Inferring strategies in repeated games: The French Defence

In this work we consider the discounted repeated prisoners dilemma (PD). The discount factor (δ) is commonly interpreted as the constant continuation probability of having another round, or as the players' common discount rate on future payoff streams.

We define the one-shot PD in the following form:

$$\begin{pmatrix} 1-c & -c \\ 1 & 0 \end{pmatrix} \tag{1}$$

where c is the cost of cooperation assumed to be $0 < c < 1$.

We restrict ourselves to memory-one strategies, a set of strategies that take into account the outcome of one previous round. Such strategies can be represented as a 5-tuple, $p = (p_0, p_{CC}, p_{CD}, p_{DC}, p_{DD})$. The entry p_0 denotes the probability to cooperate in the first round. The entry p_{ij} denotes the probability to cooperate in the next round. This probability depends on the player's action i and the co-player's action j in the previous round. The set of pure memory-one strategies, denoted by Δ, contains 32 elements from $d_0 = (0,0,0,0,0)$ to $d_{31} = (1,1,1,1,1)$.

Let's assume that player A plays $\mathbf{d_a} = (p_0, p_{CC}, p_{CD}, p_{DC}, p_{DD})$ and player B plays $\mathbf{d_b} = (q_0, q_{CC}, q_{CD}, q_{DC}, q_{DD})$, the repeated interaction between the players can be described as a Markovian process (with the transition matrix M), and we can explicitly obtain the stationary probability distribution,

$$\mathbf{v}(\mathbf{d_a}, \mathbf{d_b}, \delta) = (v_{CC}, v_{CD}, v_{DC}, v_{DD})$$

where u_{ij} mean the long-term average probability to observe player A and B choosing i and j respectively. We calculate the stationary probability distribution as follows,

$$\mathbf{v}(\mathbf{d_a}, \mathbf{d_b}, \delta) = (1-\delta)\mathbf{v_0}\left(I^4 - \delta M\right), \text{ where}$$

$$\mathbf{v_0} = (p_0 q_0, p_0(1-q_0), (1-p_0)q_0, (1-p_0)(1-q_0)), \text{ and}$$

$$M=\begin{bmatrix} p_{CC}q_{CC} & p_{CC}\left(1-q_{CC}\right) & q_{CC}\left(1-p_{CC}\right) & \left(1-p_{CC}\right)\left(1-q_{CC}\right) \\ p_{CD}q_{DC} & p_{CD}\left(1-q_{DC}\right) & q_{DC}\left(1-p_{CD}\right) & \left(1-p_{CD}\right)\left(1-q_{DC}\right) \\ p_{DC}q_{CD} & p_{DC}\left(1-q_{CD}\right) & q_{CD}\left(1-p_{DC}\right) & \left(1-p_{DC}\right)\left(1-q_{CD}\right) \\ p_{DD}q_{DD} & p_{DD}\left(1-q_{DD}\right) & q_{DD}\left(1-p_{DD}\right) & \left(1-p_{DD}\right)\left(1-q_{DD}\right) \end{bmatrix}. \tag{2}$$

The average payoff of player A against player B can be calculated as

$$\pi(\mathbf{d_a}, \mathbf{d_b}, \delta) = \mathbf{v}(\mathbf{d_a}, \mathbf{d_b}, \delta) \cdot (1-c, -c, 1, 0).$$

In table 1 we list the best response to each strategy in Δ. To calculate the best response of a given strategy, $\mathbf{d_b}$, we calculate the long-term average payoff of $\mathbf{d_a}$ against it for every $a \in [0, \dots, 31]$ and choose a such that the payoff is maximized.

From table 1 we can observe that each strategy has multiple best responses. In some cases the best responses will depend on the relationship between c and δ. For example in the case of $\mathbf{d_2}$, the best responses depend on whether c is smaller or greater that δ. We do not specify what happens when c and δ are equal, because then the list of best responses is the union of $c < \delta$ and $c > \delta$. In same cases, however, we specify the best responses in case of equality. For example for $\mathbf{d_{26}}$. That is because now the best responses are the best responses for $c < \delta$ and $c > \delta$, and the strategies specified for $c = \delta$.

opponent strategy	best response	best response payoff	Misc.
d_0	$d_0^\dagger,\ d_4,\ d_8,\ d_{12}$	$1-\delta$	ALLD (D)
d_1	$d_0,\ d_4,\ d_8,\ d_{12}$	$\frac{\delta}{(\delta+1)}$	
d_2	$\left\{\begin{array}{ll} d_0, d_2^\dagger, d_4, d_6, d_8, d_{10}, d_{12}, d_{14}, & c>\delta \\ d_{18}, d_{19}, d_{26}, d_{27}, & c<\delta \\ d_1, d_3, d_9, d_{11}, d_{16}, d_{17}, d_{24}, d_{25}, & c=\delta \end{array}\right\}$	$\left\{\begin{array}{c} -\frac{c-\delta}{\delta+1} \\ 0 \\ \frac{\delta(-c+\delta)}{\delta^2+\delta+1} \end{array}\right\}$	
d_3	$d_0,\ d_4,\ d_8,\ d_{12}$	$\frac{\delta}{\delta+1}$	
d_4	$d_0, d_2, d_4^\dagger, d_6, d_8, d_{10}, d_{12}, d_{14}$	0	
d_5	$d_0,\ d_4,\ d_8,\ d_{12}$	δ	
d_6	$\left\{\begin{array}{ll} d_0, d_2, d_4, d_6^\dagger, d_8, d_{10}, d_{12}, d_{14}, & c>\frac{\delta}{1-\delta} \\ d_{16}, d_{17}, d_{24}, d_{25}, & c<\frac{\delta}{1-\delta} \end{array}\right\}$	$\left\{\begin{array}{c} 0 \\ c\,(\delta-1)+\delta \end{array}\right\}$	
d_7	$d_0,\ d_4,\ d_8,\ d_{12}$	δ	
d_8	$d_0, d_2, d_4, d_6, d_8^\dagger, d_{10}, d_{12}, d_{14}$	0	GT (D)
d_9	$\left\{\begin{array}{ll} d_0, d_4, d_8, d_{12}, & c>\frac{\delta}{\delta+1} \\ d_9^\dagger, d_{11}, d_{13}, d_{15} & c<\frac{\delta}{\delta+1} \end{array}\right\}$	$\left\{\begin{array}{c} \frac{\delta}{\delta+1} \\ -\delta\,(c-1) \end{array}\right\}$	WSLS (D)
d_{10}	$\left\{\begin{array}{ll} d_0, d_2, d_4, d_6, d_8, d_{10}^\dagger, d_{12}, d_{14}, & c>\delta \\ d_{28}, d_{29}, d_{30}, d_{31}, & c<\delta \\ \triangle, & c=\delta \end{array}\right\}$	$\left\{\begin{array}{c} -c+\delta \\ 0 \end{array}\right\}$	TFT (D)
d_{11}	$\left\{\begin{array}{ll} d_0, d_4, d_8, d_{12}, & c>\frac{\delta}{\delta+1} \\ d_9, d_{11}^\dagger, d_{13}, d_{15} & c<\frac{\delta}{\delta+1} \end{array}\right\}$	$\left\{\begin{array}{c} \frac{\delta}{\delta+1} \\ -\delta\,(c-1) \end{array}\right\}$	
d_{12}	$d_0, d_2, d_4, d_6, d_8, d_{10}, d_{12}^\dagger, d_{14}$	0	
d_{13}	$d_0,\ d_4,\ d_8,\ d_{12}$	δ	
d_{14}	$\left\{\begin{array}{ll} d_0, d_2, d_4, d_6, d_8, d_{10}, d_{12}, d_{14}^\dagger, & c>\frac{\delta}{1-\delta} \\ d_{16}, d_{17}, d_{24}, d_{25}, & c<\frac{\delta}{1-\delta} \end{array}\right\}$	$\left\{\begin{array}{c} c\,(\delta-1)+\delta \\ 0 \end{array}\right\}$	
d_{15}	$d_0,\ d_4,\ d_8,\ d_{12}$	δ	ALLC (D)
d_{16}	$d_0,\ d_4,\ d_8,\ d_{12}$	1 - δ	ALLD (C)
d_{17}	$d_0,\ d_4,\ d_8,\ d_{12}$	$\frac{1}{1+\delta}$	
d_{18}	$\left\{\begin{array}{ll} d_0, d_4, d_8, d_{12}, & c>\delta \\ d_2, d_3, d_{10}, d_{11}, & c<\delta \\ d_1, d_9, & c=\delta \end{array}\right\}$	$\left\{\begin{array}{c} 1-\delta \\ \frac{1-c\delta}{1+\delta} \\ \frac{-(c\delta^2-1)}{(\delta^2+\delta+1)} \end{array}\right\}$	
d_{19}	$d_0,\ d_4,\ d_8,\ d_{12}$	$\frac{1}{1+\delta}$	
d_{20}	$d_0, d_1, d_4, d_5, d_8, d_9, d_{12}, d_{13}$	1	
d_{21}	$d_0, d_1, d_4, d_5, d_8, d_9, d_{12}, d_{13}$	1	
d_{22}	$d_0, d_1, d_4, d_5, d_8, d_9, d_{12}, d_{13}$	1	
d_{23}	$d_0, d_1, d_4, d_5, d_8, d_9, d_{12}, d_{13}$	1	
d_{24}	$\left\{\begin{array}{ll} d_0, d_4, d_8, d_{12}, & c>\delta \\ d_{24}^\dagger, d_{25}, d_{26}, d_{27}, d_{28}, d_{29}, d_{30}, d_{31} & c<\delta \\ d_{16}, d_{20}, & c=\delta \end{array}\right\}$	$\left\{\begin{array}{c} \frac{1}{\delta+1} \\ 1-c \\ (\delta-1)\,(c-\delta-1) \end{array}\right\}$	GT (C)
d_{25}	$\left\{\begin{array}{ll} d_0, d_4, d_8, d_{12}, & c>\frac{\delta}{1+\delta} \\ d_{24}, d_{25}^\dagger, d_{26}, d_{27}, d_{28}, d_{29}, d_{30}, d_{31} & c<\frac{\delta}{1+\delta} \end{array}\right\}$	$\left\{\begin{array}{c} 1-\delta \\ 1-c \end{array}\right\}$	WSLS (C)
d_{26}	$\left\{\begin{array}{ll} d_0, d_4, d_8, d_{12}, & c>\delta \\ d_{24}, d_{25}, d_{26}^\dagger, d_{27}, d_{28}, d_{29}, d_{30}, d_{31}, & c<\delta \\ \triangle, & c=\delta \end{array}\right\}$	$\left\{\begin{array}{c} 1-\delta \\ 1-c \\ \frac{-(c\delta^2-1)}{(\delta^2+\delta+1)} \end{array}\right\}$	TFT (C)
d_{27}	$\left\{\begin{array}{ll} d_0, d_4, d_8, d_{12}, & c>\frac{\delta}{1+\delta} \\ d_{24}, d_{25}, d_{26}, d_{27}^\dagger, d_{28}, d_{29}, d_{30}, d_{31} & c<\frac{\delta}{1+\delta} \end{array}\right\}$	$\left\{\begin{array}{c} 1-\delta \\ 1-c \end{array}\right\}$	
d_{28}	$d_0, d_1, d_4, d_5, d_8, d_9, d_{12}, d_{13}$	1	
d_{29}	$d_0, d_1, d_4, d_5, d_8, d_9, d_{12}, d_{13}$	1	
d_{30}	$d_0, d_1, d_4, d_5, d_8, d_9, d_{12}, d_{13}$	1	
d_{31}	$d_0, d_1, d_4, d_5, d_8, d_9, d_{12}, d_{13}$	1	ALLC (C)

Table 1: **Best responses among memory-one strategies.**