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Vince: to Riggins
Geraint: also, to Riggins

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This is the foreword

Preface

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Getting Started

CHAPTER 1

Introduction

THANK you for starting to read this book. This book aims to bring together two fascinating topics:

- Problems that can be solved using mathematics;
- Software that is free to use and change.

What we mean by both of those things will become clear through reading this chapter and the rest of the book.

1.1 WHO IS THIS BOOK FOR?

Anyone who is interested in using mathematics and computers to solve problems will hopefully find this book helpful.

If you are a student of a mathematical discipline, a graduate student of a subject like operational research, a hobbyist who enjoys solving the travelling salesman problem or even if you get paid to do this stuff: this book is for you. We will introduce you to the world of open source software that allows you to do all these things freely.

If you are a student learning to write code, a graduate student using databases for their research, an enthusiast who programmes applications to help coordinate the neighbourhood watch, or even if you get paid to write software: this book is for you. We will introduce you to a world of problems that can be solved using your skill sets.

It would be helpful for the reader of this book to:

- Have access to a computer and be able to connect to the internet (at least once) to be able to download the relevant software.
- Be prepared to read some mathematics. Technically you do not need to understand the specific mathematics to be able to use the tools in this book. The topics covered use some algebra, calculus and probability.

1.2 WHAT DO WE MEAN BY APPLIED MATHEMATICS?

We consider this book to be a book on applied mathematics. This is not however a universal term, for some applied mathematics is the study of mechanics and involves modelling projectiles being fired out of canons. We will use the term a bit more freely here and mean any type of real world problem that can be tackled using mathematical tools. This is sometimes referred to as operational research, operations research, mathematical modelling or indeed just mathematics.

One of the authors, Vince, used mathematics to plan the sitting plan at his wedding. Using a particular area of mathematics call graph theory he was able to ensure that everyone sat next to someone they liked and/or knew.

The other author, Geraint, used mathematics to find the best team of Pokemon. Using an area of mathematics call linear programming which is based on linear algebra he was able to find the best makeup of pokemon.

Here, applied mathematics is the type of mathematics that helps us answer questions that the real world asks.

1.3 WHAT IS OPEN SOURCE SOFTWARE

Strictly speaking open source software is software with source code that anyone can read, modify and improve. In practice this means that you do not need to pay to use it which is often one of the first attractions. This financial aspect can also be one of the reasons that someone will not use a particular piece of software due to a confusion between cost and value: if something is free is it really going to be any good?

In practice open source software is used all of the world and powers some of the most important infrastructure around. For example, one should never use any cryptographic software that is not open source: if you cannot open up and read things than you should not trust it (this is indeed why most cryptographic systems used are open source).

Today, open source software is a lot more than a licensing agreement: it is a community of practice. Bugs are fixed faster, research is implemented immediately and knowledge is spread more widely thanks to open source software. Bugs are fixed faster because anyone can read and inspect the source code. Most open source software projects also have a clear mechanisms for communicating with the developers and even reviewing and accepting code contributions from the general public. Research is implemented immediately because when new algorithms are discovered they are often added directly to the software by the researchers who found them. This all contributes to the spread of knowledge: open source software is the modern should of giants that we all stand on.

Open source software is software that, like scientific knowledge is not restricted in its use.

1.4 HOW TO GET THE MOST OUT OF THIS BOOK

The book itself is open source. You can find the source files for this book online at `github.com/drvinceknight/ampwoss`. There will will also find a number of *Jupyter notebooks* and *R markdown files* that include code snippets that let you follow along.

We feel that you can choose to read the book from cover to cover, writing out the code examples as you go; or it could also be used as a reference text when faced with particular problem and wanting to know where to start.

The book is made up of 10 chapters that are paired in two 4 parts. Each part corresponds to a particular area of mathematics, for example “Emergent Behaviour”. Two chapters are paired together for each chapter, usually these two chapters correspond to the same area of mathematics but from a slightly different scale that correspond to different ways of tackling the problem.

Every chapter has the following structure:

1. Introduction - a brief overview of a given problem type. Here we will describe the problem at hand in general terms.

2. An Example problem. This will provide a tangible example problem that offers the reader some intuition for the rest of the discussion.

3. Solving with Python. We will describe the mathematical tools available to us in a programming language called Python to solve the problem.

4. Solving with R. Here we will do the same with the R programming language.

5. Brief theoretic background with pointers to reference texts. Some readers might like to delve in to the mathematics of the problem a bit further, we will include those details here.

6. Examples of research using these methods. Finally, some readers might even be interested in finding out a bit more of what mathematicians are doing on these problems. Often this will include some descriptions of the problem considered but perhaps at a much larger scale than the one presented in the example.

For a given reader, not all sections of a chapter will be of interest. Perhaps a reader is only interested in R and finding out more about the research. Please do take from the book what you find useful.

CHAPTER 2

Software

THIS book will involve using software, the particular interface to software we will use is to write code. There are numerous reasons why this is the correct way to do things but one of them is reproducibility.

This chapter will go over the basics of getting your computer set up to use the software discussed in this book: the programming languages R and Python. It will also briefly discuss using the command line: a particular interface to your whole computer. Finally it will give a brief introduction to R and Python.

This chapter (and indeed this whole book) is not a place to learn R and Python completely. We will cover specific tasks and how to carry them out in each language, but we will not cover the every intricacy of each language. There are numerous sources (books, websites, courses) that are available to do that. A lot of these places would argue that you should not learn multiple programming languages from one book, and instead concentrate on a single skill at a time. We agree, and the single skill to concentrate on with this book is the use of software to solve applied mathematical problems. The particular software itself is not the most important component.

2.1 SOFTWARE INSTALLATION

There are a number of different places from which you can buy your vegetables, you can grow them yourself, you can go to a market and pick fresh fruit from specific stalls, you can go to a supermarket and buy a bag of a collection of vegetables and in some places you can even get a box of vegetables regularly posted to you. Software is similar, there are a variety of places from which you can get it and a number of different forms in which it can be obtained.

If you're comfortable with using R and Python then you probably do not need to read this section and you might even use different so called "distributions" of each piece of software, but for the purpose of this book here is where we will be getting what we need:

- Python: we will use the Anaconda distribution: `https://www.anaconda.com/distribution/`
- R: we will be getting this directly from the Comprehensive R Archive Network (commonly referred to as CRAN): `https://cran.r-project.org`. We will also use another piece of software called Rstudio: `https://rstudio.com`.

2.1.1 Installing Python

Installing Python and all the software we need around it is done by downloading and running the installer for the Anaconda distribution.

1. Go to this webpage: `https://www.anaconda.com/download/`.
2. Identify and download the version of Python 3 for your operating system (Windows, Mac OSX, Linux). Run the installer.

2.1.2 Installing R

There are actually two pieces of software we need to install to use R for the purposes of this book, first the R language itself and second an application with which we will write R code.

1. Go to this webpage: `https://cran.r-project.org`.
2. Identify and download the latest version of R for your operating system (Windows, Mac OSX, Linux). Run the installer.
3. Go to this webpage: `https://rstudio.com`.
4. Identify and download the latest version of Rstudio for your operating system (Windows, Mac OSX, Linux). Run the installer.

2.2 USING THE COMMAND LINE

There are various interfaces to using a computer, the most common one is to use a mouse and keyboard and click on programmes we want to use. Another approach is to use what is called a command line interface this is where we do not interact graphically with a computer but we type in specific commands.

We can use our command line to navigate the various directories on our computer.

There are two types of operating systems that we consider here:

- Windows
- Nix: this includes OSX (the Mac operating system) and Linux

Not all commands are the same on each type of operating system.

So let us start by opening our command line interface:

- Windows: after having installed Anaconda look to open the Anaconda Prompt. There are a number of other command line interfaces available but this is the one we recommend for the purposes of this book.
- Nix: look to open the Terminal.

This should open something that looks like and somewhat resembles a black box with some text in it. This is where we will write our commands to the computer.

For example to list the contents of the directory we are currently in:

On nix:

Cli input

```
ls
```

On Windows

Cli input

```
dir
```

It is also possible to get the name of the directory we are currently in:
On nix:

Cli input

```
pwd
```

On Windows

Cli input

```
cd
```

Finally we can also use the command line to move to another directory. The command for this are the same on Nix and on Windows.

Cli input

```
cd <name_of_subdirectory>
```

The command line is an important tool to learn to use when doing tasks:

- If we want to scale the tasks, a commonly heard phrase is that 'mouse clicks do not scale' highlighting that to repeat a task many times when using a graphical interface is inefficient.

- If we want someone else to be able to repeat the tasks, we can use screenshots of graphical interfaces but there will always be a level of ambiguity whereas the commands used in the command line are precise.

We can use our two programming languages right within the command line interface (we will actually be using a different tool that we will describe shortly).

To use Python, simply type the following and press Enter:

Cli input

```
python
```

This should make something like the following appear:

Cli output

```
Python 3.7.1 | packaged by conda-forge | (default, Nov 13 2018, 10:30:07)
[Clang 4.0.1 (tags/RELEASE_401/final)] :: Anaconda, Inc. on darwin
Type "help", "copyright", "credits" or "license" for more information.
>>>
```

The >>> is a prompt ready to accept a Python command. Let us start with the following:

Python input

```
>>> 2 + 2
```

When you press Enter, this will give:

Python output

```
4
```

This particular way of using Python is called a REPL which stands for: 'Read Eval Print Loop' which indicates that it takes a command, evaluates it and waits for the next one.

To quit Python's REPL type the following (note that `()`, more about that later):

Python input

```
>>> quit()
```

We can do the same for R. To start R's REPL, in your command line type the following and press Enter:

Cli input

```
R
```

This should make something like the following appear:

Cli output

```
R version 3.5.1 (2018-07-02) -- "Feather Spray"
Copyright (C) 2018 The R Foundation for Statistical Computing
Platform: x86_64-apple-darwin13.4.0 (64-bit)

R is free software and comes with ABSOLUTELY NO WARRANTY.
You are welcome to redistribute it under certain conditions.
Type 'license()' or 'licence()' for distribution details.

  Natural language support but running in an English locale

R is a collaborative project with many contributors.
Type 'contributors()' for more information and
'citation()' on how to cite R or R packages in publications.

Type 'demo()' for some demos, 'help()' for on-line help, or
'help.start()' for an HTML browser interface to help.
Type 'q()' to quit R.

>
```

The > is a prompt ready to accept an R command. Let us start with the following:

R input

```
> 2 + 2
```

When you press Enter, this will give:

R output

```
4
```

To quit R's REPL type the following:

R input

```
> q()
```

This will bring up a further prompt asking you to save some information about what you just did. You can type `n` for now:

R input

```
> Save workspace image? [y/n/c]: n
```

These two REPLs are not unique and also not the most efficient way of using the languages, however they can at times be useful if you just want to type a very short command or perhaps check something quickly.

Another approach is to save a collection of commands in a plain text file and pass it to the interpreter at the command line.

For example, if we had a number of Python commands in `main.py` we could run this at the command line using:

Cli input

```
python main.py
```

Similarly for a file with a number of R commands `main.R`:

Cli input

```
Rscript main.R
```

These are just a few of many ways to use Python and R. An important notion to understand is that Python and R are not the particular tools that we use to interface to them. On a day to day basis the authors of this book will use both of the above approaches as well as the next ones, we recommend readers take time to experiment and understand the particular use cases for which each tool works best for them.

The two tools we recommend to use in this book are:

- For Python: the Jupyter notebook, a tool that behaves similarly to a REPL, runs in the web browser and is very popular in research.
- For R: RStudio, an integrated development environment with a lot of helpful features.

The best way to start the Jupyter notebook is to type the following in your command line:

Cli input

```
jupyter notebook
```

This will create a *notebook server* that runs on your computer and should open a page that looks like Note that despite running in a web browser this does not need the internet to run.

We can create a new notebook and write and run code in the *cells*.

To start Rstudio, locate the application on your computer and double click on it. This will open an application that looks like

Rstudio includes its on REPL, so we can type and run single commands there but we can also write in a file that we can run

In the next sections we will cover some basics of Python and R.

2.3 BASIC PYTHON

This section gives a very brief overview of some introductory aspects of Python, there are excellent resources available for learning Python and we recommend the reader goes there if they feel they need an in depth understanding of the language

In the previous section, we saw how to get Python to perform a single calculation:

Python input

```
print(3 + 5)
```

which will give:

Python output

```
8
```

We can also assign values to a variable:

Python input

```
a = 3
b = 5
c = a + b
print(c)
```

This makes a point at 3 etc...

which will give:

Python output

```
8
```

There are a number of different types of variables in Python, here is a very brief list of some of them:

- Integers – `int` – for example `2`, `4`, `-459060`.
- Floats – `float` – for example `2.0`, `3.4`, `-3.459060`.
- Strings – `str` – for example `"two"`, `"hello world"`, `"3450"`.
- Booleans – `bool` – for example `True` or `False`.

Based on the values of a variable it is possible to construct Booleans:

Python input

```
is_a_larger_than_b = a > b
```

The variable `is_a_larger_than_b` will be the boolean variable `False`.

This is an important concept as boolean variable allow us to use conditional statements that let us write code that does specific things based on the value of variables. For example the following code will add 5 to the smallest variable:

Python input

```
a = 3
b = 5
if a < b:
    a = a + 3
elif a > b:
    b = b + 5
else:
    a = a + 3
    b = b + 3
print(a, b)
```

which gives:

Python output

```
6 5
```

If you are experimenting by typing the code as you go change the value of `a` or `b` to see how the behaviour changes. What happens if they are equal?

It is also possible to use these conditional statements to repeat code. For example the following code will repeatedly add 1 to the smallest variable until it becomes equal to the largest one:

Python input

```
a = 3
b = 5
while a != b:
    if a < b:
        a = a + 1
    else:
        b = b + 1
```

It is important to be able to reuse code, this is done using a programming concept called a *function*, which acts similarly to a mathematical function.

The following code, creates a function that takes two variables as input and outputs the largest number and the smallest increased by 3.

Python input

```
def add_3_to_smallest(a, b):
    """This function adds 3 to the smallest of a or b."""
    if a < b:
        return a + 3, b
    return a, b + 3
```

Once we have defined the function, the following is how we use it:

Python input

```
print(add_3_to_smallest(a=5, b=-42))
```

which gives:

Python output

```
(5, -39)
```

Python has a type of variable that is in fact a collection of pointers to other variables. This is called a list. Here for example is a collection of strings:

Python input

```python
tennis_players = [
    "Federer",
    "S. Williams",
    "V. Williams",
    "King",
]
```

There are a number of things that can be done with lists but one particular aspect is that they are a sub type of something called an iterable in Python which means we can iterate over them. We do this in Python using a `for` loop. For example, the following code will iterate over the list and print all the values:

Python input

```python
for name in tennis_players:
    print(name)
```

which gives:

Python output

```
Federer
S. Williams
V. Williams
King
```

We will often want to iterate over a set of integers, Python has a `range` command that can create such a set with ease. The following code will print every 3 integers from 30 to 50:

Python input

```python
for integer in range(30, 50, 3):
    print(integer)
```

which will give:

Python output

```
30
33
36
39
42
45
48
```

A final important aspect of Python is that of libraries. The code examples above are from the so called 'standard library' but Python has numerous libraries specific to given problems. A lot of these libraries came bundled with the anaconda distribution but if you want to download one that is not you can always do so as long as you have an internet connection.

For example, to download a library for studying queueing systems `ciw` open your command line interface and type the following:

Cli input

```
pip install ciw
```

Once you restart your python interpreter, for example if you are using a Jupyter notebook then restart the Kernel, you can then run the following to make `ciw` available to you:

Python input

```python
import ciw
```

2.4 BASIC R

This section gives a very brief overview of some introductory aspects of R, there are excellent resources available for learning R [1] and we recommend the reader goes there if they feel they need an in depth understanding of the language

In the previous section, we saw how to get R to perform a single calculation:

R input

```r
print(3 + 5)
```

which will give:

R output

```
[1] 8
```

We can also assign values to a variable:

R input

```r
a <- 3
b <- 5
c <- a + b
print(c)
```

which will give:

R output

```
[1] 8
```

An important difference between R and Python is that in R the base structure is in fact a vector, even if it only contains a single variable. We can use the `c` command to *concatenate* these base structures together:

R input

```r
print(c(a, 4))
```

giving:

R output

```
[1] 3 4
```

There are a number of different types of variables in R, here is a very brief list of some of them:

- Integers – `integer` – for example `2`, `4`, `-459060`.
- Floats – `double` – for example `2.0`, `3.4`, `-3.459060`.
- Strings – `character` – for example `"two"`, `"hello world"`, `"3450"`.
- Booleans – `logical` – for example `TRUE` or `FALSE`.

Based on the values of a variable it is possible to construct Booleans:

R input

```
is_a_larger_than_b <- a > b
```

The variable `is_a_larger_than_b` will be the boolean variable `FALSE`.

This is an important concept as boolean variable allow us to use conditional statements that let us write code that does specific things based on the value of variables. For example the following code will add 5 to the smallest variable:

R input

```
a <- 3
b <- 5
if (a < b) {
  a <- a + 3
} else if (a > b) {
  b <- b + 3
} else {
  a <- a + 3
  b <- b + 3
}
print(c(a, b))
```

which gives:

R output

```
[1] 6 5
```

If you are experimenting by typing the code as you go, change the value of `a` or `b` to see how the behaviour changes. What happens if they are equal?

R is a so called "vectorized" language which means that there is often a more appropriate approach to doing things repeatedly using vectors. This applies to the `if` statement in that there exists a `ifelse` statement that applies to vectors of booleans. For example:

R input

```
booleans <- c(FALSE, TRUE, FALSE, FALSE)
print(ifelse(booleans, "cat", "dog"))
```

which gives:

R output

```
[1] "dog" "cat" "dog" "dog"
```

It is also possible to use conditional statements to repeat code. For example the following code will repeatedly add 1 to the smallest variable until it becomes equal to the largest one:

R input

```
a <- 3
b <- 5
while (a != b) {
  if (a < b) {
    a <- a + 1
  }
  else {
    b <- b + 1
  }
}
```

It is important to be able to reuse code, this is done using a programming concept called a *function*, which acts similarly to a mathematical function.

The following code creates a function that takes two variables as input and outputs the largest number and the smallest increased by 3.

R input

```
add_3_to_smallest <- function(a, b) {
  # This function adds 3 to the smallest of a or b.
  if (a < b) {
    return(c(a + 3, b))
  }
  else {
    return(c(a, b + 3))
  }
}
```

Note that R will implicitly return the last computed expression without the need for a `return` statement. So the above can also be written as:

R input

```
add_3_to_smallest <- function(a, b) {
  # This function adds 3 to the smallest of a or b.
  if (a < b) {
    c(a + 3, b)
  }
  else {
    c(a, b + 3)
  }
}
```

Once we have defined the function, the following is how we use it:

R input

```
print(add_3_to_smallest(a = 5, b = -42))
```

which gives:

R output

```
[1]   5 -39
```

It is possible to iterate over elements inside R vectors:

R input

```
tennis_players <- c("Federer",
                    "S. Williams",
                    "V. Williams",
                    "King")
```

The following will print all the names contained in the vector:

R input

```
for (name in tennis_players) {
    print(name)
}
```

which gives:

R output

```
[1] "Federer"
[1] "S. Williams"
[1] "V. Williams"
[1] "King"
```

We will often want to iterate over a vector of integers, R has a `seq` command that can create such a vector with ease. The following code will print every 3 integers from 30 to 50:

R input

```
for (i in seq(30, 50, 3)) {
  print(i)
}
```

which will give:

R output

```
[1] 30
[1] 33
[1] 36
[1] 39
[1] 42
[1] 45
[1] 48
```

A final important aspect of R is that of packages. The code examples above are from the so called 'base R' but R has numerous packages specific to given problems. If you want to download and use one you can always do so as long as you have an internet connection.

For example, to download a very common collection of data science tools called `tidyverse` we use the following line of code inside of an R session:

R input

```
install.packages("simmer")
```

Once this package is installed it is loaded using

R input

```
library(simmer)
```

2.5 A NOTE ON HOW CODE IS DISPLAYED IN THIS BOOK

FURTHER READING

Becskei, A. and Serrano, L. (2000). Engineering stability in gene networks by autoregulation. *Nature,* 405: 590–593.

Rosenfeld, N., Elowitz, M.B., and Alon, U. (2002). Negative auto-regulation speeds the response time of transcription networks. *J. Mol. Biol.*, 323: 785–793.

Savageau, M.A. (1976). *Biochemical Systems Analysis: A study of Function and Design in Molecular Biology.* Addison-Wesley. Chap. 16.

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II

Probabilistic Modelling

CHAPTER 3

Markov Chains

MANY real world situations have some level of unpredictability through randomness: the flip of a coin, the number of orders of coffee in a shop, the winning numbers of the lottery. However, mathematics can in fact let us make predictions about what we expect to happen. One tool used to understand randomness is Markov chains, an area of mathematics sitting at the intersection of probability and linear algebra.

3.1 PROBLEM

Consider a barber shop. The shop owners have noticed that customers will not wait if there is no room in their waiting room and will choose to take their business elsewhere. The Barber shop would like to make an investment so as to avoid this situation. They know the following information:

- They currently have 2 barber chairs (and 2 barbers).
- They have waiting room for 4 people.
- They usually have 10 customers arrive per hour.
- Each Barber takes about 15 minutes to serve a customer so they can serve 4 customers an hour.

This is represented diagrammatically in Figure 3.1.

They are planning on reconfiguring space to either have 2 extra waiting chairs or another barber's chair and barber.

The mathematical tool used to model this situation is a Markov Processes.

3.2 THEORY

A Markov Process is a model of a sequence of random events that is defined by a collection of **states** and rules that define how to move between these states.

For example, in the barber shop a single number is sufficient to describe the status of the shop. If that number is 1 this implies that 1 customer is currently having their

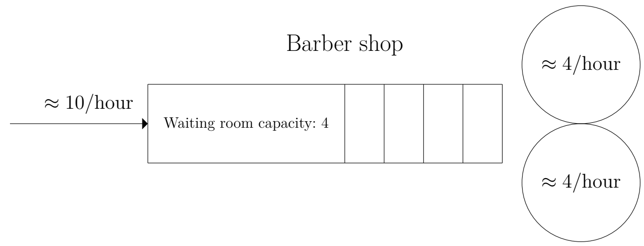


Figure 3.1 Diagrammatic representation of the barber shop as a queuing system.

hair cut. If that number is 5 this implies that 2 customers are being served and 3 are waiting. Thus the state space for this particular Markov Process is:

$$S = \{0, 1, 2, 3, 4, 5, 6\} \tag{3.1}$$

As customers arrive and leave the system goes between states as shown in Figure 3.2.

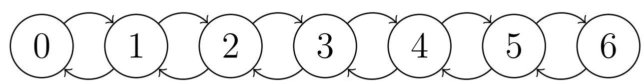


Figure 3.2 Diagrammatic representation of the state space

The rules that govern how to move between these states can be defined in two ways:

- Using probabilities of changing state (or not) in a well defined time period. This is called a discrete Markov process.
- Using rates of change from one state to another. This is called a continuous time Markov process.

For our barber shop we will consider it as a continuous Markov process as shown in Figure 3.3

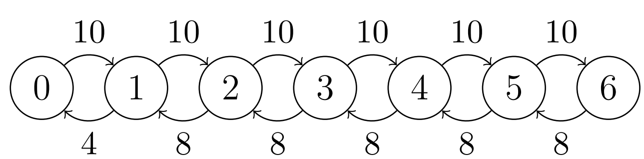


Figure 3.3 Diagrammatic representation of the state space and the transition rates

Note that a Markov process assumes the rates follow an exponential distribution. One interesting property of this distribution is that it is considered memoryless which means that if a customer has been having their hair cut for 5 minutes this does not change the rate at which their service ends. This distribution is quite common in the real world and therefore a common assumption.

These states and rates can be represented mathematically using a transition matrix Q where Q_{ij} represents the rate of going from state i to state j. In this case we have:

$$Q = \begin{pmatrix} -10 & 10 & 0 & 0 & 0 & 0 & 0 \\ 4 & -14 & 10 & 0 & 0 & 0 & 0 \\ 0 & 8 & -18 & 10 & 0 & 0 & 0 \\ 0 & 0 & 8 & -18 & 10 & 0 & 0 \\ 0 & 0 & 0 & 8 & -18 & 10 & 0 \\ 0 & 0 & 0 & 0 & 8 & -18 & 10 \\ 0 & 0 & 0 & 0 & 0 & 8 & -8 \end{pmatrix} \tag{3.2}$$

You will see that Q_{ii} are negative and ensure the rows of Q sum to 0. This gives the total rate of change leaving state i.

We can use Q to understand the probability of being in a given state after t time unis. This is can be represented mathematically using a matrix P_t where $(P_t)_{ij}$ is the probability of being in state j after t time units having started in state i. We can use Q to calculate P_t using the matrix exponential:

$$P_t = e^{Qt} \tag{3.3}$$

What is also useful is understanding the long run behaviour of the system. This allows us to answer questions such as "what state are we most likely to be in on average?" or "what is the probability of being in the last state on average?".

This long run probability distribution over the state can be represented using a vector π where π_i represents the probability of being in state i. This vector is in fact the solution to the following matrix equation:

$$\pi Q = 0 \tag{3.4}$$

In the upcoming sections we will demonstrate all of the above concepts.

3.3 SOLVING WITH PYTHON

The first step we will take is to write a function to obtain the transition rates between two given states:

Python input

```
def get_transition_rate(
    in_state,
    out_state,
    waiting_room=4,
    number_of_barbers=2,
    arrival_rate=10,
    service_rate=4,
):
    """Return the transition rate for two given states.

    Args:
        in_state: an integer denoting the current state
        out_state: an integer denoting the next state
        waiting_room: an integer denoting the size of the
                      waiting room (default: 4)
        number_of_barbers: an integer denoting the number of
                           barber and chairs (default: 2)
        arrival_rate: a real number denoting the number of
                      individuals per unit time that arrive at
                      the barber shop (default: 10)
        service_rate: a real number denoting the number of
                      individuals per unit time that a single
                      barber can serve (default: 4)

    Returns:
        A real.
    """
    capacity = waiting_room + number_of_barbers
    delta = out_state - in_state

    if delta == 1 and in_state < capacity:
        return arrival_rate

    if delta == -1:
        return min(in_state, number_of_barbers) * service_rate

    return 0
```

Next, we write a function that creates an entire transition rate matrix Q for a given problem. We will use the `numpy` to handle all the linear algebra and the `itertools` library for some iterations:

Python input

```
import itertools
import numpy as np
```

Now we define the function:

Python input

```python
def get_transition_rate_matrix(
    waiting_room=4,
    number_of_barbers=2,
    arrival_rate=10,
    service_rate=4,
):
    """Return the transition matrix Q.

    Args:
        waiting_room: an integer denoting the size of the
                      waiting room (default: 4)
        number_of_barbers: an integer denoting the number of
                           barber and chairs (default: 2)
        arrival_rate: a real number denoting the number of
                      individuals per unit time that arrive at
                      the barber shop (default: 10)
        service_rate: a real number denoting the number of
                      individuals per unit time that a single
                      barber can serve (default: 4)

    Returns:
        A matrix.
    """
    capacity = waiting_room + number_of_barbers
    state_pairs = itertools.product(
        range(capacity + 1), repeat=2
    )
    flat_transition_rates = [
        get_transition_rate(
            in_state=in_state,
            out_state=out_state,
            waiting_room=waiting_room,
            number_of_barbers=number_of_barbers,
            arrival_rate=arrival_rate,
            service_rate=service_rate,
        )
        for in_state, out_state in state_pairs
    ]
    transition_rates = np.reshape(
        flat_transition_rates, (capacity + 1, capacity + 1)
    )
    np.fill_diagonal(
        transition_rates, -transition_rates.sum(axis=1)
    )

    return transition_rates
```

Using this we can obtain the matrix Q for our default system:

Python input

```
Q = get_transition_rate_matrix()
print(Q)
```

which gives:

Python output

```
[[-10  10   0   0   0   0   0]
 [  4 -14  10   0   0   0   0]
 [  0   8 -18  10   0   0   0]
 [  0   0   8 -18  10   0   0]
 [  0   0   0   8 -18  10   0]
 [  0   0   0   0   8 -18  10]
 [  0   0   0   0   0   8  -8]]
```

We can take the matrix exponential as discussed above. To do this, we need to use the `scipy` library:

Python input

```
import scipy.linalg
```

To see what would happen after .5 time units we obtain:

Python input

```
print(scipy.linalg.expm(Q * 0.5).round(5))
```

which gives:

Python output

```
[[0.10492 0.21254 0.20377 0.17142 0.13021 0.09564 0.0815 ]
 [0.08501 0.18292 0.18666 0.1708  0.14377 0.1189  0.11194]
 [0.06521 0.14933 0.16338 0.16478 0.15633 0.14751 0.15346]
 [0.04388 0.10931 0.13183 0.15181 0.16777 0.18398 0.21142]
 [0.02667 0.07361 0.10005 0.13422 0.17393 0.2189  0.27262]
 [0.01567 0.0487  0.07552 0.11775 0.17512 0.24484 0.32239]
 [0.01068 0.03668 0.06286 0.10824 0.17448 0.25791 0.34914]]
```

To see what would happen after 500 time units we obtain:

Python input

```
print(scipy.linalg.expm(Q * 500).round(5))
```

which gives:

Python output

```
[[0.03431 0.08577 0.10722 0.13402 0.16752 0.2094  0.26176]
 [0.03431 0.08577 0.10722 0.13402 0.16752 0.2094  0.26176]
 [0.03431 0.08577 0.10722 0.13402 0.16752 0.2094  0.26176]
 [0.03431 0.08577 0.10722 0.13402 0.16752 0.2094  0.26176]
 [0.03431 0.08577 0.10722 0.13402 0.16752 0.2094  0.26176]
 [0.03431 0.08577 0.10722 0.13402 0.16752 0.2094  0.26176]
 [0.03431 0.08577 0.10722 0.13402 0.16752 0.2094  0.26176]]
```

We see that no matter what state (column) the system is in, after 500 time units the probabilities are all the same. We could in fact stop our analysis here, however our choice of 500 time units was arbitrary and might not be the correct amount for all possible scenarios, as such we will continue to aim to solve the underlying equation 3.4 directly.

To do this we will solve the underlying system using a numerically efficient algorithm called least squares optimisation (available from the `numpy` library):

Python input

```
def get_steady_state_vector(Q):
    """Return the steady state vector of any given continuous
    time transition rate matrix.

    Args:
       Q: a transition rate matrix

    Returns:
        A vector
    """
    state_space_size, _ = Q.shape
    A = np.vstack((Q.T, np.ones(state_space_size)))
    b = np.append(np.zeros(state_space_size), 1)
    x, _, _, _ = np.linalg.lstsq(A, b, rcond=None)
    return x
```

So if we now see the steady state vector for our default system:

Python input

```
print(get_steady_state_vector(Q).round(5))
```

we get:

Python output

```
[0.03431 0.08577 0.10722 0.13402 0.16752 0.2094  0.26176]
```

We can see that the shop is expected to be empty approximately 3.4% of the time and full 26.2% of the time.

The final function we will write is one that uses all of the above to just return the probability of the shop being full.

Python input

```python
def get_probability_of_full_shop(
    waiting_room=4,
    number_of_barbers=2,
    arrival_rate=10,
    service_rate=4,
):
    """Return the probability of the barber shop being full.

    Args:
        waiting_room: an integer denoting the size of
                      the waiting room (default: 4)

        number_of_barbers: an integer denoting the number
                           of barber and chairs
                           (default: 2)

        arrival_rate: a real number denoting the number of
                      individuals per unit time that arrive
                      at the barber shop (default: 10)

        service_rate: a real number denoting the number of
                      individuals per unit time that a single
                      barber can serve (default: 4)

    Returns:
        A real.
    """
    Q = get_transition_rate_matrix(
        waiting_room=waiting_room,
        number_of_barbers=number_of_barbers,
        arrival_rate=arrival_rate,
        service_rate=service_rate,
    )
    pi = get_steady_state_vector(Q)
    return pi[-1]
```

We can now confirm the previous probability calculated probability of the shop being full:

Python input

```
print(round(get_probability_of_full_shop(), 6))
```

which gives:

Python output

```
0.261756
```

If we were too have 2 extra space in the waiting room:

Python input

```
print(round(get_probability_of_full_shop(waiting_room=6), 6))
```

which gives:

Python output

```
0.23557
```

This is a slight improvement however, increasing the number of barbers has a substantial effect:

Python input

```
print(
    round(get_probability_of_full_shop(number_of_barbers=3), 6)
)
```

Python output

```
0.078636
```

In fact the only way to lower the probability of the shop being full below the 10% threshold is to have barbers work at the faster rate of 4.6 customers per time unit:

Python input

```python
print(
    round(
        get_probability_of_full_shop(
            waiting_room=20, service_rate=4.6
        ),
        6,
    )
)
```

Python output

```
0.094487
```

3.4 SOLVING WITH R

The first step we will take is write a function to obtain the transition rates between two given states:

R input

```r
#' Return the transition rate for two given states.
#'
#' @param in_state an integer denoting the current state
#' @param out_state an integer denoting the next state
#' @param waiting_room an integer denoting the size of
#'        the waiting room (default: 4)
#' @param number_of_barbers an integer denoting the number
#'        of barber and chairs (default: 2)
#' @param arrival_rate a real number denoting the number
#'        of individuals per  unit time that arrive at
#'        the barber shop (default: 10)
#' @param service_rate a real number denoting the number
#'        of individuals per unit time that a single barber
#'        can serve (default: 4)
#'
#' @return A real
get_transition_rate <- function(
                                in_state,
                                out_state,
                                waiting_room = 4,
                                number_of_barbers = 2,
                                arrival_rate = 10,
                                service_rate = 4) {
  capacity <- waiting_room + number_of_barbers
  delta <- out_state - in_state

  if (delta == 1) {
    if (in_state < capacity) {
      return(arrival_rate)
    }
  }

  if (delta == -1) {
    return(min(in_state, number_of_barbers) * service_rate)
  }
  return(0)
}
```

We will not actually use this function but a vectorized version of this:

R input

```
vectorized_get_transition_rate <- Vectorize(
  get_transition_rate,
  vectorize.args = c("in_state", "out_state")
)
```

This function can now take a vector of inputs for the `in_state` and `out_state` variables which will allow us to simplify the following code that creates the matrices:

R input

```r
#' Return the transition rate matrix Q
#'
#' @param waiting_room an integer denoting the size of the
#'        waiting room (default: 4)
#' @param number_of_barbers an integer denoting the number of
#'        barber and chairs (default: 2)
#' @param arrival_rate a real number denoting the number of
#'        individuals per #' unit time that arrive at the
#'        barber shop (default: 10)
#' @param service_rate a real number denoting the number of
#'        individuals per unit time that a single barber
#'        can serve (default: 4)
#'
#' @return A matrix
get_transition_rate_matrix <- function(
                                       waiting_room = 4,
                                       number_of_barbers = 2,
                                       arrival_rate = 10,
                                       service_rate = 4) {
  max_state <- waiting_room + number_of_barbers

  Q <- outer(0:max_state,
    0:max_state,
    vectorized_get_transition_rate,
    waiting_room = waiting_room,
    number_of_barbers = number_of_barbers,
    arrival_rate = arrival_rate,
    service_rate = service_rate
  )
  row_sums <- rowSums(Q)

  diag(Q) <- -row_sums
  Q
}
```

Using this we can obtain the matrix Q for our default system:

R input

```
Q <- get_transition_rate_matrix()
print(Q)
```

which gives:

R output

```
     [,1] [,2] [,3] [,4] [,5] [,6] [,7]
[1,]  -10   10    0    0    0    0    0
[2,]    4  -14   10    0    0    0    0
[3,]    0    8  -18   10    0    0    0
[4,]    0    0    8  -18   10    0    0
[5,]    0    0    0    8  -18   10    0
[6,]    0    0    0    0    8  -18   10
[7,]    0    0    0    0    0    8   -8
```

One immediate thing we can do with this matrix is take the matrix exponential discussed above. To do this, we need to use an R library call `expm`:

R input

```
library(expm, warn.conflicts = FALSE, quietly = TRUE)
```

To be able to make use of the nice %>% ”pipe” operator we are also going to load the `dplyr` library:

R input

```
library(dplyr, warn.conflicts = FALSE, quietly = TRUE)
```

Now if we wanted to see what would happen after .5 time units we obtain:

R input

```
print( (Q * .5) %>% expm %>% round(5))
```

which gives:

R output

```
        [,1]    [,2]    [,3]    [,4]    [,5]    [,6]    [,7]
[1,] 0.10492 0.21254 0.20377 0.17142 0.13021 0.09564 0.08150
[2,] 0.08501 0.18292 0.18666 0.17080 0.14377 0.11890 0.11194
[3,] 0.06521 0.14933 0.16338 0.16478 0.15633 0.14751 0.15346
[4,] 0.04388 0.10931 0.13183 0.15181 0.16777 0.18398 0.21142
[5,] 0.02667 0.07361 0.10005 0.13422 0.17393 0.21890 0.27262
[6,] 0.01567 0.04870 0.07552 0.11775 0.17512 0.24484 0.32239
[7,] 0.01068 0.03668 0.06286 0.10824 0.17448 0.25791 0.34914
```

After 500 time units we obtain:

R input

```
print( (Q * 500) %>% expm %>% round(5))
```

which gives:

R output

```
        [,1]    [,2]    [,3]    [,4]    [,5]   [,6]    [,7]
[1,] 0.03431 0.08577 0.10722 0.13402 0.16752 0.2094 0.26176
[2,] 0.03431 0.08577 0.10722 0.13402 0.16752 0.2094 0.26176
[3,] 0.03431 0.08577 0.10722 0.13402 0.16752 0.2094 0.26176
[4,] 0.03431 0.08577 0.10722 0.13402 0.16752 0.2094 0.26176
[5,] 0.03431 0.08577 0.10722 0.13402 0.16752 0.2094 0.26176
[6,] 0.03431 0.08577 0.10722 0.13402 0.16752 0.2094 0.26176
[7,] 0.03431 0.08577 0.10722 0.13402 0.16752 0.2094 0.26176
```

We see that no matter what state (row) we are in, after 500 time units the probabilities are all the same. We could in fact stop our analysis here, however our choice of 500 time units was arbitrary and might not be the correct amount for all possible scenarios, as such we will continue to aim to solve the underlying equation 3.4 directly.

To be able to do this, we will make use of the versatile `pracma` package which includes a number of numerical analysis functions for efficient computations.

R input

```r
library(pracma, warn.conflicts = FALSE, quietly = TRUE)

#' Return the steady state vector of any given continuous time
#' transition rate matrix
#'
#' @param Q a transition rate matrix
#'
#' @return A vector
get_steady_state_vector <- function(Q){
  state_space_size <- dim(Q)[1]
  A <- rbind(t(Q), 1)
  b <- c(integer(state_space_size), 1)
  mldivide(A, b)
}
```

This is making use of `pracma`'s `mldivide` function which chooses the best numerical algorithm to find the solution to a given matrix equation $Ax = b$.

So if we now see the steady state vector for our default system:

R input

```r
print(get_steady_state_vector(Q))
```

we get:

R output

```
           [,1]
[1,] 0.03430888
[2,] 0.08577220
[3,] 0.10721525
[4,] 0.13401906
[5,] 0.16752383
[6,] 0.20940479
[7,] 0.26175598
```

We can see that the shop is expected to be empty approximately 3.4% of the time and full 26.2% of the time.

The final piece of this puzzle is to create a single function that uses all of the above to just return the probability of the shop being full.

R input

```r
#' Return the probability of the barber shop being full
#'
#' @param waiting_room an integer denoting the size of the
#'        waiting room (default: 4)
#' @param number_of_barbers an integer denoting the number of
#'        barber and chairs (default: 2)
#' @param arrival_rate a real number denoting the number of
#'        individuals per #' unit time that arrive at the
#'        barber shop (default: 10)
#' @param service_rate a real number denoting the number of
#'        individuals per unit time that a single barber can
#'        serve (default: 4)
#'
#' @return A real
get_probability_of_full_shop <- function(
                                  waiting_room = 4,
                                  number_of_barbers = 2,
                                  arrival_rate = 10,
                                  service_rate = 4) {
  pi <- get_transition_rate_matrix(
    waiting_room = waiting_room,
    number_of_barbers = number_of_barbers,
    arrival_rate = arrival_rate,
    service_rate = service_rate
  ) %>%
    get_steady_state_vector()

  capacity <- waiting_room + number_of_barbers
  pi[capacity + 1]
}
```

Now we can run this code efficiently with both scenarios:

R input

```r
print(get_probability_of_full_shop(waiting_room = 6))
```

which decreases the probability of a full shop to:

R output

```
[1] 0.2355699
```

but adding another barber and chair:

R input

```
print(get_probability_of_full_shop(number_of_barbers = 3))
```

gives:

R output

```
[1] 0.0786359
```

In fact even with room for 20 people to wait the only way 2 barbers would be able to have a less than 10% of the shop being full is to find a way to each serve .6 more of a customer per hour:

R input

```
print(
    get_probability_of_full_shop(
        waiting_room = 20,
        service_rate = 4.6)
        )
```

R output

```
[1] 0.09448688
```

3.5 RESEARCH

TBA

CHAPTER 4

Discrete Event Simulation

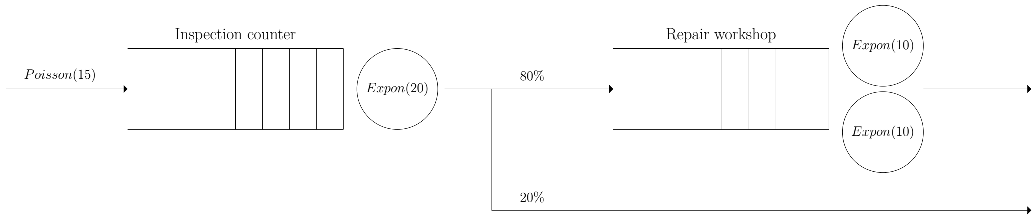
Complex situations further compounded by randomness appear throughout our daily lives. For example, data flowing through a computer network, patients being treated at an emergency services, and daily commutes to work. Mathematics can be used to understand these complex situations so as to make predications which in turn can be used to make improvements. One tool used to do this is to let a computer create a dynamic virtual representation of the scenario in question, the particular type we are going to cover here is called Discrete Event Simulation.

4.1 PROBLEM

Consider the following situation: a bicycle repair shop would like reconfigure their set-up in order to guarantee that all bicycles processed by the repair shop take a maximum of 30 minutes. Their current set-up is as follows:

- Bicycles arrive randomly at the shop at a rate of 15 per hour.
- They wait in line to be seen at an inspection counter, manned by one member of staff who can inspect one bicycle at a time. On average an inspection takes around 3 minutes.
- After inspection it is found that around 20% of bicycles do not need repair, and they are then ready for collection.
- After inspection is is found that around 80% of bicycles go on to be repaired. These then wait in line outside the repair workshop, which is manned by two members of staff who can each repair one bicycle at a time. On average a repair takes around 6 minutes.
- After repair the bicycles are ready for collection.

A diagram of the system is shown below:



We can also assume that there is infinite capacity at the bicycle repair shop for waiting bicycles. The shop will hire and extra member of staff in order to meet their target of a maximum throughout of 30 minutes. They would like to know if they should work on the inspection counter or in the repair workshop?

4.2 THEORY

A number of the events that govern the behaviour of the bicycle shop above are probabilistic. For example the times that bicycles arrive at the shop, the duration of the inspection and repairs, and whether the bicycle would need to go on to be repaired or not. When a number of these probabilistic events are arranged in a complex system such as the bicycle shop, using analytical methods to manipulate these probabilities becomes difficult. One method to deal with this is *simulation*.

Consider one probabilistic event, rolling a die. A die has six sides numbered 1 to 6, each side is equally likely to land. Therefore the probability of rolling a 1 is $\frac{1}{6}$, the probability of rolling a 2 is $\frac{1}{6}$, and so on. This means that that if we roll the die a large number of times, we would except $\frac{1}{6}$ of those rolls to be a 1. This is called the *law of large numbers*.

Now imagine we have an event in which we do not know the analytical probability of it occurring. Consider rolling a weighted die, in this case a die in which the probability of obtaining one number is much greater than the others. How can we estimate the probability of obtaining a 5 on this die?

Rolling the weighted die once does not give us much information. However due to the law of large numbers, we can roll this die a number of times, and find the proportion of those rolls which gave a 5. The more times we roll the die, the closer this proportion approaches the underlying probability of obtaining a 5.

For a complex system such as the bicycle shop, we would like to estimate the proportion of bicycles that take longer than 30 minutes to be processed. As it is a complex system it is difficult to work this out analytically. So we would like to observe this system a number of times and record the overall proportions of bicycles spending longer than 30 minutes in the shop, which will converge to the true underlying proportion. However unlike rolling a weighted die, it it costly to observe this shop over a number of days with identical conditions. In this case it is costly in terms of time, as the repair shop already exists. However some scenarios, for example the scenario where the repair shop hires and additional member of staff, do not yet exist, so observing this this would be costly in terms of money also. We can however build a virtual representation of this complex system on a computer, and observe a virtual day of work much more quickly and much less costly on the computer, similar to a video game.

In order to do this, the computer needs to be able to generate random outcomes of each of the smaller events that make up the large complex system. Generating random events are essentially doing things to random numbers, that need to be generated.

Computers are deterministic, therefore true randomness is not always possible. They can however generate pseudorandom numbers: sequences of numbers that look like random numbers, but are entirely determined from the previous numbers in the sequence. Most programming languages have methods of doing this.

In order to simulate an event we can again manipulate the law of large numbers. Let $X \sim U(0, 1)$, a uniformly pseudorandom variable between 0 and 1. Let R be the outcome of a roll of an unbiased die. Then R can be defined as:

$$R = \begin{cases} 1 & \text{if } 0 \leq X < \frac{1}{6} \\ 2 & \text{if } \frac{1}{6} \leq X < \frac{2}{6} \\ 3 & \text{if } \frac{2}{6} \leq X < \frac{3}{6} \\ 4 & \text{if } \frac{3}{6} \leq X < \frac{4}{6} \\ 5 & \text{if } \frac{4}{6} \leq X < \frac{5}{6} \\ 6 & \text{if } \frac{5}{6} \leq X < 1 \end{cases} \tag{4.1}$$

The bicycle repair shop is a system made up of interactions of a number of other simpler random events. This can be thought of as many interactions of random variables, each generated using pseudorandom numbers.

In this case the fundemental random event that need to be generated are:

- time each bicycle arrives to the repair shop,
- the time each bicycle spends at the inspection counter,
- whether each bicycle needs to go on the the repair workshop,
- the time each those bicycles spends at the repair shop.

As the simulation progresses these events will be generated, and will interact together as described in Section 4.1. The proportion of customers spending longer than 30 minutes in the shop can then be counted. This proportion itself is a random variable, and so just like the weighted die, running this simulation once does not give us much information. But we can run the simulation many times and take an average proportion, to smooth out any variability.

The process outlined above is a particular implementation of Monte Carlo simulation called *discrete event simulation*, which generates pseudorandom numbers and observes their interactions. In practice there are two main approaches to simulating complex probabilistic systems such as this one: the *event scheduling* approach, and *process based* simulation. It just so happens that the main implementations in Python and R use each of these approaches, so you will see both approaches used here.

4.2.1 Event Scheduling Approach

When using the event scheduling approach, we can think of the 'virtual representation' of the system as being the facilities that the bicycles use, and let entities (the bicycles) interact with these facilities. It is these facilities that determine how the entities behave.

In a simulation that uses an event scheduling approach, a key concept is that events occur that cause further events to occur in the future, either immediately or after a delay, such as after some time in service. In the bicycle shop examples of such events include a bicycle joining a queue, a bicycle beginning service, and a bicycle finishing service. At each event the event list is updated, and the clock then jumps forward to the next event in this updated list.

4.2.2 Process Based Simulation

When using process based simulation, we can think of the 'virtual representation' of the system as being the sequence of actions that each entity (the bicycles) must take, and these sequences of actions might contain delays as a number of entities seize and release a finite amount of resources. It is the sequence of actions that determine how the entities behave.

For the bicycle repair shop an example of one possible sequence of actions would be:

arrive → *seize inspection counter* → *delay* → *release inspection counter* → *seize repair shop* → *delay* → *release repair shop* → *leave*

The scheduled delays in this sequence of events correspond to the time spend being inspected and the time spend being repaired. Waiting in line for service at these facilities are not included in the sequence of events; that is implicit by the 'seize' and 'release' actions, as an entity will wait for a free resource before seizing one. Therefore in process based simulations, in addition to defining a sequence of events, resource types and their numbers also need to be defined.

4.3 SOLVING WITH PYTHON

In this book we will use the Ciw library in order to conduct discrete event simulation. Ciw uses the event scheduling approach, which means we must define the system's facilities, and then let customers loose to interact with them.

In this case there are two facilities to define: the inspection desk and the repair workshop. Let's order these as so. For each of these we need to define:

- the distribution of times between consecutive bicycles arriving,
- the distribution of times the bicycles spend in service,
- the number of servers available,
- the probability of routing to each of the other facilities after service.

In this case we will assume that the time between consecutive arrivals follow a exponential distribution, and that the service times also follow exponential distributions. These are common assumptions for this sort of queueing system.

In Ciw, these are defined in a Network object, created using the `ciw.create_network` function. The code below uses this function to create a Network object that defines the current bicycle repair shop:

Python input

```
import ciw

N = ciw.create_network(
    arrival_distributions=[ciw.dists.Exponential(15), ciw.dists.NoArrivals()],
    service_distributions=[ciw.dists.Exponential(20), ciw.dists.Exponential(10)],
    number_of_servers=[1, 2],
    routing=[[0.0, 0.8], [0.0, 0.0]]
)
```

This function takes arguments that are used to define each of the four properties we listed above. Each of these arguments take in a list of properties, for each facility respectively in their order. Arguments asking for for distributions take in Ciw objects defining those distributions, `ciw.dists.Exponential` for exponential distributions, and `ciw.dists.NoArrivals` for when there are no external arrivals at the repair workshop (all bicycles arriving there are routed from the inspection desk). The `number_of_servers` argument takes a list of integers (1 server at the inspection desk, 2 at the repair workshop). The `routing` argument defines how bicycles are routed after service: from the inspection desk there is zero probability of returning to the inspection desk, and a probability of 0.8 of being routed to the repair workshop; from the repair workshop bicycles are not routed anywhere, they leave the system.

Now this we have defined the system, we need to use this to build the virtual representation of the system: in Ciw this is a Simulation object:

Python input

```
ciw.seed(0)
Q = ciw.Simulation(N)
```

Notice here a random seed is set. This is because there is some element of randomness when initialising this object, and in order to ensure reproducible results we force the pseudorandom number generator to produce the same sequence of pseudorandom numbers each time.

Now we have a virtual representation of the system, we can run this for one eight hour working day:

Python input

```
Q.simulate_until_max_time(8)
```

Note that the simulation always begins with an empty system, so the first bicycle to arrive will never wait for service. Depending on the situation this may be an unwanted feature, though not in this case as it is reasonable to assume that the bicycle shop will begin the day with no customers. Now we'll count the number of bicycles that have finished service, and count the number of those whose entire journey through the system lasted longer than 0.5 hours:

Python input

```
left_individuals = Q.nodes[-1].all_individuals
number_over_30mins = 0
for ind in left_individuals:
    throughput = 0
    for record in ind.data_records:
        throughput += record.exit_date - record.arrival_date
    if throughput > 0.5:
        number_over_30mins += 1
print(number_over_30mins / len(left_individuals))
```

The first line here obtains a list of all the individuals who have left the system, that is reach the final (index -1) node. Each individual will have a data record for each facility they visited. This contains many pieces of information about their time at that facility, the ones relevant here are their `arrival_date` and their `exit_date`, the difference of which gives their total time spent at that facility. Summing the total time spent at every facility gives their overall throughput in the system.

This piece of code gives

Python output

```
0.26126126126126126
```

meaning 26.13% of all bicycles spent longer than half an hour at the repair shop. However this particular day may have contained a number of extreme events.

For a more accurate proportion this experiment should be repeated, and an average proportion taken. In order to do so, let's write a function that performs the above experiment, so that we can eventually repeat the function call.

Python input

```python
def find_percentage_over_30mins(n_inspectors, n_repairers, seed):
    """Returns the percentage of bicycles spending over 30 minutes
    at the repair shop in one run of the simulation.

    Args:
        n_inspectors: the number of servers at the inspection desk
        n_repairers: the number of servers in the repair workshop
        seed: the seed for the pseudorandom number generator

    Returns:
        A real.
    """
    N = ciw.create_network(
        arrival_distributions=[ciw.dists.Exponential(15), ciw.dists.NoArrivals()],
        service_distributions=[ciw.dists.Exponential(20), ciw.dists.Exponential(10)],
        number_of_servers=[n_inspectors, n_repairers],
        routing=[[0.0, 0.8], [0.0, 0.0]]
    )

    ciw.seed(seed)
    Q = ciw.Simulation(N)
    Q.simulate_until_max_time(24)

    left_individuals = Q.nodes[-1].all_individuals
    number_over_30mins = 0
    for ind in left_individuals:
        throughput = 0
        for record in ind.data_records:
            throughput += record.exit_date - record.arrival_date
        if throughput > 0.5:
            number_over_30mins += 1
    return number_over_30mins / len(left_individuals)
```

This can be used to find the average proportion over 100 trials:

Python input

```
percentage_over_30 = []
for trial in range(100):
    percentage_over_30.append(find_percentage_over_30mins(1, 2, trial))
print(sum(percentage_over_30) / len(percentage_over_30))
```

which gives:

Python output

```
0.15935355368513382
```

that is, on average 15.94% of bicycles will spend longer than 30 minutes at the repair shop.

Now consider the two possible future scenarios we wish top compare: hiring an extra member of staff to serve at the inspection desk, or hiring an extra member of staff at the repair workshop. Which scenario yields a smaller proportion of bicycles spending over 30 minutes at the shop? Let's investigate. First look at the situation where the additional member of staff works at the inspection desk:

Python input

```
percentage_over_30 = []
for trial in range(100):
    percentage_over_30.append(find_percentage_over_30mins(2, 2, trial))
print(sum(percentage_over_30) / len(percentage_over_30))
```

which gives:

Python output

```
0.038476805648229126
```

that is 3.85% of bicycles.

Now look at the situation where the additional member of staff works at the repair workshop:

Python input

```
percentage_over_30 = []
for trial in range(100):
    percentage_over_30.append(find_percentage_over_30mins(1, 3, trial))
print(sum(percentage_over_30) / len(percentage_over_30))
```

which gives:

Python output

```
0.10359146418929761
```

that is 10.36% of bicycles.

Therefore an additional member of staff at the inspection desk would be more beneficial than an additional member of staff at the repair workshop.

4.4 SOLVING WITH R

In this book we will use the Simmer package in order to conduct discrete event simulation. Simmer uses the process based approach, which means we must define the each bicycle's sequence of actions, and then generate a number of bicycles with these sequences.

In Simmer these sequences of actions are made up of called trajectories. The diagram below shows the branched trajectories than a bicycle would take at the repair shop:

These are defined in simmer by:

R input

```
library(simmer)

bicycle <-
  trajectory("Inspection") %>%
  seize("Inspector") %>%
  timeout(function() {rexp(1, 20)}) %>%
  release("Inspector") %>%
  branch(
    function() (runif(1) < 0.8), continue=c(F),
    trajectory("Repair") %>%
      seize("Repairer") %>%
      timeout(function() {rexp(1, 10)}) %>%
      release("Repairer"),
    trajectory("Out")
  )
```

Here we define a `bicycle` object which is made up of Simmer `trajectory`s. You will see that this sequence of event matches those in Figure ??. First the bicycle seizes an "Inspector" resource (yet to be defined), pauses for some service time, sampled from an exponential distribution with parameter 20, then releases the "Inspector" resource, so that resource is free to be seized by another bicycle. Then trajectory branches on the condition that an uniformly pseudorandom variable lies above or lies 0.8: if below begin the "Repair" sub-trajectory, otherwise the "Out" sub-trajectory which does not have any actions. The "Repair trajectory first seizes a "Repairer" resource (again, yet to be defined), pauses for some service time, sampled from an exponential distribution with parameter 10, then releases the "Repairer" resource. Once there are no actions left in the sequence the bicycle leaves the system.

These trajectories are not very useful alone, we are yet to define the resources used, or a way to generate bicycles with these trajectories. This is done in the code below, where a `repair_shop` is defined:

R input

```
repair_shop <-
  simmer("Repair Shop") %>%
  add_resource("Inspector", 1) %>%
  add_resource("Repairer", 2) %>%
  add_generator("Bicycle", bicycle, function() {rexp(1, 15)})
```

Here we have added one resource labelled "Inspector", and two resources labelled

"Repairer". Finally we have added a generator which generates a number of the `bicycle` objects that we defined earlier. This generator also takes in a function to sample the delays between generating new `bicycle` objects, that is the inter-arrival times.

Now we have a virtual representation of the system, we can run this for one eight hour working day:

R input

```
set.seed(seed)
repair_shop %>% run(until=8)
```

Note that the simulation always begins with an empty system, so the first bicycle to arrive will never wait for service. Depending on the situation this may be an unwanted feature, though not in this case as it is reasonable to assume that the bicycle shop will begin the day with no customers. Now we'll count the number of bicycles that have finished service, and count the number of those whose entire journey through the system lasted longer than 0.5 hours:

R input

```
recs <- repair_shop %>% get_mon_arrivals()
throughput = recs$end_time - recs$start_time
print(mean(throughput > 0.5))
```

The first line here obtains a data frame of information about all the arrivals to the system. This data frame will have columns containing a limited number of pieces of information about the arrivals, however much more information can be recorded and monitored by adding monitoring actions to the trajectories. The relevant columns here are the `start_time` and `end_time`, the difference of which gives their total time spent in the system, the throughput. A new vector is created containing throughput information. We then calculate the proportion of entries in that vector over half an hour by taking the mean of a vector of boolean values checking this condition (this works because the booleans **`TRUE`** and RFALSE have numeric values of 1 and 0 respectively).

This piece of code gives

R output

```
[1] 0.02777778
```

meaning 27.78% of all bicycles spent longer than half an hour at the repair shop.

However this particular day may have contained a number of extreme events. For a more accurate proportion this experiment should be repeated, and an average proportion taken. In order to do so, let's write a function that performs the above experiment, so that we can eventually repeat the function call.

R input

```r
#' Returns the percentage of bicycles spending over 30 minutes
#' at the repair shop in one run of the simulation.
#'
#' @param n_inspectors the number of servers at the inspection desk
#' @param n_repairers the number of servers in the repair workshop
#' @param seed the seed for the pseudorandom number generator
#'
#' @return A real
find_percentage_over_30mins <- function(n_inspectors, n_repairers, seed) {
  bicycle <-
    trajectory("Inspection") %>%
    seize("Inspector") %>%
    timeout(function() {rexp(1, 20)}) %>%
    release("Inspector") %>%
    branch(
      function() (runif(1) < 0.8), continue=c(F),
      trajectory("Repair") %>%
        seize("Repairer") %>%
        timeout(function() {rexp(1, 10)}) %>%
        release("Repairer"),
      trajectory("Out")
    )

  repair_shop <-
    simmer("Repair Shop") %>%
    add_resource("Inspector", n_inspectors) %>%
    add_resource("Repairer", n_repairers) %>%
    add_generator("Bicycle", bicycle, function() {rexp(1, 15)})

  set.seed(seed)
  repair_shop %>% run(until=8)
  recs <- repair_shop %>% get_mon_arrivals()
  throughput = recs$end_time - recs$start_time
  return(mean(throughput > 0.5))
}
```

This can be used to find the average proportion over 100 trials:

R input

```
percentage_over_30 <- c()
for (seed in 1:100) {
  percentage_over_30[seed] <- find_percentage_over_30mins(1, 2, seed)
}
print(mean(percentage_over_30))
```

which gives:

R output

```
[1] 0.1551579
```

that is, on average 15.52% of bicycles will spend longer than 30 minutes at the repair shop.

Now consider the two possible future scenarios we wish top compare: hiring an extra member of staff to serve at the inspection desk, or hiring an extra member of staff at the repair workshop. Which scenario yields a smaller proportion of bicycles spending over 30 minutes at the shop? Let's investigate. First look at the situation where the additional member of staff works at the inspection desk:

R input

```
percentage_over_30 <- c()
for (seed in 1:100) {
  percentage_over_30[seed] <- find_percentage_over_30mins(2, 2, seed)
}
print(mean(percentage_over_30))
```

which gives:

R output

```
[1] 0.04115338
```

that is 4.12% of bicycles.

Now look at the situation where the additional member of staff works at the repair workshop:

R input

```
percentage_over_30 <- c()
for (seed in 1:100) {
  percentage_over_30[seed] <- find_percentage_over_30mins(1, 3, seed)
}
print(mean(percentage_over_30))
```

which gives:

R output

```
[1] 0.1000899
```

that is 10.01% of bicycles.

Therefore an additional member of staff at the inspection desk would be more beneficial than an additional member of staff at the repair workshop.

4.5 RESEARCH

TBA

Bibliography

[1] Hadley Wickham. *Advanced r.* Chapman and Hall/CRC, 2014.