

## Inferring strategies in repeated games: The French Defence

In this work we consider the discounted repeated prisoners dilemma (PD). The discount factor ( $\delta$ ) is commonly interpreted as the constant continuation probability of having another round, or as the players' common discount rate on future payoff streams.

We define the one-shot PD in the following form:

$$\begin{pmatrix} 1-c & -c \\ 1 & 0 \end{pmatrix} \quad (1)$$

where  $c$  is the cost of cooperation assumed to be  $0 < c < 1$ .

We restrict ourselves to memory-one strategies, a set of strategies that take into account the outcome of one previous round. Such strategies can be represented as a 5-tuple,  $p = (p_0, p_{CC}, p_{CD}, p_{DC}, p_{DD})$ . The entry  $p_0$  denotes the probability to cooperate in the first round. The entry  $p_{ij}$  denotes the probability to cooperate in the next round. This probability depends on the player's action  $i$  and the co-player's action  $j$  in the previous round. The set of pure memory-one strategies, denoted by  $\Delta$ , contains 32 elements from  $d_0 = (0, 0, 0, 0, 0)$  to  $d_{31} = (1, 1, 1, 1, 1)$ .

Let's assume that player A plays  $\mathbf{d}_a = (p_0, p_{CC}, p_{CD}, p_{DC}, p_{DD})$  and player B plays  $\mathbf{d}_b = (q_0, q_{CC}, q_{CD}, q_{DC}, q_{DD})$ , the repeated interaction between the players can be described as a Markovian process (with the transition matrix  $M$ ), and we can explicitly obtain the stationary probability distribution,

$$\mathbf{v}(\mathbf{d}_a, \mathbf{d}_b, \delta) = (v_{CC}, v_{CD}, v_{DC}, v_{DD})$$

where  $u_{ij}$  mean the long-term average probability to observe player A and B choosing  $i$  and  $j$  respectively. We calculate the stationary probability distribution as follows,

$$\mathbf{v}(\mathbf{d}_a, \mathbf{d}_b, \delta) = (1 - \delta)\mathbf{v}_0 (I^4 - \delta M), \text{ where}$$

$$\mathbf{v}_0 = (p_0 q_0, p_0(1 - q_0), (1 - p_0)q_0, (1 - p_0)(1 - q_0)), \text{ and}$$

$$M = \begin{bmatrix} p_{CC}q_{CC} & p_{CC}(1-q_{CC}) & q_{CC}(1-p_{CC}) & (1-p_{CC})(1-q_{CC}) \\ p_{CD}q_{DC} & p_{CD}(1-q_{DC}) & q_{DC}(1-p_{CD}) & (1-p_{CD})(1-q_{DC}) \\ p_{DC}q_{CD} & p_{DC}(1-q_{CD}) & q_{CD}(1-p_{DC}) & (1-p_{DC})(1-q_{CD}) \\ p_{DD}q_{DD} & p_{DD}(1-q_{DD}) & q_{DD}(1-p_{DD}) & (1-p_{DD})(1-q_{DD}) \end{bmatrix}. \quad (2)$$

The average payoff of player A against player B can be calculated as

$$\pi(\mathbf{d}_a, \mathbf{d}_b, \delta) = \mathbf{v}(\mathbf{d}_a, \mathbf{d}_b, \delta) \cdot (1 - c, -c, 1, 0).$$

In table 1 we list the best response to each strategy in  $\Delta$ . To calculate the best response of a given strategy,  $\mathbf{d}_b$ , we calculate the long-term average payoff of  $\mathbf{d}_a$  against it for every  $a \in [0, \dots, 31]$  and choose  $a$  such that the payoff is maximized.

From table 1 we can observe that each strategy has multiple best responses. In some cases the best responses will depend on the relationship between  $c$  and  $\delta$ . For example in the case of  $\mathbf{d}_2$ , the best responses depend on whether  $c$  is smaller or greater than  $\delta$ . We do not specify what happens when  $c$  and  $\delta$  are equal, because then the list of best responses is the union of  $c < \delta$  and  $c > \delta$ . In some cases, however, we specify the best responses in case of equality. For example for  $\mathbf{d}_{26}$ . That is because now the best responses are the best responses for  $c < \delta$  and  $c > \delta$ , and the strategies specified for  $c = \delta$ .

opponent strategy	best response	best response payoff	Misc.
$d_0$	$d_0^\dagger, d_2, d_4, d_6, d_8, d_{10}, d_{12}, d_{14}$	0	ALLD (D)
$d_1$	$d_0, d_4, d_8, d_{12}$	$\frac{\delta}{\delta+1}$	
$d_2$	$\left\{ \begin{array}{ll} d_0, d_2^\dagger, d_4, d_6, d_8, d_{10}, d_{12}, d_{14}, & c > \delta \\ d_{18}, d_{19}, d_{26}, d_{27}, & c < \delta \\ d_1, d_3, d_9, d_{11}, d_{16}, d_{17}, d_{24}, d_{25}, & c = \delta \end{array} \right\}$	$\left\{ \begin{array}{c} 0 \\ -\frac{c-\delta}{\delta+1} \\ 0 \end{array} \right\}$	
$d_3$	$d_0, d_4, d_8, d_{12}$	$\frac{\delta}{\delta+1}$	
$d_4$	$d_0, d_2, d_4^\dagger, d_6, d_8, d_{10}, d_{12}, d_{14}$	0	
$d_5$	$d_0, d_4, d_8, d_{12}$	$\delta$	
$d_6$	$\left\{ \begin{array}{ll} d_0, d_2, d_4, d_6^\dagger, d_8, d_{10}, d_{12}, d_{14}, & c < \frac{-\delta}{1-\delta} \\ d_{16}, d_{17}, d_{24}, d_{25}, & c > \frac{-\delta}{1-\delta} \end{array} \right\}$	$\left\{ \begin{array}{c} 0 \\ c(\delta - 1) + \delta \end{array} \right\}$	
$d_7$	$d_0, d_4, d_8, d_{12}$	$\delta$	
$d_8$	$d_0, d_2, d_4, d_6, d_8^\dagger, d_{10}, d_{12}, d_{14}$	0	GT (D)
$d_9$	$\left\{ \begin{array}{ll} d_0, d_4, d_8, d_{12}, & c > \frac{\delta}{\delta+1} \\ d_9^\dagger, d_{11}, d_{13}, d_{15} & c < \frac{\delta}{\delta+1} \end{array} \right\}$	$\left\{ \begin{array}{c} \frac{\delta}{\delta+1} \\ \delta(1 - c) \end{array} \right\}$	WSLS (D)
$d_{10}$	$\left\{ \begin{array}{ll} d_0, d_2, d_4, d_6, d_8, d_{10}^\dagger, d_{12}, d_{14}, & c > \delta \\ d_{28}, d_{29}, d_{30}, d_{31}, & c < \delta \\ \Delta, & c = \delta \end{array} \right\}$	$\left\{ \begin{array}{c} 0 \\ \delta - c \\ 0 \end{array} \right\}$	TFT (D)
$d_{11}$	$\left\{ \begin{array}{ll} d_0, d_4, d_8, d_{12}, & c > \frac{\delta}{\delta+1} \\ d_9, d_{11}^\dagger, d_{13}, d_{15} & c < \frac{\delta}{\delta+1} \end{array} \right\}$	$\left\{ \begin{array}{c} \frac{\delta}{\delta+1} \\ \delta(1 - c) \end{array} \right\}$	
$d_{12}$	$d_0, d_2, d_4, d_6, d_8, d_{10}, d_{12}^\dagger, d_{14}$	0	
$d_{13}$	$d_0, d_4, d_8, d_{12}$	$\delta$	
$d_{14}$	$\left\{ \begin{array}{ll} d_0, d_2, d_4, d_6, d_8, d_{10}, d_{12}, d_{14}^\dagger, & c < \frac{-\delta}{1-\delta} \\ d_{16}, d_{17}, d_{24}, d_{25}, & c > \frac{-\delta}{1-\delta} \end{array} \right\}$	$\left\{ \begin{array}{c} 0 \\ c(\delta - 1) + \delta \end{array} \right\}$	
$d_{15}$	$d_0, d_4, d_8, d_{12}$	$\delta$	ALLC (D)
$d_{16}$	$d_0, d_4, d_8, d_{12}$	$1 - \delta$	ALLD (C)
$d_{17}$	$d_0, d_4, d_8, d_{12}$	$\frac{1}{1+\delta}$	
$d_{18}$	$\left\{ \begin{array}{ll} d_0, d_4, d_8, d_{12}, & c > \delta \\ d_2, d_3, d_{10}, d_{11}, & c < \delta \\ d_1, d_9, & c = \delta \end{array} \right\}$	$\left\{ \begin{array}{c} 1 - \delta \\ \frac{1-c\delta}{1+\delta} \\ 1 - \delta \end{array} \right\}$	
$d_{19}$	$d_0, d_4, d_8, d_{12}$	$\frac{1}{1+\delta}$	
$d_{20}$	$d_0, d_1, d_4, d_5, d_8, d_9, d_{12}, d_{13}$	1	
$d_{21}$	$d_0, d_1, d_4, d_5, d_8, d_9, d_{12}, d_{13}$	1	
$d_{22}$	$d_0, d_1, d_4, d_5, d_8, d_9, d_{12}, d_{13}$	1	
$d_{23}$	$d_0, d_1, d_4, d_5, d_8, d_9, d_{12}, d_{13}$	1	
$d_{24}$	$\left\{ \begin{array}{ll} d_0, d_4, d_8, d_{12}, & c > \delta \\ d_{24}^\dagger, d_{25}, d_{26}, d_{27}, d_{28}, d_{29}, d_{30}, d_{31} & c < \delta \\ d_{16}, d_{20}, & c = \delta \end{array} \right\}$	$\left\{ \begin{array}{c} 1 - \delta \\ 1 - c \\ 1 - \delta \end{array} \right\}$	GT (C)
$d_{25}$	$\left\{ \begin{array}{ll} d_0, d_4, d_8, d_{12}, & c > \frac{\delta}{1+\delta} \\ d_{24}, d_{25}^\dagger, d_{26}, d_{27}, d_{28}, d_{29}, d_{30}, d_{31} & c < \frac{\delta}{1+\delta} \end{array} \right\}$	$\left\{ \begin{array}{c} \frac{1}{1+\delta} \\ 1 - c \end{array} \right\}$	WSLS (C)
$d_{26}$	$\left\{ \begin{array}{ll} d_0, d_4, d_8, d_{12}, & c > \delta \\ d_{24}, d_{25}, d_{26}^\dagger, d_{27}, d_{28}, d_{29}, d_{30}, d_{31}, & c < \delta \\ \Delta, & c = \delta \end{array} \right\}$	$\left\{ \begin{array}{c} 1 - \delta \\ 1 - c \\ 1 - \delta \end{array} \right\}$	TFT (C)
$d_{27}$	$\left\{ \begin{array}{ll} d_0, d_4, d_8, d_{12}, & c > \frac{\delta}{1+\delta} \\ d_{24}, d_{25}, d_{26}, d_{27}^\dagger, d_{28}, d_{29}, d_{30}, d_{31} & c < \frac{\delta}{1+\delta} \end{array} \right\}$	$\left\{ \begin{array}{c} \frac{1}{1+\delta} \\ 1 - c \end{array} \right\}$	
$d_{28}$	$d_0, d_1, d_4, d_5, d_8, d_9, d_{12}, d_{13}$	1	
$d_{29}$	$d_0, d_1, d_4, d_5, d_8, d_9, d_{12}, d_{13}$	1	
$d_{30}$	$d_0, d_1, d_4, d_5, d_8, d_9, d_{12}, d_{13}$	1	
$d_{31}$	$d_0, d_1, d_4, d_5, d_8, d_9, d_{12}, d_{13}$	1	ALLC (C)

Table 1: Best responses among memory-one strategies.