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CS 109

Problem Set #2
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Problem Set #2 (Solved)

With problems by Mehran Sahami and Chris Piech

Important: If you want to view and work on the document in the \LaTeX system, click on this [url](#), create a copy of the project, and you're ready to go!

For each problem, briefly explain/justify how you obtained your answer. Brief explanations of your answer are necessary to get full credit for a problem even if you have the correct numerical answer. The explanations help us determine your understanding of the problem whether or not you got the correct answer. Moreover, in the event of an incorrect answer, we can still try to give you partial credit based on the explanation you provide. It is fine for your answers to include summations, products, factorials, exponentials, or combinations; you don't need to calculate those all out to get a single numeric answer.

Written Problems

1. Say in Silicon Valley, 35% of engineers program in Java and 28% of the engineers who program in Java also program in C++. Furthermore, 40% of engineers program in C++.
 - a. What is the probability that a randomly selected engineer programs in Java and C++?
 - b. What is the conditional probability that a randomly selected engineer programs in Java given that they program in C++?

Theory background: Conditional probability, Chain rule, Bayes' Theorem

Solution:

a. $P(\text{Java and } C++) = P(C++|\text{Java}) \cdot P(\text{Java}) = 0.28 \cdot 0.35 = 0.098$

b. $P(\text{Java}|C++) = \frac{P(C++|\text{Java}) \cdot P(\text{Java})}{P(C++)} = \frac{0.098}{0.4} = 0.245$

2. A website wants to detect if a visitor is a robot. They give the visitor three CAPTCHA tests that are hard for robots but easy for humans. If the visitor fails one of the tests, they are flagged as a robot. The probability that a human succeeds at a single test is 0.95, while a robot only succeeds with probability 0.15. Assume all tests are independent.
- If a visitor is actually a robot, what is the probability they get flagged (the probability they fail at least one test)?
 - If a visitor is human, what is the probability they get flagged?
 - The percentage of visitors on the site that are robots is 5%. Suppose a visitor gets flagged. Using your answers from part (a), what is the probability that the visitor is a robot?

Theory background: Probability of AND with independent events, Identity 1, Bayes' Theorem

Solution:

a. $P(\text{robot fails at least one test}) = 1 - P(\text{robot succeeds all tests}) = 1 - 0.15^3 \approx 0.9966$

b. $P(\text{human fails at least one test}) = 1 - P(\text{human succeeds all tests}) = 1 - 0.95^3 \approx 0.14$

c.

$$P(\text{robot}|\text{get flagged}) = P(\text{robot}|\text{failed at least one test})$$

$$\begin{aligned}
 &= \frac{P(\text{robot fails at least one test}) \cdot P(\text{robot})}{P(\text{robot fails at least one test}) \cdot P(\text{robot}) + P(\text{human fails at least one test}) \cdot P(\text{human})} \\
 &= \frac{0.9966 \cdot 0.05}{0.9966 \cdot 0.05 + 0.14 \cdot 0.95} \\
 &= \frac{0.04983}{0.18283} \\
 &\approx 0.2725
 \end{aligned}$$

3. Say all computers either run operating system W or X. A computer running operating system W is twice as likely to get infected with a virus as a computer running operating system X. If 70% of all computers are running operating system W, what percentage of computers infected with a virus are running operating system W?

Theory background: Identity 1, Bayes' Theorem

Solution:

Given: $P(W) = 0.7$, $P(X) = 1 - P(W) = 0.3$, $P(Infected|X) = \alpha$, $P(Infected|W) = 2\alpha$

$$P(W|Infected) = \frac{P(Infected|W) \cdot P(W)}{P(Infected)} = \frac{P(Infected|W) \cdot P(W)}{P(Infected|W) \cdot P(W) + P(Infected|X) \cdot P(X)} = \frac{2\alpha \cdot 0.7}{2\alpha \cdot 0.7 + \alpha \cdot 0.3} \approx 0.8235$$

4. Consider a hash table with 15 buckets, of which 9 are empty (have no strings hashed to them) and the other 6 buckets are non-empty (have at least one string hashed to each of them already). Now, 2 new strings are independently hashed into the table, where each string is equally likely to be hashed into any bucket. Later, another 2 strings are hashed into the table (again, independently and equally likely to get hashed to any bucket). What is the probability that both of the final 2 strings are each hashed to empty buckets in the table?

Theory background: Probability of AND with dependent and independent events, Probability of OR, Law of total probability

Solution:

Important assumption for the sake of easier computation: two final strings can go also to the same empty bucket (no condition that they must be in different buckets)

The experiment has two stages: first stage, distribute first 2 strings, second stage - distribute second 2 strings; the first stage influences the second stage in terms of how many empty buckets remain for the second stage. There are three possible scenarios for the first stage: a) 2 strings occupies two nonempty buckets (7 buckets remain for the second stage); b) 2 strings occupies one non-empty buckets (8 buckets remain for the second stage); c) 2 strings occupies zero non-empty buckets (9 buckets remain for the second stage).

By the total law of probability: $P(\text{final 2 to empty}) = \sum P(\text{final 2 to empty} \mid \text{fist stage scenario}) \cdot P(\text{fist stage scenario})$

- a. 7 buckets remain empty for the final 2 strings

$$P(\text{scenario a}) = \frac{9}{15} \cdot \frac{8}{15} = \frac{8}{25}$$

$$P(\text{final 2 to empty} \mid \text{scenario a}) = \left(\frac{7}{15}\right)^2$$

- b. 8 buckets remain empty for the final 2 strings

$$P(\text{scenario b}) = P(\text{both go to the same empty bucket}) + P(\text{one goes to empty, another to the non-empty}) = \left(\frac{9}{15} \cdot \frac{1}{15}\right) + \left(\frac{9}{15} \cdot \frac{6}{15}\right)^2 = \frac{13}{25}$$

$$P(\text{final 2 to empty} \mid \text{scenario b}) = \left(\frac{8}{15}\right)^2$$

- c. 9 buckets remain empty for the final 2 strings

$$P(\text{scenario c}) = \frac{6}{15} \cdot \frac{6}{15} = \frac{4}{25}$$

$$P(\text{final 2 to empty} \mid \text{scenario c}) = \left(\frac{9}{15}\right)^2$$

$$P(\text{final 2 to empty}) = \frac{8 \cdot 49 + 13 \cdot 64 + 4 \cdot 81}{25 \cdot 15^2} \approx 0.2752$$

5. Five servers are located in a computer cluster. After one year, each server independently is still working with probability p , and otherwise fails (with probability $1 - p$).
- What is the probability that *at least* 1 server is still working after one year?
 - What is the probability that *exactly* 2 servers are still working after one year?
 - What is the probability that *at least* 2 servers are still working after one year?

Theory background: Identity 1, Probability of AND with independent events, Many coin flips

Solution:

a. $P(\text{at least 1 server works}) = 1 - P(\text{all servers don't work}) = 1 - (1 - p)^5$

b. $P(\text{exactly 2 servers}) = \binom{5}{2} \cdot p^2 \cdot (1 - p)^3$

c. $P(\text{at least 2 servers}) = P(\text{more than 1 server}) = \sum_{i=2}^5 \binom{5}{i} \cdot p^i \cdot (1 - p)^{5-i}$

There is another solution approach *for c* which you are free to experiment:
 $P(\text{at least 2 servers}) = 1 - P(\text{no server works or exactly one sever works})$

6. The Superbowl institutes a new way to determine which team receives the kickoff first. The referee chooses with equal probability one of three coins. Although the coins look identical, they have probability of heads 0.1, 0.5 and 0.9, respectively. Then the referee tosses the chosen coin 3 times. If more than half the tosses come up heads, one team will kick off; otherwise, the other team will kick off. If the tosses resulted in the sequence H, T, H, what is the probability that the fair coin was actually used?

Theory background: Identity 1, General law of total probability, Bayes' Theorem

Solution:

$$P(seq) = P(seq|1 \text{ coin}) \cdot P(1 \text{ coin}) + P(seq|2 \text{ coin}) \cdot P(2 \text{ coin}) + P(seq|3 \text{ coin}) \cdot P(3 \text{ coin}) = \frac{(0.1 \cdot 0.9 \cdot 0.1) + (0.5 \cdot 0.5 \cdot 0.5) + (0.9 \cdot 0.1 \cdot 0.9)}{3} \approx 0.0717$$

$$P(2 \text{ coin}|seq) = \frac{P(seq|2 \text{ coin}) \cdot P(2 \text{ coin})}{P(seq)} \approx 0.5811$$

7. A robot can either be in one of two locations: L_1 (which does not have a window) or L_2 (which has a window). The robot doesn't know exactly where it is and it represents this uncertainty by keeping track of two probabilities: $P(L_1)$ and $P(L_2)$. Based on all past observations, the robot thinks that there is a 0.7 probability it is in L_1 and a 0.3 probability that it is in L_2 .

The robot has a single camera. The robot observes a window through its camera, and although there is only a window in L_2 , it can't conclude with certainty that it is in fact in L_2 , since its image recognition algorithm is not perfect. The probability of observing a window given there is no window at its location is 0.2, and the probability of observing a window given there is a window is 0.9. After incorporating the observation of a window, what are the robot's new probabilities for being in L_1 and L_2 , respectively?

Theory background: Identity 1, Law of total probability, Bayes' Theorem

Solution:

Given: $P(L_1) = 0.7$, $P(L_2) = 0.3$, $P(W|L_1) = 0.2$, $P(W|L_2) = 0.9$

$$P(\text{Window}) = P(W|L_1) \cdot P(L_1) + P(W|L_2) \cdot P(L_2) = 0.41$$

$$P(L_1|W) = \frac{P(W|L_1) \cdot P(L_1)}{P(W)} \approx 0.341$$

$$P(L_2|W) = 1 - P(L_1|W) \approx 0.659$$

8. After a long night of programming, you have built a powerful, but slightly buggy, email spam filter. When you don't encounter the bug, the filter works very well, always marking a spam email as SPAM and always marking a non-spam email as GOOD. Unfortunately, your code contains a bug that is encountered 10% of the time when the filter is run on an email. When the bug is encountered, the filter always marks the email as GOOD. As a result, emails that are actually spam will be erroneously marked as GOOD when the bug is encountered. Let p denote the probability that an email is actually non-spam, and let q denote the conditional probability that an email is non-spam given that it is marked as GOOD by the filter.

- a. Determine q in terms of p .
- b. Using your answer from part (a), explain mathematically whether q or p is greater. Also, provide an intuitive justification for your answer.

Theory background: Identity 1, Law of total probability, Bayes' Theorem

Solution:

a.

$$\begin{aligned}
 q = P(\text{non-spam} | \text{GOOD}) &= \frac{P(\text{non-spam, GOOD})}{P(\text{GOOD})} \\
 &= \frac{P(\text{GOOD} | \text{non-spam}) \cdot P(\text{non-spam})}{P(\text{GOOD} | \text{bug}^c) \cdot P(\text{bug}^c) + P(\text{GOOD} | \text{bug}) \cdot P(\text{bug})} \\
 &= \frac{1 \cdot p}{p \cdot 0.9 + 1 \cdot 0.1} \\
 &= \frac{p}{0.9p + 0.1}
 \end{aligned}$$

- b. Approach: find out if $p - q$ greater or lower than 0, don't forget that $0 < p < 1$
 Answer: $q > p$

Internal justification: the portion of non-spam email in GOOD pool is understandably higher than the portion of non-spam email in all-email pool, since the GOOD pool consist of all true non-spam emails and just a small fraction of spam emails

9. The color of a person's eyes is determined by a pair of eye-color genes, as follows:

- if both of the eye-color genes are blue-eyed genes, then the person will have blue eyes
- if one or more of the genes is a brown-eyed gene, then the person will have brown eyes

A newborn child independently receives one eye-color gene from each of its parents, and the gene it receives from a parent is equally likely to be either of the two eye-color genes of that parent. Suppose William and both of his parents have brown eyes, but William's sister (Claire) has blue eyes. (We assume that blue and brown are the only eye-color genes.)

- a. What is the probability that William possesses a blue-eyed gene?
- b. Suppose that William's wife has blue eyes. What is the probability that their first child will have blue eyes?

Theory background: Probability of AND with independent and dependent events, Probability of OR

Solution:

- a. The information about the parents and the daughter gives a very important conclusion: both parents have one brown and one blue gene. Given that there are four equally likely outcomes for Williams, in regards to his eye-color:

First case - the father gives brown gene and the mother gives the brown gene

Second case - the father gives blue gene and the mother gives the brown gene

Third case - the father gives brown gene and the mother gives the blue gene

Fourth case - the father gives blue gene and the mother gives the blue gene

However, we already know that William does not have blue eyes, which means that the sample space shrinks to first three cases. Among these three cases there are two (second and third) that satisfies the condition '*William possesses a blue-eyed gene*', thus $P(\text{William has blue gene}) = \frac{2}{3}$

- b. $P(\text{child will have blue}) = P(\text{receives blue from W and receives blue from W's wife}) = P(\text{receives blue from William}) \cdot P(\text{receives blue from William's wife})$

$P(\text{receives blue from William's wife}) = 1$

$P(\text{receives blue from William}) = P(\text{W gives blue} \mid \text{W has blue gene}) \cdot$

$P(\text{W has blue gene}) = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$

$P(\text{child will have blue}) = \frac{1}{3}$

10. Your colleagues in a comp-bio lab have sequenced DNA from a large population in order to understand how a gene (G) influences two particular traits (T_1 and T_2). They find that $P(G) = 0.6$, $P(T_1 | G) = 0.7$, and $P(T_2 | G) = 0.9$. They also observe that if a subject does not have the gene G , they express neither T_1 nor T_2 . The probability of a patient having both T_1 and T_2 given that they have the gene G is 0.63.
- Are T_1 and T_2 conditionally independent given G ?
 - Are T_1 and T_2 conditionally independent given G^c ?
 - What is $P(T_1)$?
 - What is $P(T_2)$?
 - Are T_1 and T_2 independent?

Theory background: Independence, Conditional independence, Law of total probability
Solution:

- yes, since $P(T_1, T_2 | G) = P(T_1 | G) \cdot P(T_2 | G) = 0.63$
- yes, since $P(T_1, T_2 | G^c) = P(T_1 | G^c) \cdot P(T_2 | G^c) = 0$
- $P(T_1) = P(T_1 | G) \cdot P(G) + P(T_1 | G^c) \cdot P(G^c) = 0.42$
- $P(T_2) = P(T_2 | G) \cdot P(G) + P(T_2 | G^c) \cdot P(G^c) = 0.54$
- no since $P(T_1, T_2) \neq P(T_1) \cdot P(T_2)$
 $P(T_1, T_2) = P(T_1, T_2 | G) \cdot P(G) + P(T_1, T_2 | G^c) \cdot P(G^c) = 0.63 \cdot 0.6 = 0.378$
 $P(T_1) \cdot P(T_2) = 0.2268$

Coding Problems

11. **[Coding]** After the Ebola outbreak of 2015, there was an urgent need to learn more about the virus. You have been asked to uncover how a particular group of bat genes impact an important trait: whether the bat can carry Ebola. Nobody knows the underlying mechanism; it is up to you to hypothesize what is going on. For 100,000 independently sampled bats you have collected data of whether or not five genes are expressed, and whether or not the bat can carry Ebola.¹ If a gene is expressed, it can affect both the probability of other genes being expressed and the probability of the trait being expressed. You can find the data in a file called `bats.csv`. Each row in the file corresponds to **one bat** and has 6 Booleans:

- Boolean 1: Whether the 1st gene is expressed in the bat (G_1)
- Boolean 2: Whether the 2nd gene is expressed in the bat (G_2)
- Boolean 3: Whether the 3rd gene is expressed in the bat (G_3)
- Boolean 4: Whether the 4th gene is expressed in the bat (G_4)
- Boolean 5: Whether the 5th gene is expressed in the bat (G_5)
- Boolean 6: Whether the trait is expressed in the bat; i.e., the bat can carry Ebola (T)

Write a program to analyze the data you have collected. Report the following:

- a. What is the probability of the trait being expressed $P(T)$?
- b. For each gene i calculate and report $P(G_i)$.
- c. For each gene i decide whether or not you think that it would be reasonable to assume that G_i is independent of T . Support your argument with numbers. Remember that our probabilities are based on 100,000 bats, not infinite bats, and are therefore just estimates of the true probabilities.
- d. For each gene i that is not assumed to be independent of T , calculate $P(T | G_i)$.
- e. Give your best interpretation of the results from (a) to (d).
- f. Let K be the number of genes expressed. What is $E[K]$?
- g. **Above and Beyond:** For extra credit, try and find conditional independence relationships between the genes and the trait. Incorporate this information to improve your hypothesis of how the five genes relate to whether or not a bat can carry Ebola.

- a. $P(T) = 0.30079$
- b. $P(G_1) = 0.70228, P(G_2) = 0.30076, P(G_3) = 0.5009, P(G_4) = 0.80162, P(G_5) = 0.32705$
- c. For each G checked if $P(T \text{ and } G) = P(T) * P(G)$, in case of G_1 and G_2 , it was really close (rounding errors), given that there are just 100000 cases come to conclusion that G_1 and G_2 are independent of T
- d. $P(T|G_3) = 0.5831902575364344, P(T|G_4) = 0.37053716224645095, P(T|G_5) = 0.8999847118177648$
- e. All genes increase the probability of the trait T . However, the data indicates that the genes G_1 and G_2 have the lowest influence on T (possibly even negligible), while genes G_3 and G_5 exert the highest influence on the expression of T .

¹Humane note: bats can carry Ebola, but it causes them no harm. No fake bats were hurt in the making of this problem. Why are bats immune to the harmful effects? Open question!

f. $E[k] = 2.63261$

g. See inline comments in the script **ebola.py** in coding_solutions folder

12. **[Coding]** Your cell phone is constantly trying to keep track of where you are. At any given point in time, for all nearby locations, your phone stores a probability that you are in that location. Right now your phone believes that you are in one of 16 different locations arranged in a grid with the following probabilities (see the figure on the left):

Prior belief of location				$P(\text{Observe two bars of signal} \mid \text{Location})$			
0.05	0.10	0.05	0.05	0.75	0.95	0.75	0.05
0.05	0.10	0.05	0.05	0.05	0.75	0.95	0.75
0.05	0.05	0.10	0.05	0.01	0.05	0.75	0.95
0.05	0.05	0.10	0.05	0.01	0.01	0.05	0.75

Your phone connects to a known cell tower and records two bars of signal. For each grid location L_i , you can calculate the probability of observing two bars from this particular tower, assuming that cell phone is in location L_i (see the figure on the right). That calculation is based on knowledge of the dynamics of this particular cell tower and stochasticity of signal strength.

As an example: the value of 0.05 in the highlighted cell on the left figure means that you believed there was a 0.05 probability that the user was in the bottom right grid cell prior to observing the cell tower signal. The value of 0.75 in the highlighted cell on the right figure means that you think the probability of observing two bars, given the user was in the bottom right grid cell, is 0.75.

Write a program to calculate, for each of the 16 locations, the new probability that the user is in each location given the cell tower observation. The matrices are provided on the website on the Problem Set #2 page. The grid in the left figure is stored in a file called `prior.csv`, and the grid in the right figure is stored in a file called `conditional.csv`. Report the probabilities of all 16 cells and provide the code for your program.

Please refer to **phone.py** and **updated_loc_probs.txt** in the `coding_solutions` folder.