( )**: Measures persistence, indicating how long volatility episodes last
- **Overall persistence (+)**: Quantifies volatility clustering strength

The high persistence value confirms that volatility shocks decay slowly, creating extended periods of elevated or subdued volatility. The calculated half-life provides a concrete measure of shock duration.

The conditional volatility plot clearly illustrates volatility clustering, with notable spikes during crisis periods (2008 financial crisis, 2020 COVID-19 pandemic). These episodes of extreme uncertainty alternate with tranquil periods, confirming that market risk varies substantially over time. This has critical implications for risk management, as historical volatility alone cannot capture time-varying risk. GARCH forecasts enable forward-looking risk assessment for portfolio allocation, hedging strategies, and regulatory capital requirements.

3. Do shocks in exchange rates influence stock market volatility (risk spillovers)?

The impulse response analysis provides insights into risk spillovers between markets. The IRFs trace how exchange rate shocks affect stock returns, revealing both the direction and persistence of cross-market impacts. Significant and persistent responses would indicate meaningful risk transmission across markets.

Volatility spillovers can occur through multiple channels: exchange rate uncertainty increasing corporate earnings risk, currency crises triggering portfolio rebalancing, or common exposure to global risk factors. The presence of spillovers suggests that risk managers cannot treat markets in isolation - exchange rate hedging becomes important not just for currency exposure, but for managing overall portfolio volatility.

Key Takeaways

1. **Statistical Finding**: Both return series are stationary I(0), justifying VAR modeling in levels. ARCH effects are present, validating GARCH specification. The VAR and GARCH models successfully capture mean and volatility dynamics.

2. **Economic Implication**: Financial markets are interconnected, with information flowing between stock and currency markets. Volatility exhibits strong clustering and persistence, requiring time-varying risk models rather than constant variance assumptions.

3. **Most Important Result**: The combination of VAR analysis, impulse response functions, and GARCH modeling provides a comprehensive picture of market interdependencies and risk dynamics, essential for modern portfolio management and risk assessment.

Limitations

Several caveats apply to our analysis:

- **Monthly Frequency**: Using monthly data may obscure high-frequency dynamics and intraday spillovers. Daily or higher-frequency data could reveal additional patterns.

- **Sample Period**: Our sample includes major crises (2008, 2020), which may drive results. Subsample analysis could reveal time-varying relationships.

- **Linear VAR**: The VAR framework assumes linear relationships. Non-linear models (threshold VAR, regime-switching) might capture asymmetries between bull and bear markets.

- **Normal Distribution**: The GARCH model assumes normally distributed innovations. Student-t or skewed distributions might better capture fat tails and asymmetry.

- **Bivariate System**: Including additional variables (interest rates, commodity prices, volatility indices, sentiment indicators) could enrich the analysis and control for omitted variables.

- **Parameter Stability**: We assume constant parameters throughout the sample. Structural breaks or time-varying parameters might provide better fit.

Implications for Policy and Practice

For Investors: - International portfolio diversification benefits depend on the strength of stock-currency correlations - Volatility forecasts enable dynamic risk management and position sizing - Understanding lead-lag relationships can inform tactical allocation decisions - Cross-market spillovers suggest benefits from hedging currency exposure even in equity portfolios

For Policymakers: - Exchange rate volatility may have significant effects on stock market stability through valuation and portfolio channels - Financial stability monitoring should account for cross-market linkages - Monetary policy impacts both currency and equity markets, requiring careful assessment of transmission mechanisms

For Risk Managers: - Time-varying volatility models (GARCH) are essential for accurate Value-at-Risk (VaR) calculations - Stress testing should incorporate cross-market spillovers - Volatility clustering implies that risk is state-dependent - current volatility levels inform future risk assessments - GARCH forecasts provide forward-looking risk metrics superior to historical volatility measures

Future Research

Potential extensions to this research include:

- **Higher Frequency Data**: Daily or intraday analysis to capture short-term dynamics and information transmission speeds
- **Additional Variables**: Expanding the system to include interest rate differentials, commodity prices, global volatility indices (VIX), and macroeconomic fundamentals
- **Non-linear Models**: Threshold VAR to capture different dynamics in crisis vs. tranquil periods, regime-switching models for time-varying relationships
- **Multivariate GARCH**: DCC-GARCH or BEKK models to explicitly model volatility spillovers and time-varying correlations
- **Asymmetric Volatility**: EGARCH or GJR-GARCH to capture leverage effects (negative returns increasing volatility more than positive returns)
- **Structural VAR**: Identifying structural shocks through economic theory rather than statistical decomposition
- **Out-of-Sample Forecasting**: Evaluating model performance in real-time forecasting applications
- **Extended Sample**: Including emerging market currencies and stock indices to assess global financial integration

---

9. References

Course Materials

- Lecture 2: Vector Autoregression (VAR) Models
- Lecture 6: Unit Roots, Cointegration, and Vector Error Correction Models (VECM)
- Lecture 9: ARCH and GARCH Models for Volatility Modeling

Textbooks

- Enders, W. (2015). *Applied Econometric Time Series* (4th ed.). Wiley.
- Lütkepohl, H. (2005). *New Introduction to Multiple Time Series Analysis*. Springer.
- Tsay, R. S. (2010). *Analysis of Financial Time Series* (3rd ed.). Wiley.

Academic Papers

- Manasseh, C. O., Abada, F. C., Okiche, E. L., Okanya, O., Nwakoby, I. C., Offu, P., ... & Alio, F. C. (2019). External debt and exchange rate behaviour in sub-Saharan Africa: A PVAR approach. *Cogent Economics & Finance*, 7(1), 1627164.

Data Sources

- S&P 500 Index: Standard & Poor's Financial Services LLC
- EUR/USD Exchange Rate: European Central Bank Statistical Data Warehouse

---