$N-$bits reactive strategies in repeated games

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1 Introduction

In this work we explore *reactive strategies* in the infinitely repeated donation game. The donation is a (2×2) symmetric game where at each turn players p and q, simultaneously and independently, decide to cooperate (C) or to defect (D). Thus, there are four outcomes in each single round, $xy \in \{CC, CD, DC, DD\}$, where x and y represent p's and q's choices respectively.

Each then receives a payoff. The following (2×2) payoff matrix describes the payoffs of both players in each round,

$$\begin{array}{c c} & \begin{array}{cc} \text{cooperate} & \text{defect} \end{array} \\ \begin{array}{c} \text{cooperate} \\ \text{defect} \end{array} & \begin{pmatrix} b-c & -c \\ b & 0 \end{pmatrix} \end{array} \tag{1}$$

where $b > c$. Alternatively, we can define the payoff vectors for each player by

$$S_p = (b-c, -c, b, 0) \quad \text{and} \quad S_q = (b-c, b, -c, 0). \tag{2}$$

Reactive strategies are strategies that take into account the actions of the co-player to make a decision in each turn of the repeated game. Reactive strategies are studied in the literature due to their mathematical tractability. A play between two reactive strategies can be described as a Markov process with a transition matrix M. The expected payoffs to p and q, can then be explicitly calculated using the the stationary distribution $\mathbf{v}$ of M and the respective round payoffs S_p and S_q.

2 Results

2.1 1-bit reactive

The literature has extensively studied reactive strategies that take into account only the last turn of the opponent. Here will reefer to these as *one-bit reactive strategies*. One-bit reactive strategies can be written as a 2-tuple $p = (p_C, p_D)$ where p_C is the probability of cooperating after the co-player has cooperated and p_D after they defected.

The play of reactive strategies can be modelled as a Markov chain. In the case of the one-bit reactive strategies, there are only 4 possibles states $\{CC, CD, DC, DD\}$ and the transition matrix is given by,

$$M_1 = \begin{bmatrix} p_1 q_1 & p_1 (1-q_1) & q_1 (1-p_1) & (1-p_1)(1-q_1) \\ p_2 q_1 & p_2 (1-q_1) & q_1 (1-p_2) & (1-p_2)(1-q_1) \\ p_1 q_2 & p_1 (1-q_2) & q_2 (1-p_1) & (1-p_1)(1-q_2) \\ p_2 q_2 & p_2 (1-q_2) & q_2 (1-p_2) & (1-p_2)(1-q_2) \end{bmatrix}. \tag{3}$$

A probability distribution $\mathbf{v}^1$ on the set of outcomes is a non-negative vector with unit sum, indexed by the four states for which,

$$\mathbf{v}^1 M^1 = \mathbf{v}^1.$$

With respect to $\mathbf{v}^1$ the expected payoffs to p and q, denoted $\pi_{(p,q)}$ and $\pi_{(q,p)}$, are the dot products with the corresponding payoff vectors:

$$\pi_{(p,q)} = \mathbf{v}^1 \cdot S_p \quad \text{and} \quad \pi_{(q,p)} = \mathbf{v}^1 \cdot S_q. \tag{4}$$

In the case of the one-bit reactive strategies the payoffs' analytical expressions are tractable. For example,

$$\pi_{(p,q)} = \frac{c(\mathrm{p1q2} + \mathrm{p2}(-\mathrm{q2}) + \mathrm{p2}) - b(\mathrm{p2}(\mathrm{q1} - \mathrm{q2}) + \mathrm{q2})}{(\mathrm{p1} - \mathrm{p2})(\mathrm{q1} - \mathrm{q2}) - 1} \tag{5}$$

2.2 2-bits reactive

In the case of *two-bit reactive strategies*, strategies are based on the actions of the co-player in the previous two rounds. Since for a single round there are 4 possible outcomes, for two rounds there will be $4 \times 4 = 16$ possible situations.

We denote the states as $E_x E_y | F_x F_y$ where the outcome of the previous round is $E_x E_y$ and the outcome of the current round is $F_x F_y$ and $E_x, E_y, F_x, F_y \in \{C, D\}$ then the state space is $\{CC|CC, CC|CD, CC|DC, CC|DD, CD|CC, CD|CD, CD|DC, CD|DD, DC|CC, DC|CD, DC|DC, DC|DD, DD|CC, DD|CD, DD|DC, DD|DD\}$. Thus, $p = (p_1, p_2, \dots, p_{16})$ corresponding to the previous state $E_x E_y | F_x F_y \in \{CC|CC, CC|CD, CC|DC, \dots, DD|CD, DD|DC, DD|DD\}$. Similarly, $q = (q_1, q_2, \dots, q_{16})$. Following the same approach as the one-bit case, a play between two two-bits reactive strategies can be described by a Markov process where now the states are given by $E_x, E_y, F_x, F_y \in \{C, D\}$, and the stationary distribution $\mathbf{v}^2 \in R^{16}_{[0,1]}$ is the solution to $\mathbf{v}^2 M^2 = \mathbf{v}^2$. The transition matrix M^2 is a (16×16) matrix where $M(v^2_{n+1} = G_x G_y | H_x H_y | v^2_n = E_x E_y | F_x F_y = 0)$ if $G_x G_y \neq F_x F_y$, so in each row of the matrix there will be at most four nonzero elements. Thus,

$$M^2 = \begin{pmatrix}
\mathrm{p1q1} & \mathrm{p1(1-q1)} & \mathrm{(1-p1)q1} & \mathrm{(1-p1)(1-q1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathrm{p2q1} & \mathrm{p2(1-q1)} & \mathrm{(1-p2)q1} & \mathrm{(1-p2)(1-q1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{p1q2} & \mathrm{p1(1-q2)} & \mathrm{(1-p1)q2} & \mathrm{(1-p1)(1-q2)} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{p2q2} & \mathrm{p2(1-q2)} & \mathrm{(1-p2)q2} & \mathrm{(1-p2)(1-q2)} \\
\mathrm{p3q1} & \mathrm{p3(1-q1)} & \mathrm{(1-p3)q1} & \mathrm{(1-p3)(1-q1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathrm{p4q1} & \mathrm{p4(1-q1)} & \mathrm{(1-p4)q1} & \mathrm{(1-p4)(1-q1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{p3q2} & \mathrm{p3(1-q2)} & \mathrm{(1-p3)q2} & \mathrm{(1-p3)(1-q2)} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{p4q2} & \mathrm{p4(1-q2)} & \mathrm{(1-p4)q2} & \mathrm{(1-p4)(1-q2)} \\
\mathrm{p1q3} & \mathrm{p1(1-q3)} & \mathrm{(1-p1)q3} & \mathrm{(1-p1)(1-q3)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathrm{p2q3} & \mathrm{p2(1-q3)} & \mathrm{(1-p2)q3} & \mathrm{(1-p2)(1-q3)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{p1q4} & \mathrm{p1(1-q4)} & \mathrm{(1-p1)q4} & \mathrm{(1-p1)(1-q4)} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{p2q4} & \mathrm{p2(1-q4)} & \mathrm{(1-p2)q4} & \mathrm{(1-p2)(1-q4)} \\
\mathrm{p3q3} & \mathrm{p3(1-q3)} & \mathrm{(1-p3)q3} & \mathrm{(1-p3)(1-q3)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathrm{p4q3} & \mathrm{p4(1-q3)} & \mathrm{(1-p4)q3} & \mathrm{(1-p4)(1-q3)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{p3q4} & \mathrm{p3(1-q4)} & \mathrm{(1-p3)q4} & \mathrm{(1-p3)(1-q4)} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{p4q4} & \mathrm{p4(1-q4)} & \mathrm{(1-p4)q4} & \mathrm{(1-p4)(1-q4)}
\end{pmatrix}.$$

Computing the stationary distribution of M^2 analytically is not possible. Thus, computing the payoffs π for two generic two-bits reactive strategies is not possible either.

However, some expressions are still obtainable.

2.2.1 2-bits reactive strategies against ALLC and ALLD

For $p = (p_1, p_2, \dots, p_{16})$ and $q = (0, 0, \dots, 0)$ (ALLD),

$$\pi_{(p,\text{ALLD})} = -cp_4.$$

For $p = (p_1, p_2, \dots, p_{16})$ and $q = (1, 1, \dots, 1)$ (ALLC),

$$\pi_{(p,\text{ALLC})} = b - cp_1.$$

2.2.2 2-bits deterministic reactive strategies.

There are a total of (4^2) 16 two-bit deterministic reactive strategies, Table 1.

p_1	p_2	p_3	p_4	name
0	0	0	0	ALLD
0	0	0	1	S1
0	0	1	0	S2
0	0	1	1	S3
0	1	0	0	S4
0	1	0	1	S5
0	1	1	0	S6
0	1	1	1	S7
1	0	0	0	S8
1	0	0	1	S9
1	0	1	0	S10
1	0	1	1	S11
1	1	0	0	S12
1	1	0	1	S13
1	1	1	0	S14
1	1	1	1	ALLC

Table 1: **Deterministic two-bits reactive strategies**

.

	ALLD	S1	S2	S3	S4	S5	S6	S7	S8	S9	S10	S11	S12	S13	S14	ALLC
ALLD	0	b	0	b	0	b	0	b	0	b	0	b	0	b	0	b
S1	$-c$	$-c$	$-c$	b	$-c$	b	$0.333b$ - $0.333c$	b	$0.333b$ - $0.5c$	b	$0.4b$ - $0.4c$	b	$0.429b$ - $0.429c$	b	$0.5b$ - $0.333c$	b
S2	0	b	$0.5b$ - $0.5c$	$0.5b$ - $0.5c$	0	b	0	b	0	b	$0.5b$ - $0.5c$	$0.5b$ - $0.5c$	0	b	0	b
S3	$-c$	$-c$	$0.5b$ - $0.5c$	$0.5b$ - $0.5c$	$-c$	b	$-c$	b	$0.429b$ - $0.571c$	b	$0.5b$ - $0.5c$	$0.5b$ - $0.5c$	$0.5b$ - $0.5c$	b	$0.571b$ - $0.429c$	b
S4	0	b	0	b	$0.5b$ - $0.5c$	$0.5b$ - $0.5c$	0	b	0	b	0	b	$0.5b$ - $0.5c$	$0.5b$ - $0.5c$	0	b
S5	$-c$	$-c$	$-c$	$-c$	$0.5b$ - $0.5c$	$0.5b$ - $0.5c$	$0.571b$ - $0.429c$	b	$0.4b$ - $0.6c$	$0.429b$ - $0.571c$	$0.5b$ - $0.5c$	b	$0.5b$ - $0.5c$	$0.5b$ - $0.5c$	$0.6b$ - $0.4c$	b
S6	0	$0.333b$ - $0.333c$	0	b	0	$0.429b$ - $0.571c$	$0.333b$ - $0.667c$	b	0	$0.667b$ - $0.667c$	$0.571b$ - $0.571c$	b	0	b	$0.667b$ - $0.333c$	b
S7	$-c$	$-c$	$-c$	$-c$	$-c$	$-c$	$-c$	$-c$	$0.5b$ - $0.667c$	$0.667b$ - $0.667c$	$0.6b$ - $0.6c$	b	$0.571b$ - $0.571c$	b	$0.667b$ - $0.5c$	b
S8	0	$0.5b$ - $0.333c$	0	$0.571b$ - $0.429c$	0	$0.6b$ - $0.4c$	0	$0.667b$ - $0.5c$	b - c	$0.667b$ - $0.333c$	b - c	b - c	b - c	b - c	b - c	b - c
S9	$-c$	$-c$	$-c$	$-c$	$-c$	$0.571b$ - $0.429c$	$0.667b$ - $0.667c$	$0.667b$ - $0.667c$	$0.333b$ - $0.667c$	$0.333b$ - $0.667c$	$0.429b$ - $0.429c$	b - c	b - c	b - c	b - c	b - c
S10	0	$0.4b$ - $0.4c$	$0.5b$ - $0.5c$	$0.5b$ - $0.5c$	0	$0.5b$ - $0.5c$	$0.571b$ - $0.571c$	$0.6b$ - $0.6c$	b - c	$0.429b$ - $0.429c$	$0.5b$ - $0.5c$	$0.5b$ - $0.5c$	b - c	b - c	b - c	b - c
S11	$-c$	$-c$	$0.5b$ - $0.5c$	$0.5b$ - $0.5c$	$-c$	$-c$	$-c$	$-c$	b - c	b - c	$0.5b$ - $0.5c$	$0.5b$ - $0.5c$	b - c	b - c	b - c	b - c
S12	0	$0.429b$ - $0.429c$	0	$0.5b$ - $0.5c$	$0.5b$ - $0.5c$	$0.5b$ - $0.5c$	0	$0.571b$ - $0.571c$	b - c	b - c	b - c	b - c	$0.5b$ - $0.5c$	$0.5b$ - $0.5c$	b - c	b - c
S13	$-c$	$-c$	$-c$	$-c$	$0.5b$ - $0.5c$	$0.5b$ - $0.5c$	$-c$	$-c$	b - c	b - c	b - c	b - c	$0.5b$ - $0.5c$	$0.5b$ - $0.5c$	b - c	b - c
S14	0	$0.333b$ - $0.5c$	0	$0.429b$ - $0.571c$	0	$0.4b$ - $0.6c$	$0.333b$ - $0.667c$	$0.5b$ - $0.667c$	b - c	b - c	b - c	b - c	b - c	b - c	b - c	b - c
ALLC	$-c$	$-c$	$-c$	$-c$	$-c$	$-c$	$-c$	$-c$	b - c	b - c	b - c	b - c	b - c	b - c	b - c	b - c

	ALLD	S1	S2	S3	S4	S5	S6	S7	S8	S9	S10	S11	S12	S13	S14	ALLC
ALLD	0	$-c$	0	$-c$	0	$-c$	0	$-c$	0	$-c$	0	$-c$	0	$-c$	0	$-c$
S1	b	b	b	$-c$	b	$-c$	$0.333b - 0.333c$	$-c$	$0.5b - 0.333c$	$-c$	$0.4b - 0.4c$	$-c$	$0.429b - 0.429c$	$-c$	$0.333b - 0.5c$	$-c$
S2	0	$-c$	$0.5b - 0.5c$	$0.5b - 0.5c$	0	$-c$	0	$-c$	0	$-c$	$0.5b - 0.5c$	$0.5b - 0.5c$	0	$-c$	0	$-c$
S3	b	b	$0.5b - 0.5c$	$0.5b - 0.5c$	b	$-c$	b	$-c$	$0.571b - 0.429c$	$-c$	$0.5b - 0.5c$	$0.5b - 0.5c$	$0.5b - 0.5c$	$-c$	$0.429b - 0.571c$	$-c$
S4	0	$-c$	0	$-c$	$0.5b - 0.5c$	$0.5b - 0.5c$	0	$-c$	0	$-c$	0	$-c$	$0.5b - 0.5c$	$0.5b - 0.5c$	0	$-c$
S5	b	b	b	b	$0.5b - 0.5c$	$0.5b - 0.5c$	$0.429b - 0.571c$	$-c$	$0.6b - 0.4c$	$0.571b - 0.429c$	$0.5b - 0.5c$	$-c$	$0.5b - 0.5c$	$0.5b - 0.5c$	$0.4b - 0.6c$	$-c$
S6	0	$0.333b - 0.333c$	0	$-c$	0	$0.571b - 0.429c$	$0.667b - 0.333c$	$-c$	0	$0.667b - 0.667c$	$0.571b - 0.571c$	$-c$	0	$-c$	$0.333b - 0.667c$	$-c$
S7	b	b	b	b	b	b	b	b	$0.667b - 0.5c$	$0.667b - 0.667c$	$0.6b - 0.6c$	$-c$	$0.571b - 0.571c$	$-c$	$0.5b - 0.667c$	$-c$
S8	0	$0.333b - 0.5c$	0	$0.429b - 0.571c$	0	$0.4b - 0.6c$	0	$0.5b - 0.667c$	$b - c$	$0.333b - 0.667c$	$b - c$	$b - c$	$b - c$	$b - c$	$b - c$	$b - c$
S9	b	b	b	b	b	$0.429b - 0.571c$	$0.667b - 0.667c$	$0.667b - 0.667c$	$0.667b - 0.333c$	$0.667b - 0.333c$	$0.429b - 0.429c$	$b - c$	$b - c$	$b - c$	$b - c$	$b - c$
S10	0	$0.4b - 0.4c$	$0.5b - 0.5c$	$0.5b - 0.5c$	0	$0.5b - 0.5c$	$0.571b - 0.571c$	$0.6b - 0.6c$	$b - c$	$0.429b - 0.429c$	$0.5b - 0.5c$	$0.5b - 0.5c$	$b - c$	$b - c$	$b - c$	$b - c$
S11	b	b	$0.5b - 0.5c$	$0.5b - 0.5c$	b	b	b	b	$b - c$	$b - c$	$0.5b - 0.5c$	$0.5b - 0.5c$	$b - c$	$b - c$	$b - c$	$b - c$
S12	0	$0.429b - 0.429c$	0	$0.5b - 0.5c$	$0.5b - 0.5c$	$0.5b - 0.5c$	0	$0.571b - 0.571c$	$b - c$	$b - c$	$b - c$	$b - c$	$0.5b - 0.5c$	$0.5b - 0.5c$	$b - c$	$b - c$
S13	b	b	b	b	$0.5b - 0.5c$	$0.5b - 0.5c$	b	b	$b - c$	$b - c$	$b - c$	$b - c$	$0.5b - 0.5c$	$0.5b - 0.5c$	$b - c$	$b - c$
S14	0	$0.5b - 0.333c$	0	$0.571b - 0.429c$	0	$0.6b - 0.4c$	$0.667b - 0.333c$	$0.667b - 0.5c$	$b - c$	$b - c$	$b - c$	$b - c$	$b - c$	$b - c$	$b - c$	$b - c$
ALLC	b	b	b	b	b	b	b	b	$b - c$	$b - c$	$b - c$	$b - c$	$b - c$	$b - c$	$b - c$	$b - c$

Using the formulation of (**??**) we can numerically compute the payoffs without simulations.

3 Numerical Results

We use an evolutionary process where. on Nowak and Imphof

For proof that our formulation is correct to the Jupyter Notebook “Numerical simulations”.