Reactive strategies with longer memory

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1 Formal Model

We consider infinitely repeated games among two players, player 1 and player 2. Each round, they engage in the donation game with payoff matrix

$$\begin{pmatrix} b-c & -c \\ b & 0 \end{pmatrix}. \tag{1}$$

Here b and c denote the benefit and the cost of cooperation, respectively. We assume $b > c > 0$ throughout. Therefore, payoff matrix (1) is a special case of the prisoner's dilemma with payoff matrix,

$$\begin{pmatrix} R & S \\ T & P \end{pmatrix}, \tag{2}$$

where $T > R > S > P$ and $2R > T + S$. Here, R is the reward payoff of mutual cooperation, T is the temptation to defect payoff, S is the sucker's payoff, and P is the punishment payoff for mutual defection.

We assume in the following, that the players' decisions only depend on the outcome of the previous n rounds. To this end, an *n-history for player $i \in \{1, 2\}$* is a string $h^i = (a^i_{-n}, \dots, a^i_{-1}) \in \{C, D\}^n$. An entry a^i_{-k} corresponds to player i's action k rounds ago. Let H^i denote the space of all n-histories for $i \in \{1, 2\}$. Set H^i contains $|H^i| = 2^n$ elements. A pair $h = (h^1, h^2)$ is called an *n-history of the game*. We use $H = H^1 \times H^2$ to denote the space of all such histories. This set contains $|H| = 2^{2n}$ elements.

A *memory-n* strategy is a vector $\mathbf{m} = (m_h)_{h\in H} \in [0, 1]^{2n}$. Each entry m_h corresponds to the player's cooperation probability in the next round, depending on the outcome of the previous n rounds. One special case of memory$-n$ strategies are the round$-k-$repeat strategies for some $1 \le k \le n$. Player 1 uses a *round-k-repeat strategy* $\mathbf{m}^{k-\text{Rep}}$ if in any given round, the player chooses the same action as k rounds ago. That is, if the game's n-history is such that,

$$\begin{cases} m^{k-\text{Rep}}_h = 1, & \text{if } a^1_{-k} = C \\ m^{k-\text{Rep}}_h = 0, & \text{if } a^1_{-k} = D. \end{cases}$$

Two additional special cases of memory$-n$ strategies that we will be discussing in this work are, reactive$-n$ and self-reactive-n strategies. A *reactive$-n$ strategy* for player 1 is denoted by a vector $\mathbf{p} = (p_h)_{h\in H^2} \in [0, 1]^n$. Each entry p_h corresponds to the player's cooperation probability in the next round, based on the co-player's actions in the previous n rounds. Therefore, reactive-n strategies exclusively rely on the co-player's n-history, independent of the focal player's own actions during the past n rounds. On the other hand, *self-reactive-n* strategies only consider the focal player's own n-history, and ignore the co-player's. Formally, a self-reactive-n strategy for player 1 is a vector $\tilde{\mathbf{p}} = (\tilde{p}_h)_{h \in H^1} \in [0,1]^n$. Each entry $\tilde{p}_h$ corresponds to the player's cooperation probability in the next, depending on the player's own actions in the previous n rounds. From hereon, we will use the notations $\mathbf{m}, \mathbf{p},$ and $\tilde{\mathbf{p}}$ to denote a memory-n, a reactive-n, and a self-reactive-n strategy.

Let players 1 and 2 use memory$-n$ strategies $\mathbf{m}$ and $\mathbf{m}'$. Then one can represent the interaction as a Markov chain. The set of states of the chain is the space of all possible histories H. The $2^{2n} \times 2^{2n}$ transition matrix M, describes the transition probabilities between all possible histories. For $h = (h^1, h^2), \tilde{h} = (\tilde{h}^1, \tilde{h}^2) \in H$,

$$M_{h,\tilde{h}} = \begin{cases} m_h \cdot m'_h & \text{if } a^1_{-k} = C \text{ and } a^2_{-k} = C \\ m_h \cdot (1 - m'_h) & \text{if } a^1_{-k} = C \text{ and } a^2_{-k} = D \\ (1 - m_h) \cdot m'_h & \text{if } a^1_{-k} = D \text{ and } a^2_{-k} = C \\ (1 - m_h) \cdot (1 - m'_h) & \text{if } a^1_{-k} = D \text{ and } a^2_{-k} = D \\ 0, & \text{if } ((a^1_{-(n-1)}, \ldots, a^1_{-1}), (a^2_{-(n-1)}, \ldots, a^2_{-1})) \neq ((\tilde{a}^1_{-n}, \ldots, \tilde{a}^1_{-2}), (\tilde{a}^2_{-n}, \ldots, \tilde{a}^2_{-2})) \end{cases}$$

The final case ensures that the only reachable states are those in which the outcomes of these states match the previous outcomes in the 'next' state.

Let $\mathbf{v} = (v_h)_{h \in H}$ be an invariant distribution of this Markov chain. Based on the invariant distribution $\mathbf{v}$, we can also compute the players' payoffs. To this end, let $\mathbf{S}^k = (S^k_h)_{h \in H}$ denote the vector that returns for each h the one-shot payoff that player 1 obtained k rounds ago,

$$S^k_h = \begin{cases} b - c & \text{if } a^1_{-k} = C \text{ and } a^2_{-k} = C \\ -c & \text{if } a^1_{-k} = C \text{ and } a^2_{-k} = D \\ b & \text{if } a^1_{-k} = D \text{ and } a^2_{-k} = C \\ 0 & \text{if } a^1_{-k} = D \text{ and } a^2_{-k} = D \end{cases} \tag{3}$$

Then we can define player 1's repeated-game payoff $s_{\mathbf{m},\mathbf{m}'}$ as

$$s_{\mathbf{m},\mathbf{m}'} = \mathbf{v} \cdot \mathbf{S}^1 = \mathbf{v} \cdot \mathbf{S}^2 = \ldots = \mathbf{v} \cdot \mathbf{S}^n. \tag{4}$$

The equalities $\mathbf{v} \cdot \mathbf{S}^1 = \ldots = \mathbf{v} \cdot \mathbf{S}^n$ correspond to the intuition that it does not matter which of the past n rounds we use to define average payoffs. The payoff $s_{\mathbf{m}',\mathbf{m}}$ of player 2 can be defined analogously.

Let's provide definitions for some additional terms that will be used in this manuscript.

Definition 1.1 (Nash Strategies.)**.** A strategy $\mathbf{m}$ for player 1, is a *Nash strategy* if,

$$s_{\mathbf{m}',\mathbf{m}} \leq s_{\mathbf{m},\mathbf{m}} \; \forall \; m' \in [0,1]^{2n}. \tag{5}$$

Definition 1.2 (Nice Strategies.)**.** A player's strategy is *nice*, if the player is never the first to defect. A nice strategy against itself receives the mutual cooperation payoff, $(b - c)$.

Definition 1.3 (Partner Strategies.)**.** A *partner strategy* is a strategy which is both nice and Nash.

Partners strategies are of interest because they are strategies that strive to achieve the mutual cooperation payoff of $(b-c)$ with their co-player. However, if the co-player doesn't cooperate, they are prepared to penalize them with lower payoffs. Partner strategies, by definition, are best responses to themselves [Hilbe et al., 2015]. All partner strategies are Nash strategies, but not all Nash strategies are partner strategies.

2 An Extension of Akin's Lemma

The work of [Akin, 2016] focuses on the case of memory-one strategies, thus for $n = 1$. A memory-one strategy of player p is represented by the vector $\mathbf{m} = (m_{CC}, m_{CD}, m_{DC}, m_{DD})$, and and when played against a co-player with strategy $\mathbf{m'}$, the resulting stationary distribution is denoted as $\mathbf{v} = (v_{CC}, v_{CD}, v_{DC}, v_{DD})$. Akin's lemma states the following,

Assume that player 1 uses the memory-one strategy $\mathbf{m} = (m_{CC}, m_{CD}, m_{DC}, m_{DD})$, and q uses a strategy that leads to a sequence of distributions $\{\mathbf{v}^k, k = 1, 2, ...\}$ with $\mathbf{v}^k$ representing the distribution over the states in the k^{th} round of the game. Let $\mathbf{v}$ be the associated stationary distribution, then,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathbf{v}^k \cdot (\mathbf{m} - \mathbf{m}^{1-\text{Rep}}) = 0, \text{ and therefore } \mathbf{v}\cdot(\mathbf{m} - (1,1,0,0)) = 0. \tag{6}$$

With the same method as in [Akin, 2016], one can show *Akin's Lemma*: For each k with $1 \leq k \leq n$, the invariant distribution $\mathbf{v}$ satisfies the following relationship,

$$\mathbf{v}\cdot(\mathbf{m} - \mathbf{m}^{k-\text{Rep}}) = \sum_{h\in H} v_h(m_h - m_h^{k-\text{Rep}}) = 0. \tag{7}$$

The intuition for this result is that $\mathbf{v}\cdot\mathbf{m}$ and all $\mathbf{v}\cdot\mathbf{m}^{k-\text{Rep}}$ are just different (but equivalent) expressions for player p's average cooperation rate. For example, $\mathbf{v}\cdot\mathbf{m}$ corresponds to a setup in which one first draws a history h according to the invariant distribution $\mathbf{v}$; then one takes player p's probability m_h to cooperate in the next round; the expectation of this procedure is $\sum_{h\in H} v_h m_h$.

Proof. The probability that p cooperates in the n^{th} round, denoted by $\hat{v}_{\text{C}}^{(n)}$, is $\hat{v}_{\text{C}}^{(n)} = \hat{v}_1^{(n)} + \hat{v}_2^{(n)} + \hat{v}_5^{(n)} + \hat{v}_6^{(n)} + \hat{v}_9^{(n)} + \hat{v}_{10}^{(n)} + \hat{v}_{13}^{(n)} + \hat{v}_{14}^{(n)} = \hat{\mathbf{v}}\cdot\hat{\mathbf{e}}_{12}$ where $\hat{\mathbf{e}}_{12} = (1,1,1,1,0,0,0,0,1,1,1,1,0,0,0,0)$. The probability that p cooperates in the $(n+1)^{th}$ round, denoted by $\hat{v}_{\text{C}}^{(n+1)} = \hat{\mathbf{v}}^{(n)}\cdot\hat{\mathbf{p}}$. Hence,

$$\hat{v}_{\text{C}}^{(n+1)} - \hat{v}_{\text{C}}^{(n)} = \hat{\mathbf{v}}^{(\mathbf{n})}\cdot\hat{\mathbf{p}} - \hat{\mathbf{v}}\cdot\hat{\mathbf{e}}_{12} = \hat{\mathbf{v}}^{(\mathbf{n})}\cdot(\hat{\mathbf{p}} - \hat{\mathbf{e}}_{12}) = \hat{\mathbf{v}}^{(n)}\cdot\tilde{\mathbf{p}}.$$

This implies,

$$\sum_{k=1}^{n}\hat{\mathbf{v}}^{(k)}\cdot\tilde{\mathbf{p}} = \sum_{k=1}^{n}\hat{v}_{\text{C}}^{(k+1)} - \hat{v}_{\text{C}}^{(k)} \quad\Rightarrow\quad \sum_{k=1}^{n}\hat{\mathbf{v}}^{(k)}\cdot\tilde{\mathbf{p}} = \hat{v}_{\text{C}}^{(n+1)} - \hat{v}_{\text{C}}^{(1)}. \tag{8}$$

As the right side has absolute value at most 1,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \hat{\mathbf{v}}^{(k)} \cdot \tilde{\mathbf{p}} = 0. \tag{9}$$

□

Zero-determinant strategies. Based on Akin's Lemma, we can derive a theory of zero-determinant strategies analogous to the case of memory-one strategies. In the following, we say a memory-n strategy $\mathbf{m}$ is a zero-determinant strategy if there are k_1, k_2, k_3 and α, β, γ such that $\mathbf{m}$ can be written as

$$\mathbf{m} = \alpha \mathbf{S}^{k_1} + \beta \tilde{\mathbf{S}}^{k_2} + \gamma \mathbf{1} + \mathbf{m}^{k-\mathrm{Rep}}, \tag{10}$$

where $\mathbf{1}$ is the vector for which every entry is 1. By Akin's Lemma and the definition of payoffs,

$$0 = \mathbf{v} \cdot (\mathbf{m} - \mathbf{m}^{k-\mathrm{Rep}}) = \mathbf{v} \cdot (\alpha \mathbf{S}^{k_1} + \beta \tilde{\mathbf{S}}^{k_2} + \gamma \mathbf{1}) = \alpha s_{\mathbf{m},\mathbf{m}'} + \beta s_{\mathbf{m}',\mathbf{m}} + \gamma. \tag{11}$$

That is, payoffs satisfy a linear relationship.

One interesting special case arises if $k_1 = k_2 = k_3 =: k$ and $\alpha = -\beta = 1/(b+c)$ and $\gamma = 0$. In that case, the formula (8) yields the strategy

$$m_h = \begin{cases} 1 & \text{if } a^q_{-k} = C \\ 0 & \text{if } a^q_{-k} = D \end{cases} \tag{12}$$

That is, this strategy implements Tit-for-Tat (for $k = 1$) or delayed versions thereof (for $k > 1$). These strategies are partners strategies that also satisfy a stronger relationship. By Eq. (9), the enforced payoff relationship is $s_{\mathbf{m},\mathbf{m}'} = s_{\mathbf{m}',\mathbf{m}}$.

Another interesting special case arises if $k_1 = k_2 = k_3 =: k$ and $\alpha = 0$, $\beta = -1/b$, $\gamma = 1 - c/b$. In that case Eq. (8) yields the strategy

$$m_h = \begin{cases} 1 & \text{if } a^q_{-k} = C \\ 1 - c/b & \text{if } a^q_{-k} = D \end{cases} \tag{13}$$

That is, the generated strategy is GTFT (if $k = 1$), or delayed versions thereof (for $k > 1$). By Eq. (9), the enforced payoff relationship is $s_{\mathbf{m}',\mathbf{m}} = b - c$. In particular, these strategies are partner strategies.

3 Reactive Partner Strategies

A *reactive*$-n$ *strategy* is denoted by a vector $\mathbf{p} = (p_h)_{h \in H^q} \in [0,1]^{2n}$. Each entry p_h corresponds to the player's cooperation probability in the next round, based on the co-player's action(s) in the previous n rounds. Therefore, n-bit reactive strategies exclusively rely on the co-player's n-history, remaining unaffected by the focal player's own actions during the past n rounds. From this point onward, we distinguish between memory-n strategies and reactive-n strategies, using notations $\mathbf{m}$ and $\mathbf{p}$ respectively for each set of strategies.

To begin, let's introduce some additional notation. Suppose player p adopts are reactive$-n$ strategy $\mathbf{p}$, and suppose player q adopts an arbitrary memory-n strategy. Let $\mathbf{v} = (v_h)_{h \in H}$ be an invariant of the game between the two players. We define the following marginal distributions with respect to the possible n-histories of player q,

$$v_h^q = \sum_{h^p \in H^p} v_{(h^p, h^q)} \; \forall \; h^q \in H^q. \tag{14}$$

These entries describe how often we observe player q to choose actions h^q, in n consecutive rounds (irrespective of the actions of player p). Note that,

$$\sum_{h \in H^q} v_h^q = 1. \tag{15}$$

Similarly, the cooperation rate of player q can also be defined irrespective of the actions of player p. Let H_k^q be the subset of H^q, for which,

$$H_k^q = \{h \in H^q : h_{-k} = C\}. \tag{16}$$

Let $\rho_{\mathbf{m}}$ be the cooperation rate of player q playing an arbitrary memory-n strategy $\mathbf{m}$ when playing against player p with a reactive strategy,

$$\rho_{\mathbf{m}} = \sum_{h \in H_1^q} v_h^q \; = \; \sum_{h \in H_2^q} v_h^q \; = \; \dots \; = \sum_{h \in H_n^q} v_h^q. \tag{17}$$

Equality (15) correspond to the intuition that it does not matter which of the past n rounds player q cooperated to define the cooperation rate.

We can also express player p's average cooperation rate $\rho_{\mathbf{p}}$ in terms of v_h^q by noting that,

$$\rho_{\mathbf{p}} \quad = \quad \sum_{h \in H^q} v_h^q \cdot p_h. \tag{18}$$

Because we consider simple donation games, we note that these two quantities, $\rho_{\mathbf{m}} \;\; and \;\; \rho_{\mathbf{p}}$, are sufficient to define the payoffs of the two players,

$$\begin{aligned} s_{\mathbf{p},\mathbf{m}} &= b\, \rho_{\mathbf{m}} - c\, \rho_{\mathbf{p}} \\ s_{\mathbf{m},\mathbf{q}} &= b\, \rho_{\mathbf{p}} - c\, \rho_{\mathbf{m}}. \end{aligned} \tag{19}$$

3.1 Sufficiency of Self reactive strategies

To characterize all partner reactive-n strategies, one would usually need to check against all pure memory-n strategies McAvoy and Nowak [2019]. However, we demonstrate that when player p uses a reactive-n strategy, it is sufficient to check only against self-reactive-n strategies. This is a direct outcome of Lemma 3.1.

Self-reactive-n strategies are also a subset of memory-n strategies. They only consider the focal player's own n-history, and ignore the co-player's. Formally, a self-reactive-n strategy is a vector $\tilde{\mathbf{p}} = (\tilde{p}_h)_{h \in H^p} \in [0,1]^{2n}$. Each entry $\tilde{p}_h$ corresponds to the player's cooperation probability in the next, depending on the player's own action(s) in the previous n rounds.

Lemma 3.1. Let $\mathbf{p}$ be an reactive$-n$ strategy for player p. Then, for any memory$-n$ strategy $\mathbf{m}$ used by player q, player p's score is exactly the same as if q had played a specific self-reactive memory-n strategy $\tilde{\mathbf{p}}$.

Proof. □

Note that Lemma 3.1 aligns with the previous result by Press and Dyson [2012]. They discussed the case where one player uses a memory-one strategy and the other player employs a longer memory strategy. They demonstrated that the payoff of the player with the longer memory is exactly the same as if the player had employed a specific shorter-memory strategy, disregarding any history beyond what is shared with the short-memory player. The result here follows a similar intuition: if there is a part of history that one player does not observe, then the co-player gains nothing by considering the history not shared with the short-memory player.

More specifically, the play of a self-reactive player solely relies on their own previous actions. Hence, describing the self-reactive player's play can be achieved through a Markov process with a $2^n \times 2^n$ transition matrix $\tilde{M}$ instead. The stationary distribution $\tilde{\mathbf{v}}$ of $\tilde{M}$ has the following property:

$$\tilde{u}_h = u_h^q \ \forall \ h \in H^q. \tag{20}$$

From hereupon we will use the notation $\mathbf{m}, \mathbf{p}$, and $\tilde{\mathbf{p}}$ to denote memory-n, reactive-n, and self-reactive-n strategies.

3.2 Reactive-Two Partner Strategies

In this section, we focus on the case of $n = 2$. Reactive-two strategies are denoted as a vector $\mathbf{p} = (p_{CC}, p_{CD}, p_{DC}, p_{DD})$ where p_{CC} is the probability of cooperating in this turn when the co-player cooperated in the last 2 turns, p_{CD} is the probability of cooperating given that the co-player cooperated in the second to last turn and defected in the last, and so forth. A nice reactive-two strategy is represented by the vector $\mathbf{p} = (1, p_{CD}, p_{DC}, p_{DD})$.

Theorem 3.2 ("Reactive-Two Partner Strategies")**.** A reactive-two strategy $\mathbf{p}$, is a partner strategy if and only if, it's nice ($p_{CC} = 1$) and the remaining entries satisfy the conditions:

$$p_{DD} < 1 - \frac{c}{b} \ \ \textit{and} \ \ \frac{p_{CD} + p_{DC}}{2} < 1 - \frac{1}{2} \cdot \frac{c}{b}. \tag{21}$$

There are two independent proves of Theorem 3.2. The first prove is in line with the work of [Akin, 2016], and the second one relies on Lemma 3.1. Here, we discuss both.

Proof One. Suppose player p adopts a reactive-two strategy $\mathbf{p} = (p_{CC}, p_{CD}, p_{DC}, p_{DD})$. Moreover, suppose player q adopts an arbitrary memory-2 strategy $\mathbf{m}$. Let $\mathbf{v} = (v_h)_{h \in H}$ be an invariant distribution of the game between the two players.

We define the following four marginal distributions with respect to the possible two-histories of player q,

$$\begin{aligned} v^q_{CC} &= \sum_{h^p \in H^p} v_{(h^p, CC)} \\ v^q_{CD} &= \sum_{h^p \in H^p} v_{(h^p, CD)} \\ v^q_{DC} &= \sum_{h^p \in H^p} v_{(h^p, DC)} \\ v^q_{DD} &= \sum_{h^p \in H^p} v_{(h^p, DD)}. \end{aligned} \tag{22}$$

These four entries describe how often we observe player q to choose actions CC, CD, DC, DD in two consecutive rounds (irrespective of the actions of player p). We can define player q's average cooperation rate $\rho_{\mathbf{m}}$ as

$$\rho_{\mathbf{m}} := v^q_{CC} + v^q_{CD} = v^q_{CC} + v^q_{DC}. \tag{23}$$

Here, the second equality holds because it does not matter whether we define player q's cooperation rate based on the first or the second round of each 2-history. In particular, we can use this equality to conclude

$$v^q_{CD} = v^q_{DC}. \tag{24}$$

Similarly, we can express player p's average cooperation rate $\rho_{\mathbf{p}}$ in terms of v^q_{CC}, v^q_{CD}, v^q_{DC}, v^q_{DD} by noting that

$$\begin{aligned} \rho_{\mathbf{p}} &= v^q_{CC}\, p_{CC} + v^q_{CD}\, p_{CD} + v^q_{DC}\, p_{DC} + v^q_{DD}\, p_{DD} \\ &= v^q_{CC}\, p_{CC} + v^q_{CD}\, (p_{CD} + p_{DC}) + v^q_{DD}\, p_{DD}. \end{aligned} \tag{25}$$

Here, the second equality is due to Eq. (22).

Proof. A reactive-two strategy $\mathbf{p} = (p_{CC}, p_{CD}, p_{DC}, p_{DD})$ can only be a Nash equilibrium if *no* other strategy yields a larger payoff, in particular neither AllD nor the Alternator strategy must yield a larger payoff, where AllD= $(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)$ and Alternator= $(0,0,1,1,0,0,1,1,0,0,1,1,0,0,1,1)$. Thus, $\mathbf{p}$ can only form a Nash equilibrium if

$$\pi(\text{AllD}, \mathbf{p}) \leq b - c \quad \text{and} \quad \pi(\text{Alternator}, \mathbf{p}) \leq b - c,$$

or equivalently, if

$$p_{DD} \leq 1 - \frac{c}{b} \quad \text{and} \quad p_{CD} + p_{DC} \leq 1 + \frac{b-c}{c}. \tag{26}$$

Now, suppose player q has some strategy $\mathbf{m}$ such that $s_{\mathbf{m},\mathbf{p}} > b-c$. It follows that

$$
\begin{aligned}
0 \quad & < \quad s_{\mathbf{m},\mathbf{p}} - (b-c) \\
& \overset{Eq.\ (17)}{=} \quad b\rho_{\mathbf{p}} - c\rho_{\mathbf{m}} - (b-c) \\
& \overset{Eqs.\ (21),(23),(13)}{=} \quad b\left(v^q_{CC}p_{CC} + v^q_{CD}(p_{CD}+p_{DC}) + v^q_{DD}p_{DD}\right) - c\left(v^q_{CC}+v^q_{CD}\right) - (b-c)\left(v^q_{CC}+2v^q_{CD}+v^q_{DD}\right) \\
& = \quad v^q_{CC}\, b\,(p_{CC}-1) + v^q_{CD}\Big(b(p_{CD}+p_{DC})+c-2b\Big) + v^q_{DD}\Big(bp_{DD}-(b-c)\Big).
\end{aligned}
\tag{27}
$$

Condition (25) can hold only if,

$$
b\,(p_{CD}+p_{DC})+c-2b > 0, \qquad b\,p_{DD} - (b-c) > 0. \tag{28}
$$

Thus, Eq. (24) reassures that $\mathbf{p}$ is Nash strategy, and given that $p_{CC} = 1$, it is a partner strategy.

□

Proof Two. Suppose player p adopts a nice reactive-two strategy $\mathbf{p} = (1, p_{CD}, p_{DC}, p_{DD})$. For $\mathbf{p}$ to be a Nash strategy,

$$
s_{\mathbf{m},\mathbf{p}} \leq (b-c), \tag{29}
$$

must hold against all pure memory-2 strategies ($\mathbf{m} \in \{0,1\}^{4^2}$). Due to Lemma 3.1, it is sufficient to check only against pure self-reactive strategies, and in the case of $n = 2$ there can be only 16 such strategies. We refer to them as $\tilde{\mathbf{q}}^i$ for $i \in 1, \ldots, 16$. The strategies are as follow,

- $\tilde{\mathbf{q}}^0 = (0, 0, 0, 0)$
- $\tilde{\mathbf{q}}^1 = (0, 0, 0, 1)$
- $\tilde{\mathbf{q}}^2 = (0, 0, 1, 0)$
- $\tilde{\mathbf{q}}^3 = (0, 0, 1, 1)$
- $\tilde{\mathbf{q}}^4 = (0, 1, 0, 0)$
- $\tilde{\mathbf{q}}^5 = (0, 1, 0, 1)$
- $\tilde{\mathbf{q}}^6 = (0, 1, 1, 0)$
- $\tilde{\mathbf{q}}^7 = (0, 1, 1, 1)$
- $\tilde{\mathbf{q}}^8 = (1, 0, 0, 0)$
- $\tilde{\mathbf{q}}^9 = (1, 0, 0, 1)$
- $\tilde{\mathbf{q}}^{10} = (1, 0, 1, 0)$
- $\tilde{\mathbf{q}}^{11} = (1, 0, 1, 1)$
- $\tilde{\mathbf{q}}^{12} = (1, 1, 0, 0)$
- $\tilde{\mathbf{q}}^{13} = (1, 1, 0, 1)$
- $\tilde{\mathbf{q}}^{14} = (1, 1, 1, 0)$
- $\tilde{\mathbf{q}}^{15} = (1, 1, 1, 1)$

Proof. Suppose player p plays a nice reactive-two strategy $\mathbf{p} = (1, p_{CD}, p_{DC}, p_{DD})$, and suppose the co-player q plays a pure self-reactive-two strategy $\tilde{\mathbf{q}}$. The possible payoffs for $\tilde{\mathbf{q}} \in \{\tilde{\mathbf{q}}^0, \ldots, \tilde{\mathbf{q}}^{16}\}$ are:

$$
\begin{array}{llll}
s_{\tilde{\mathbf{q}}^i,\mathbf{p}} = & b \cdot p_{DD} & for \;\; i \in & \{0,2,4,6,8,10,12,14\} \\
s_{\tilde{\mathbf{q}}^i,\mathbf{p}} = & \frac{b \cdot (p_{CD}+p_{DC}+p_{DD})}{3} - \frac{1}{3} \cdot c & for \;\; i \in & \{1,9\} \\
s_{\tilde{\mathbf{q}}^i,\mathbf{p}} = & \frac{b \cdot (p_{CD}+p_{DC}+p_{DD}+1)}{4} - \frac{1}{2} \cdot c & for \;\; i \in & \{3\} \\
s_{\tilde{\mathbf{q}}^i,\mathbf{p}} = & \frac{b \cdot (p_{CD}+p_{DC})}{2} - \frac{1}{2} \cdot c & for \;\; i \in & \{4,5,12,13\} \\
s_{\tilde{\mathbf{q}}^i,\mathbf{p}} = & \frac{b \cdot (p_{CD}+p_{DC}+1)}{3} - \frac{2}{3} \cdot c & for \;\; i \in & \{6,7\} \\
s_{\tilde{\mathbf{q}}^i,\mathbf{p}} = & b - c & for \;\; i \in & \{8,9,10,11,12,13,14,15\}
\end{array}
$$

Setting the payoff expressions of $s_{\tilde{\mathbf{q}}^i,\mathbf{p}}$ to smaller or equal to $(b-c)$ we get the following unique conditions,

$$
\begin{aligned}
p_{DD} &\leq 1 - \frac{c}{b} && (30) \\
\frac{p_{CD}+p_{DC}+p_{DD}}{3} &\leq 1 - \frac{2}{3} \cdot \frac{c}{b} && (31) \\
\frac{p_{CD}+p_{DC}}{2} &\leq 1 - \frac{1}{2} \cdot \frac{c}{b} && (32) \\
& && (33)
\end{aligned}
$$

Note that condition (30) is the sum of conditions (29) and (31). Thus, only conditions (29) and (31) are necessary.

□

3.3 Reactive-Three Partner Strategies

In this section, we focus on the case of $n = 3$. Reactive-three strategies are denoted as a vector $\mathbf{p} = (p_{CCC}, p_{CCD}, p_{CDC}, p_{CDD}, p_{DCC}, p_{DCD}, p_{DDC}, p_{DDD})$ where p_{CCC} is the probability of cooperating in round t when the co-player cooperates in the last 3 rounds, p_{CCD} is the probability of cooperating given that the co-player cooperated in the third and second to last rounds and defected in the last, and so forth. A nice reactive-three strategy is represented by the vector $\mathbf{p} = (1, p_{CCD}, p_{CDC}, p_{CDD}, p_{DCC}, p_{DCD}, p_{DDC}, p_{DDD})$.

Theorem 3.3 ("Reactive-Three Partner Strategies")**.** A reactive-three strategy $\mathbf{p}$, is a partner strategy if and only if, it's nice ($p_{CCC} = 1$) and the remaining entries satisfy the conditions:

$$
\frac{p_{CCD}+p_{CDC}+p_{DCC}}{3} < 1 - \frac{1}{3} \cdot \frac{c}{b} \qquad \frac{p_{CDD}+p_{DCD}+p_{DDC}}{3} < 1 - \frac{2}{3} \cdot \frac{c}{b} \qquad p_{DDD} < 1 - \frac{c}{b} \tag{34}
$$

$$
\frac{p_{CCD}+p_{CDD}+p_{DCC}+p_{DDC}}{4} < 1 - \frac{1}{2} \cdot \frac{c}{b} \qquad \frac{p_{CDC}+p_{DCD}}{2} < 1 - \frac{1}{2} \cdot \frac{c}{b} \tag{35}
$$

Once again, there are two independent proves of Theorem 3.3, and we will present both.

Proof One. Suppose player p adopts a reactive-three strategy $\mathbf{p}$, and suppose player q adopts an arbitrary memory-three strategy $\mathbf{m}$. Let $\mathbf{v} = (v_h)_{h \in H}$ be an invariant distribution of the game between the two players.

We define the following eight marginal distributions with respect to the possible three-histories of player q,

$$\begin{aligned}
v^q_{CCC} &= \sum_{h^p \in H^p} v_{(h^p, CCC)} \\
v^q_{CCD} &= \sum_{h^p \in H^p} v_{(h^p, CCD)} \\
v^q_{CDC} &= \sum_{h^p \in H^p} v_{(h^p, CDC)} \\
v^q_{CDD} &= \sum_{h^p \in H^p} v_{(h^p, CDD)} \\
v^q_{DCC} &= \sum_{h^p \in H^p} v_{(h^p, DCC)} \\
v^q_{DCD} &= \sum_{h^p \in H^p} v_{(h^p, DCD)} \\
v^q_{DDC} &= \sum_{h^p \in H^p} v_{(h^p, DDC)} \\
v^q_{DDD} &= \sum_{h^p \in H^p} v_{(h^p, DDD)}.
\end{aligned} \tag{36}$$

These eight entries describe how often we observe player q to choose actions CCC, CCD, CDC, CDD, DCC, DCD, DDC, DDD in three consecutive rounds (irrespective of the actions of player p). We can define player q's average cooperation rate $\rho_{\mathbf{m}}$ as

$$\rho_{\mathbf{m}} := v^q_{CCC} + v^q_{CCD} + v^q_{DCC} + v^q_{DCD}. \tag{37}$$

In the case of $n = 3$ the following equalities hold,

$$v^q_{CCD} = v^q_{DCC} \tag{38}$$

$$v^q_{DDC} = v^q_{CDD} \tag{39}$$

$$\begin{aligned}
v^q_{CCD} + v^q_{DCD} = v^q_{CDC} + v^q_{DDC} \Rightarrow \\
v^q_{CCD} = v^q_{CDC} + v^q_{CDD} - v^q_{DCD}
\end{aligned} \tag{40}$$

The average cooperation rate of p's is given by

$$\begin{aligned}
\rho_{\mathbf{p}} \quad &= \quad v^q_{CCC}\, p_{CCC} + v^q_{CCD}\, p_{CCD} + v^q_{CDC}\, p_{CDC} + v^q_{CDD}\, p_{CDD} + v^q_{DCD}\, p_{DCD} + \\
&\qquad + v^q_{DDC}\, p_{DDC} + v^q_{DDD}\, p_{DDD} \\
&\overset{Eqs.\ (36),(37)}{=} \quad v^q_{CCC}\, p_{CCC} + v^q_{CCD}\,(p_{CCD} + p_{DCC}) + v^q_{CDC}\, p_{CDC} + v^q_{CDD}\,(p_{CDD} + p_{DDC}) + \\
&\qquad v^q_{DCD}\, p_{DCD} + v^q_{DDD}\, p_{DDD}
\end{aligned} \tag{41}$$

Proof. A reactive-three strategy $\mathbf{p}$ can only be a Nash equilibrium if *no* other strategy yields a larger payoff, in particular neither AllD nor the following self-reactive-three strategies, $\tilde{\mathbf{q}}^{\mathbf{15}} = (0,0,0,0,1,1,1,1)$, $\tilde{\mathbf{q}}^{\mathbf{17}} = (0,0,0,1,0,0,0,1)$, $\tilde{\mathbf{q}}^{\mathbf{51}} = (0,0,1,1,0,0,1,1)$ and $\tilde{\mathbf{q}}^{\mathbf{102}} = (0,1,1,0,0,1,1,0)$. Thus, $\mathbf{p}$ can only form a Nash equilibrium if

$$\pi(\text{AllD}, \mathbf{p}) \leq b-c \quad \textit{and} \quad \pi(\tilde{\mathbf{q}}^{15}, \mathbf{p}) \leq b-c \quad \textit{and} \quad \pi(\tilde{\mathbf{q}}^{17}, \mathbf{p}) \leq b-c \quad \textit{and}$$

$$\pi(\tilde{\mathbf{q}}^{51}, \mathbf{p}) \leq b-c \quad \textit{and} \quad \pi(\tilde{\mathbf{q}}^{102}, \mathbf{p}) \leq b-c$$

or equivalently, if

$$\frac{p_{CCD}+p_{CDC}+p_{DCC}}{3} < 1-\frac{1}{3}\cdot\frac{c}{b} \qquad \frac{p_{CDD}+p_{DCD}+p_{DDC}}{3} < 1-\frac{2}{3}\cdot\frac{c}{b} \qquad p_{DDD} < 1-\frac{c}{b} \tag{42}$$

$$\frac{p_{CCD}+p_{CDD}+p_{DCC}+p_{DDC}}{4} < 1-\frac{1}{2}\cdot\frac{c}{b} \qquad \frac{p_{CDC}+p_{DCD}}{2} < 1-\frac{1}{2}\cdot\frac{c}{b} \tag{43}$$

Now, suppose player q has some strategy $\mathbf{m}$ and player p has a reactive-two strategy such that $s_{\mathbf{m},\mathbf{p}} > b-c$. It follows that

$$\begin{aligned}
0 \quad &\leq \quad s_{\mathbf{m},\mathbf{p}} - (b-c) \\
&\overset{Eq.\ (17)}{=} \quad b\rho_{\mathbf{p}} - c\rho_{\mathbf{m}} - (b-c) \\
&\overset{Eqs.\ (39),(13)}{=} \quad b\left(v^q_{CCC}p_{CCC}+v^q_{CCD}(p_{CCD}+p_{DCC})+v^q_{CDC}p_{CDC}+v^q_{DDC}(p_{CDD}+p_{DDC})+v^q_{DCD}p_{DCD}+v^q_{DDD}p_{DDD}\right) \\
&\qquad -c\left(v^q_{CCC}+2v^q_{CCD}+v^q_{DCD}\right) - (b-c)\left(v^q_{CCC}+2v^q_{CCD}+v^q_{CDC}+2v^q_{DDC}+v^q_{DCD}+v^q_{DDD}\right) \\
&= \quad b\,v^q_{CCC}(p_{CCC}-1) + v^q_{CCD}(b\,(p_{CCD}+p_{DCC}-2)) + v^q_{CDC}(b\,(p_{CDC}-1)+c)+ \\
&\qquad v^q_{CDD}(b\,(p_{CDD}+p_{DDC}-2)+2\,c) + v^q_{DCD}(b\,(p_{DCD}-1)) + v^q_{DDD}(b\,(p_{DDD}-1)+c) \\
&\overset{Eq.\ (38)}{=} \quad b\,v^q_{CCC}(p_{CCC}-1) + v^q_{DDD}(b\,(p_{DDD}-1)+c) + v^q_{CDC}(b\,(p_{CCD}+p_{DCC}+p_{CDC}-3)+c)+ \\
&\qquad v^q_{CDD}(b\,(p_{CDD}+p_{DDC}+p_{CCD}+p_{DCC}-4)+2\,c) + v^q_{DCD}(b\,(p_{DCD}-1) - b\,(p_{CCD}+p_{DCC})-2)
\end{aligned} \tag{44}$$

Condition (42) holds only for,

$$b\,(p_{DDD}-1)+c<0, \quad b\,(p_{CCD}+p_{DCC}+p_{CDC}-3)+c$$

$$b\,(p_{CDD}+p_{DDC}+p_{CCD}+p_{DCC}-4)+2\,c<0 \Rightarrow -b\,(p_{CCD}+p_{DCC}-2) > b\,(p_{CDD}+p_{DDC}-2)+2\,c$$

$$b\,(p_{DCD}-1)-b\,(p_{CCD}+p_{DCC})-2<0 \Rightarrow b\,(p_{DCD}+p_{CDD}+p_{DDC}-3)+2\,c<0.$$

Thus, conditions Eq. (40) reassure that $\mathbf{p}$ is Nash strategy, and given that $p_{CC}=1$, it is a partner strategy.

□

Proof Two. Consider all the pure self-reactive-three strategies. There is a total of 256 such strategies. These are given in the Section 5. The payoff expression for each of these strategies against a nice reactive-three strategies can be calculated explicitly. We use these expressions to obtain the conditions for partner strategies similar to the previous subsection.

Proof. The payoff expressions for a nice reactive-three strategy p against all pure self-reactive-three strategies are as follows,

$$
\begin{aligned}
s_{\tilde{\mathbf{q}}^i,\mathbf{p}} &= b\, p_{DDD} && \textit{for} \quad i \in [0, 2, 4, 6, \ldots, 250, 252, 254] \\
s_{\tilde{\mathbf{q}}^i,\mathbf{p}} &= \tfrac{b\cdot(p_{CDD}+p_{DCD}+p_{DDC}+p_{DDD})}{4} - \tfrac{1}{4}\cdot c && \textit{for} \quad i \in \{1, 9, 33, 41, 65, 73, 97, 105, 129, 137, 161, \\
&&& \qquad 169, 193, 201, 225, 233\} \\
s_{\tilde{\mathbf{q}}^i,\mathbf{p}} &= \tfrac{b\cdot(p_{CCD}+p_{CDD}+p_{DCC}+p_{DDC}+p_{DDD})}{5} - \tfrac{2}{5}\cdot c && \textit{for} \quad i \in \{3, 7, 35, 39, 131, 135, 163, 167\} \\
s_{\tilde{\mathbf{q}}^i,\mathbf{p}} &= \tfrac{b\cdot(p_{CDC}+p_{DCD})}{2} - \tfrac{1}{2}\cdot c && \textit{for} \quad i \in \{4-7, 12-15, 20-23, 28-31, 68-71, \\
&&& \qquad 76-79, 84-87, 92-95, 132-135, \\
&&& \qquad 140-143, 148-151, 156-159, \\
&&& \qquad 196-199, 204-207, 212-215, 220-223\} \\
s_{\tilde{\mathbf{q}}^i,\mathbf{p}} &= \tfrac{b\cdot(p_{CCD}+p_{CDD}+p_{DCC}+p_{DDC}+p_{DDD}+1)}{6} - \tfrac{1}{2}\cdot c && \textit{for} \quad i \in \{11, 15, 43, 47\} \\
s_{\tilde{\mathbf{q}}^i,\mathbf{p}} &= \tfrac{b\cdot(p_{CDD}+p_{DCD}+p_{DDC})}{3} - \tfrac{1}{3}\cdot c && \textit{for} \quad i \in \{16, 17, 24, 25, 48, 49, 56, 57, 80, 81, 88, \\
&&& \qquad 89, 112, 113, 120, 121, 144, 145, 152, 153, \\
&&& \qquad 176, 177, 184, 185, 208, 209, 216, 217, \\
&&& \qquad 240, 241, 248, 249\} \\
s_{\tilde{\mathbf{q}}^i,\mathbf{p}} &= \tfrac{b\cdot(p_{CCD}+p_{CDD}+p_{DCC}+p_{DDC})}{4} - \tfrac{1}{2}\cdot c && \textit{for} \quad i \in \{18, 19, 22, 23, 50, 51, 54, 55, 146, 147, \\
&&& \qquad 150, 151, 178, 179, 182, 183\} \\
s_{\tilde{\mathbf{q}}^i,\mathbf{p}} &= \tfrac{b\cdot(p_{CCD}+p_{CDD}+p_{DCC}+p_{DDC}+1)}{5} - \tfrac{3}{5}\cdot c && \textit{for} \quad i \in \{26, 27, 30, 31, 58, 59, 62, 63\} \\
s_{\tilde{\mathbf{q}}^i,\mathbf{p}} &= \tfrac{b\cdot(p_{CCD}+p_{CDC}+p_{CDD}+p_{DCC}+p_{DCD}+p_{DDC}+p_{DDD})}{7} - \tfrac{3}{7}\cdot c && \textit{for} \quad i \in \{37, 67, 165, 195\} \\
s_{\tilde{\mathbf{q}}^i,\mathbf{p}} &= \tfrac{b\cdot(p_{CCD}+p_{CDC}+p_{CDD}+p_{DCC}+p_{DCD}+p_{DDC}+p_{DDD}+1)}{8} - \tfrac{1}{2}\cdot c && \textit{for} \quad i \in \{45, 75\} \\
s_{\tilde{\mathbf{q}}^i,\mathbf{p}} &= \tfrac{b\cdot(p_{CCD}+p_{CDC}+p_{CDD}+p_{DCC}+p_{DCD}+p_{DDC})}{6} - \tfrac{1}{2}\cdot c && \textit{for} \quad i \in \{52, 53, 82, 83, 180, 181, 210, 211\} \\
s_{\tilde{\mathbf{q}}^i,\mathbf{p}} &= \tfrac{b\cdot(p_{CCD}+p_{CDC}+p_{CDD}+p_{DCC}+p_{DCD}+p_{DDC}+1)}{7} - \tfrac{4}{7}\cdot c && \textit{for} \quad i \in \{60, 61, 90, 91\} \\
s_{\tilde{\mathbf{q}}^i,\mathbf{p}} &= \tfrac{b\cdot(p_{CCD}+p_{CDC}+p_{DCC})}{3} - \tfrac{2}{3}\cdot c && \textit{for} \quad i \in \{96-103, 112-119, 224-231, 240-247\} \\
s_{\tilde{\mathbf{q}}^i,\mathbf{p}} &= \tfrac{b\cdot(p_{CCD}+p_{CDC}+p_{DCC}+1)}{4} - \tfrac{3}{4}\cdot c && \textit{for} \quad i \in \{104-111, 120-127\} \\
s_{\tilde{\mathbf{q}}^i,\mathbf{p}} &= (b-c) && \textit{for} \quad i \in [128, 255]
\end{aligned}
\tag{45}
$$

Setting these to smaller or equal than the mutual cooperation payoff $(b-c)$ give the following ten conditions,

$$p_{DDD} \leq 1 - \frac{c}{b}, \quad \frac{p_{CDC}+p_{DCD}}{2} \leq 1 - \frac{1}{2}\cdot\frac{c}{b}, \quad \frac{p_{CDD}+p_{DCD}+p_{DDC}}{3} \leq 1 - \frac{2}{3}\cdot\frac{c}{b}, \tag{46}$$

$$\frac{p_{CCD}+p_{CDC}+p_{DCC}}{3} \leq 1 - \frac{1}{3}\cdot\frac{c}{b}, \quad \frac{p_{CCD}+p_{CDD}+p_{DCC}+p_{DDC}}{4} \leq 1 - \frac{1}{2}\cdot\frac{c}{b} \tag{47}$$

$$\frac{p_{CDD}+p_{DCD}+p_{DDC}+p_{DDD}}{4} \leq 1 - \frac{3}{4}\cdot\frac{c}{b}, \tag{48}$$

$$\frac{p_{CCD}+p_{CDC}+p_{CDD}+p_{DCC}+p_{DCD}+p_{DDC}+p_{DDD}}{7} \leq 1 - \frac{4}{7}\cdot\frac{c}{b}, \tag{49}$$

$$\frac{p_{CCD}+p_{CDD}+p_{DCC}+p_{DDC}+p_{DDD}}{5} \leq 1 - \frac{3}{5}\cdot\frac{c}{b}, \tag{50}$$

$$\frac{p_{CCD}+p_{CDC}+p_{CDD}+p_{DCC}+p_{DCD}+p_{DDC}}{6} \leq 1 - \frac{1}{2}\cdot\frac{c}{b} \tag{51}$$

Note that only conditions (44) and (45) are unique. The remaining conditions can be derived from the sums of two or more of these conditions. $\square$

3.4 Reactive Counting Partner Strategies

A special case of reactive strategies is reactive-counting strategies. These are strategies that respond to the co-player's actions, but they do not distinguish between when cooperations/defections occurred; they solely consider the count of cooperations in the last n turns. A reactive-counting-n strategy is represented by a vector $\mathbf{r} = (r_i)_{i\in[0,n]}$, where the entry r_i indicates the probability of cooperating given that the co-player cooperated i times in the last n turns.

Reactive-Counting-Two Partner Strategies. These are denoted by the vector $\mathbf{r} = (r_2, r_1, r_0)$ where r_i is the probability of cooperating in after i cooperations in the last 2 turns. We can characterise reactive-counting-two partner strategies by setting $r_2 = 1$, and $p_{CD} = p_{DC} = r_1$ and $p_{DD} = r_0$ in conditions (19). This gives us the following result.

Lemma 3.4. A nice reactive-counting-two strategy $\mathbf{r} = (1, r_1, r_0)$ is a partner strategy if and only if,

$$r_1 < 1 - \frac{1}{2} \cdot \frac{c}{b} \quad \textit{and} \quad r_0 < 1 - \frac{c}{b}. \tag{52}$$

Reactive-Counting-Three Partner Strategies. These are denoted by the vector $\mathbf{r} = (r_3, r_2, r_1, r_0)$ where r_i is the probability of cooperating in after i cooperations in the last 3 turns. We can characterise reactive-counting-three partner strategies by setting $r_3 = 1$, and $p_{CCD} = p_{CDC} = r_2, p_{DCD} = p_{DDC} = r_1$ and $p_{DDD} = r_0$ in conditions (32). This gives us the following result.

Lemma 3.5. A nice reactive-counting-three strategy $\mathbf{r} = (1, r_2, r_1, r_0)$ is a partner strategy if and only if,

$$r_2 < 1 - \frac{1}{3} \cdot \frac{c}{b}, \quad r_1 < 1 - \frac{2}{3} \cdot \frac{c}{b} \quad \textit{and} \quad r_0 < 1 - \frac{c}{b}. \tag{53}$$

In the case of counting reactive strategies, we generalise to the case of n.

Given a reactive-counting-n strategy $\mathbf{r} = (r_n, r_{n-1}, \dots, r_0)$, in the strategy's eyes the game can end up in n states. Each u_i state represents the state that the co-player cooperated i times in the last n turns with,

$$\sum_{i=0}^{n} u_i = 1 \Rightarrow \sum_{k=0}^{n} u_{n-k} = 1. \tag{54}$$

Thus the cooperation ratio of the strategy is,

$$\rho_{\mathbf{p}} = \sum_{k=0}^{n} r_{n-k} \cdot u_{n-k}. \tag{55}$$

the probability of cooperating given that the co-player cooperated i times. The co-player can use any self-reactive-n strategy, and thus the co-player differentiates between when the last cooperation/defection occurred. However, we can still express the co-player's cooperation rate as a function of u_i. More specifically, the co-player's cooperation rate is,

$$\rho_{\tilde{\mathbf{p}}} = \sum_{k=0}^{n} \frac{n-k}{n} \cdot u_{n-k}. \tag{56}$$

With this we have all the required tools to prove the following theorem.

Theorem 3.6 ("Reactive-Counting Partner Strategies"). A reactive-counting$-n$ strategy $\mathbf{r} = (r_i)_{i\in[0,n]}$, is a partner strategy if and only if, the r_i entries satisfy the conditions:

$$r_n = 1, \text{ and } r_{n-k} < 1 - \frac{k}{n} \cdot \frac{c}{b}, \text{ for } k \in [1, n]. \tag{57}$$

Proof. Consider a set of alternating self-reactive-n strategies that defect after cooperating i times. Since $i \in [0, n]$, there can be only $n + 1$ such strategies. We will denote this set as $A = \{\mathbf{A^0}, \mathbf{A^1}, \ldots, \mathbf{A^n}\}$. The payoff of an alternating self-reactive-n against a counting-reactive$-n$ $\mathbf{r}$ is given by,

$$s_{\mathbf{A^i},\mathbf{r}} = b \cdot r_i - \frac{i}{n} \cdot c \quad for \quad i \in [0, n]. \tag{58}$$

The intuition behind Eq. (56) is that in the long term of the gave the strategies end up in a state where $\mathbf{A^i}$ has cooperated i times in the last n turns. Thus, here the co-player will cooperate, and provide the benefit b with a probability r_i. Whilst in return the alternating strategy cooperated $\frac{i}{n}$ and pays the cost. As we have already discussed previously, a strategy can only be Nash if the payoff of the co-player does not exceed $(b - c)$. Thus for $\mathbf{p}$ to be Nash against each strategy in set A (for $i \in [0, n]$),

The intuition behind Eq. (56) is that in the long term, the strategies end up in a state where $\mathbf{A^i}$ has cooperated i times in the last n turns. Thus, the co-player will cooperate and provide the benefit b with a probability r_i, while in return, the alternating strategy has cooperated $\frac{i}{n}$ times and pays the cost. As we have already discussed previously, a strategy can only be a Nash equilibrium if the payoff of the co-player does not exceed $(b - c)$. Therefore, for p to be a Nash equilibrium against each strategy in set A (for $i \in [0, n]$),

$$s_{\mathbf{A^i},\mathbf{r}} \leq b - c \tag{59}$$

$$b \cdot r_i - \frac{i}{n} \cdot c \leq b - c \tag{60}$$

$$r_i \leq 1 - \frac{i}{n} \cdot \frac{c}{b} \tag{61}$$

Now, suppose player q has some strategy $\mathbf{m}$ and player p has a reactive-counting strategy such that $s_{\mathbf{m},\mathbf{p}} > b$–$c$. It follows that

$$\begin{aligned}
0 \quad & \leq \quad s_{\mathbf{m},\mathbf{p}} - (b{-}c) \\
& \overset{Eq.\ (17)}{=} \quad b\rho_{\mathbf{p}} - c\rho_{\mathbf{m}} - (b{-}c) \\
& \overset{Eqs.\ (52),(53),(54)}{=} \quad b\sum_{k=0}^{n} r_{n-k} \cdot u_{n-k} - c\sum_{k=0}^{n} \frac{n-k}{n} \cdot u_{n-k} - (b-c)\sum_{k=0}^{n} u_{n-k} \\
& \quad u_n\Big(b\,(r_n - 1)\Big) + \sum_{k=1}^{n} u_{n-k}\Big(b\sum_{k=1}^{n} r_{n-k} - c\sum_{k=0}^{n-1} \frac{n-k}{n} - (b-c)\sum_{k=0}^{n-1} 1\Big)
\end{aligned} \tag{62}$$

This condition holds only if,

$$\Big(b\, r_{n-k} - c\,\frac{n-k}{n} - (b-c)\Big) < 0 \Rightarrow \tag{63}$$

$$b\,(r_{n-k} - 1) + (1 - \frac{n-k}{n})\,c < 0 \Rightarrow \tag{64}$$

$$r_{n-k} < 1 - \frac{n}{k}\cdot\frac{c}{b}. \tag{65}$$

for $k \in [0, n]$. Thus, any counting strategy that satisfies conditions (57) is Nash, and if it is nice, it's also a partner strategy. $\square$

4 Prisoner's Dilemma

To characterise partner strategies for the general prisoner's dilemma, we can use the method based on Lemma 3.1. Here we discuss this result in the case of $n = 2$.

There are 16 pure-self reactive strategies in $n = 2$. To calculate the explicit payoff expressions for each pure strategy against a nice reactive-two strategy $\mathbf{p} = (1, p_{CD}, p_{DC}, p_{CD})$ we use the method discussed in Section 3.1. More specifically, for a self-reactive strategy $\mathbf{q}$, we calculate where the strategy is in the long term using the transition matrix,

$$\tilde{M} = \begin{bmatrix} \tilde{p}_1 & 1-\tilde{p}_1 & 0 & 0 \\ 0 & 0 & \tilde{p}_2 & 1-\tilde{p}_2 \\ \tilde{p}_3 & 1-\tilde{p}_3 & 0 & 0 \\ 0 & 0 & \tilde{p}_4 & 1-\tilde{p}_4 \end{bmatrix} \tag{66}$$

Using the stationary vector $\tilde{\mathbf{v}}$ we can define the payoffs in the general prisoner's dilemma as follows:

$$\mathbf{s_{q,p}} = a_R \cdot R + a_S \cdot S + a_T \cdot T + a_P \cdot P, \quad where$$

$$\begin{aligned}
a_R &= \tilde{v}_{CC}\, p_{CC}\, \tilde{q}_{CC} + \tilde{v}_{CD}\, p_{CD}\, \tilde{q}_{CD} + \tilde{v}_{DC}\, p_{DC}\, \tilde{q}_{DC} + \tilde{v}_{DD}\, p_{DD}\, \tilde{q}_{DD}, \\
a_S &= \tilde{v}_{CC}\, p_{CC}\,(1-\tilde{q}_{CC}) + \tilde{v}_{CD}\, p_{CD}\,(1-\tilde{q}_{CD}) + \tilde{v}_{DC}\, p_{DC}\,(1-\tilde{q}_{DC}) + \tilde{v}_{DD}\, p_{DD}\,(1-\tilde{q}_{DD}), \\
a_T &= \tilde{v}_{CC}\,(1-p_{CC})\,\tilde{q}_{CC} + \tilde{v}_{CD}\,(1-p_{CD})\,\tilde{q}_{CD} + \tilde{v}_{DC}\,(1-p_{DC})\,\tilde{q}_{DC} + \tilde{v}_{DD}\,(1-p_{DD})\,\tilde{q}_{DD}, \\
a_P &= \tilde{v}_{CC}\,(1-p_{CC})\,(1-\tilde{q}_{CC}) + \tilde{v}_{CD}\,(1-p_{CD})\,(1-\tilde{q}_{CD}) + \tilde{v}_{DC}\,(1-p_{DC})\,(1-\tilde{q}_{DC}) + \tilde{v}_{DD}\,(1-p_{DD})\,(1-\tilde{q}_{DD}).
\end{aligned}$$

This gives the following payoff expressions:

$$
\begin{aligned}
s_{\tilde{\mathbf{q}}^i,\mathbf{p}} &= P(1-p_{DD}) + Tp_{DD} && for \quad i \in \{0,2,4,6,8,10,12,14\} \\
s_{\tilde{\mathbf{q}}^i,\mathbf{p}} &= \tfrac{-P(p_{CD}+p_{DC}-2)+Rp_{DD}-S(p_{DD}-1)+T(p_{CD}+p_{DC})}{3} && for \quad i \in \{1,9\} \\
s_{\tilde{\mathbf{q}}^i,\mathbf{p}} &= \tfrac{P(1-p_{CD})+R(p_{DC}+p_{DD})-S(p_{DC}+p_{DD}-2)+T(p_{CD}+1)}{4} && for \quad i \in \{3\} \\
s_{\tilde{\mathbf{q}}^i,\mathbf{p}} &= \tfrac{P(1-p_{DC})+Rp_{CD}-S(p_{CD}-1)+Tp_{DC}}{2} && for \quad i \in \{4,5,12,13\} \\
s_{\tilde{\mathbf{q}}^i,\mathbf{p}} &= \tfrac{R(p_{CD}+p_{DC})-S(p_{CD}+p_{DC}-2)+T}{3} && for \quad i \in \{6,7\} \\
s_{\tilde{\mathbf{q}}^i,\mathbf{p}} &= R && for \quad i \in \{8,9,10,11,12,13,14,15\}
\end{aligned}
$$

Setting the above expressions to smaller than R gives the following conditions,

$$
\begin{gathered}
(P-T)\,p_{DD} < P-R, \quad (T-P)\,(p_{CD}+p_{DC}) + (R-S)\,p_{DD} < 3R+S-2\,P, \\
(T-P)\,p_{CD} + (R-S)\,(p_{CD}+p_{DC}) < 4\,R-2\,S-P-T, \qquad (R-S)\;p_{CD} + (T-P)\;p_{DC} < 2R-S-P \\
(R-S)\,(p_{CD}+p_{DC}) < 3R-2S-T
\end{gathered}
$$

Consider the case where $T=1$ and $S=0$,

$$
\begin{gathered}
(P-1)\,p_{DD} < P-R, \quad (1-P)\,(p_{CD}+p_{DC}) + R\,p_{DD} < 3R-2\,P, \\
(1-P)\,p_{CD} + R\,(p_{CD}+p_{DC}) < 4\,R-P-1, \qquad R\,p_{CD} + (1-P)\;p_{DC} < 2R-P \\
R\,(p_{CD}+p_{DC}) < 3R-1
\end{gathered}
$$

$$s_{\tilde{\mathbf{q}}^i,\mathbf{p}} = \frac{(T-P)\,(p_{CDD}+p_{DCD}+p_{DDC})+3\,P+(R-S)\,p_{DDD}+S}{4} \qquad for \quad i \in$$

$$s_{\tilde{\mathbf{q}}^i,\mathbf{p}} = \frac{(T-P)\,p_{CDC}+P+(R-S)\,p_{DCD}+S}{2} \qquad for \quad i \in$$

$$s_{\tilde{\mathbf{q}}^i,\mathbf{p}} = -P\,(p_{DDD}-1)+Tp_{DDD} \qquad for \quad i \in$$

$$s_{\tilde{\mathbf{q}}^i,\mathbf{p}} = \frac{(T-P)\,(p_{CCD}+p_{CDD}+p_{DDC})+3\,P+(R-S)\,(p_{CDC}+p_{DCC}+p_{DCD}+p_{DDD})+4\,S+T}{8} \qquad for \quad i \in$$

$$s_{\tilde{\mathbf{q}}^i,\mathbf{p}} = \frac{(T-P)\,p_{DCC}+P+(R-S)\,(p_{CDC}+p_{CCD})+2\,S}{3} \qquad for \quad i \in$$

$$s_{\tilde{\mathbf{q}}^i,\mathbf{p}} = \frac{(T-P)\,(p_{CCD}+p_{DCC}+p_{DDC})+3\,P+(R-S)\,(p_{CDC}+p_{CDD+p_{DCD}})+3\,S}{6} \qquad for \quad i \in$$

$$s_{\tilde{\mathbf{q}}^i,\mathbf{p}} = \frac{(T-P)\,(p_{CCD}+p_{DDC})+2\,P+T+(R-S)\,(p_{CDC}+p_{CDD}+p_{DCC}+p_{DCD})+4\,S}{7} \qquad for \quad i \in$$

$$s_{\tilde{\mathbf{q}}^i,\mathbf{p}} = \frac{(T-P)\,(p_{CCD}+p_{CDD}+p_{DCC})+3\,P+(R-S)\,(p_{DDC}+p_{DDD})+2\,S}{5} \qquad for \quad i \in$$

$$s_{\tilde{\mathbf{q}}^i,\mathbf{p}} = \frac{(T-P)\,(p_{DCD}+p_{DDC})+2\,P+(R-S)\,p_{CDD}+S}{3} \qquad for \quad i \in$$

$$s_{\tilde{\mathbf{q}}^i,\mathbf{p}} = R \qquad for \quad i \in [128-255]$$

$$s_{\tilde{\mathbf{q}}^i,\mathbf{p}} = \frac{(T-P)\,p_{CCD}+P+T+(R-S)\,(p_{CDD}+p_{DCC}+p_{DDC})+S}{5} \qquad for \quad i \in$$

$$s_{\tilde{\mathbf{q}}^i,\mathbf{p}} = \frac{(T-P)\,(p_{CCD}+p_{DCC})+2\,P+(R-S)\,(p_{CDD}+p_{DDD})+2\,S}{4} \qquad for \quad i \in$$

$$s_{\tilde{\mathbf{q}}^i,\mathbf{p}} = \frac{(T-P)\,p_{CDC}+2\,P+T+(R-S)\,(p_{CCD}+p_{CDD}+p_{DCC}+p_{DDC})+4\,S}{7} \qquad for \quad i \in$$

$$s_{\tilde{\mathbf{q}}^i,\mathbf{p}} = \frac{(T-P)\,(p_{CDC}+p_{CDD}+p_{DCD})+3\,P+T+(R-S)\,(p_{CCD}+p_{DCC}+p_{DDC}+p_{DDD})+4\,S}{8} \qquad for \quad i \in$$

$$s_{\tilde{\mathbf{q}}^i,\mathbf{p}} = \frac{(T-P)\,(p_{CDC}+p_{DDC}+p_{DCD})+3\,P+T+(R-S)\,(p_{CCD}+p_{CDD}+p_{DDC})+3\,S}{6} \qquad for \quad i \in$$

$$s_{\tilde{\mathbf{q}}^i,\mathbf{p}} = \frac{(T-P)\,(p_{CCD}+p_{CDD}+p_{DCC}+p_{DDC})+4\,P+(R-S)\,(p_{CDC}+p_{DCD}+p_{DDD})+3\,S}{7} \qquad for \quad i \in$$

$$s_{\tilde{\mathbf{q}}^i,\mathbf{p}} = \frac{T+(R-S)\,(p_{CCD}+p_{CDC}+p_{DCC})+3\,S}{4} \qquad for \quad i \in$$

$$s_{\tilde{\mathbf{q}}^i,\mathbf{p}} = \frac{(T-P)\,(p_{CCD}+p_{CDD})+2\,P+T+(R-S)\,(p_{DCC}+p_{DDC}+p_{DDD})+3\,S}{6} \qquad for \quad i \in$$

$$s_{\tilde{\mathbf{q}}^i,\mathbf{p}} = \frac{(T-p)\,(p_{CDC}+p_{CDD}+p_{DCC}+p_{DCD})+4\,P+(R-S)\,(p_{CCD}+p_{DDC}+p_{DDD})+3\,S}{7} \qquad for \quad i \in$$

$$
\begin{aligned}
&(T-P)\ (p_{CDD}+p_{DCD}+p_{DDC})+(R-S)\,p_{DDD} && < 4\,R-3\,P-S\\
&(T-P)\,p_{CDC}+(R-S)\,p_{DCD} && < 2\,R-P-S\\
&-P\,(p_{DDD}-1)+T p_{DDD} && < R\\
&(T-P)\,(p_{CCD}+p_{CDD}+p_{DDC})+3\,P+(R-S)\,(p_{CDC}+p_{DCC}+p_{DCD}+p_{DDD})+4\,S+T && < 8\,R\\
&(T-P)\,p_{DCC}+P+(R-S)\,(p_{CDC}+p_{CCD})+2\,S && < 3\,R\\
&(T-P)\,(p_{CCD}+p_{DCC}+p_{DDC})+3\,P+(R-S)\,(p_{CDC}+p_{CDD+p_{DCD}})+3\,S && < 6\,R\\
&(T-P)\,(p_{CCD}+p_{DDC})+2\,P+T+(R-S)\,(p_{CDC}+p_{CDD}+p_{DCC}+p_{DCD})+4\,S && < 7\,R\\
&(T-P)\,(p_{CCD}+p_{CDD}+p_{DCC})+3\,P+(R-S)\,(p_{DDC}+p_{DDD})+2\,S && < 5\,R\\
&(T-P)\,(p_{DCD}+p_{DDC})+2\,P+(R-S)\,p_{CDD}+S && < 3\,R\\
&(T-P)\,p_{CCD}+P+T+(R-S)\,(p_{CDD}+p_{DCC}+p_{DDC})+S && < 5\,R\\
&(T-P)\,(p_{CCD}+p_{DCC})+2\,P+(R-S)\,(p_{CDD}+p_{DDD})+2\,S && < 4\,R\\
&(T-P)\,p_{CDC}+2\,P+T+(R-S)\,(p_{CCD}+p_{CDD}+p_{DCC}+p_{DDC})+4\,S && < 7\,R\\
&(T-P)\,(p_{CDC}+p_{CDD}+p_{DCD})+3\,P+T+(R-S)\,(p_{CCD}+p_{DCC}+p_{DDC}+p_{DDD})+4\,S && < 8\,R\\
&(T-P)\,(p_{CDC}+p_{DDC}+p_{DCD})+3\,P+T+(R-S)\,(p_{CCD}+p_{CDD}+p_{DDC})+3\,S && < 6\,R\\
&(T-P)\,(p_{CCD}+p_{CDD}+p_{DCC}+p_{DDC})+4\,P+(R-S)\,(p_{CDC}+p_{DCD}+p_{DDD})+3\,S && < 7\,R\\
&T+(R-S)\,(p_{CCD}+p_{CDC}+p_{DCC})+3\,S && < 4\,R\\
&(T-P)\,(p_{CCD}+p_{CDD})+2\,P+T+(R-S)\,(p_{DCC}+p_{DDC}+p_{DDD})+3\,S && < 6\,R\\
&(T-p)\,(p_{CDC}+p_{CDD}+p_{DCC}+p_{DCD})+4\,P+(R-S)\,(p_{CCD}+p_{DDC}+p_{DDD})+3\,S && < 7\,R
\end{aligned}
$$

5 Pure Self-Reactive-Three Strategies

The 256 pure self-reactive-three strategies and their vectors are as follows,

- $\tilde{\mathbf{q}}^{0}=(0,0,0,0,0,0,0,0)$
- $\tilde{\mathbf{q}}^{1}=(0,0,0,0,0,0,0,1)$
- $\tilde{\mathbf{q}}^{2}=(0,0,0,0,0,0,1,0)$
- $\tilde{\mathbf{q}}^{3}=(0,0,0,0,0,0,1,1)$
- $\tilde{\mathbf{q}}^{4}=(0,0,0,0,0,1,0,0)$
- $\tilde{\mathbf{q}}^{5}=(0,0,0,0,0,1,0,1)$
- $\tilde{\mathbf{q}}^{6}=(0,0,0,0,0,1,1,0)$
- $\tilde{\mathbf{q}}^{7}=(0,0,0,0,0,1,1,1)$
- $\tilde{\mathbf{q}}^{8}=(0,0,0,0,1,0,0,0)$
- $\tilde{\mathbf{q}}^{9}=(0,0,0,0,1,0,0,1)$
- $\tilde{\mathbf{q}}^{10}=(0,0,0,0,1,0,1,0)$
- $\tilde{\mathbf{q}}^{11}=(0,0,0,0,1,0,1,1)$
- $\tilde{\mathbf{q}}^{12}=(0,0,0,0,1,1,0,0)$
- $\tilde{\mathbf{q}}^{13}=(0,0,0,0,1,1,0,1)$
- $\tilde{\mathbf{q}}^{14}=(0,0,0,0,1,1,1,0)$
- $\tilde{\mathbf{q}}^{15}=(0,0,0,0,1,1,1,1)$
- $\tilde{\mathbf{q}}^{16}=(0,0,0,1,0,0,0,0)$
- $\tilde{\mathbf{q}}^{17}=(0,0,0,1,0,0,0,1)$
- $\tilde{\mathbf{q}}^{18}=(0,0,0,1,0,0,1,0)$
- $\tilde{\mathbf{q}}^{19}=(0,0,0,1,0,0,1,1)$
- $\tilde{\mathbf{q}}^{20}=(0,0,0,1,0,1,0,0)$
- $\tilde{\mathbf{q}}^{21}=(0,0,0,1,0,1,0,1)$
- $\tilde{\mathbf{q}}^{22}=(0,0,0,1,0,1,1,0)$
- $\tilde{\mathbf{q}}^{23}=(0,0,0,1,0,1,1,1)$
- $\tilde{\mathbf{q}}^{24}=(0,0,0,1,1,0,0,0)$
- $\tilde{\mathbf{q}}^{25}=(0,0,0,1,1,0,0,1)$
- $\tilde{\mathbf{q}}^{26}=(0,0,0,1,1,0,1,0)$
- $\tilde{\mathbf{q}}^{27}=(0,0,0,1,1,0,1,1)$
- $\tilde{\mathbf{q}}^{28}=(0,0,0,1,1,1,0,0)$
- $\tilde{\mathbf{q}}^{29}=(0,0,0,1,1,1,0,1)$
- $\tilde{\mathbf{q}}^{30}=(0,0,0,1,1,1,1,0)$
- $\tilde{\mathbf{q}}^{31}=(0,0,0,1,1,1,1,1)$
- $\tilde{\mathbf{q}}^{32}=(0,0,1,0,0,0,0,0)$
- $\tilde{\mathbf{q}}^{33}=(0,0,1,0,0,0,0,1)$
- $\tilde{\mathbf{q}}^{34}=(0,0,1,0,0,0,1,0)$
- $\tilde{\mathbf{q}}^{35}=(0,0,1,0,0,0,1,1)$
- $\tilde{\mathbf{q}}^{36}=(0,0,1,0,0,1,0,0)$
- $\tilde{\mathbf{q}}^{37}=(0,0,1,0,0,1,0,1)$
- $\tilde{\mathbf{q}}^{38}=(0,0,1,0,0,1,1,0)$
- $\tilde{\mathbf{q}}^{39}=(0,0,1,0,0,1,1,1)$
- $\tilde{\mathbf{q}}^{40}=(0,0,1,0,1,0,0,0)$
- $\tilde{\mathbf{q}}^{41}=(0,0,1,0,1,0,0,1)$
- $\tilde{\mathbf{q}}^{42}=(0,0,1,0,1,0,1,0)$
- $\tilde{\mathbf{q}}^{43}=(0,0,1,0,1,0,1,1)$
- $\tilde{\mathbf{q}}^{44}=(0,0,1,0,1,1,0,0)$
- $\tilde{\mathbf{q}}^{45}=(0,0,1,0,1,1,0,1)$
- $\tilde{\mathbf{q}}^{46}=(0,0,1,0,1,1,1,0)$
- $\tilde{\mathbf{q}}^{47}=(0,0,1,0,1,1,1,1)$

- $\tilde{\mathbf{q}}^{48} = (0, 0, 1, 1, 0, 0, 0, 0)$
- $\tilde{\mathbf{q}}^{49} = (0, 0, 1, 1, 0, 0, 0, 1)$
- $\tilde{\mathbf{q}}^{50} = (0, 0, 1, 1, 0, 0, 1, 0)$
- $\tilde{\mathbf{q}}^{51} = (0, 0, 1, 1, 0, 0, 1, 1)$
- $\tilde{\mathbf{q}}^{52} = (0, 0, 1, 1, 0, 1, 0, 0)$
- $\tilde{\mathbf{q}}^{53} = (0, 0, 1, 1, 0, 1, 0, 1)$
- $\tilde{\mathbf{q}}^{54} = (0, 0, 1, 1, 0, 1, 1, 0)$
- $\tilde{\mathbf{q}}^{55} = (0, 0, 1, 1, 0, 1, 1, 1)$
- $\tilde{\mathbf{q}}^{56} = (0, 0, 1, 1, 1, 0, 0, 0)$
- $\tilde{\mathbf{q}}^{57} = (0, 0, 1, 1, 1, 0, 0, 1)$
- $\tilde{\mathbf{q}}^{58} = (0, 0, 1, 1, 1, 0, 1, 0)$
- $\tilde{\mathbf{q}}^{59} = (0, 0, 1, 1, 1, 0, 1, 1)$
- $\tilde{\mathbf{q}}^{60} = (0, 0, 1, 1, 1, 1, 0, 0)$
- $\tilde{\mathbf{q}}^{61} = (0, 0, 1, 1, 1, 1, 0, 1)$
- $\tilde{\mathbf{q}}^{62} = (0, 0, 1, 1, 1, 1, 1, 0)$
- $\tilde{\mathbf{q}}^{63} = (0, 0, 1, 1, 1, 1, 1, 1)$
- $\tilde{\mathbf{q}}^{64} = (0, 1, 0, 0, 0, 0, 0, 0)$
- $\tilde{\mathbf{q}}^{65} = (0, 1, 0, 0, 0, 0, 0, 1)$
- $\tilde{\mathbf{q}}^{66} = (0, 1, 0, 0, 0, 0, 1, 0)$
- $\tilde{\mathbf{q}}^{67} = (0, 1, 0, 0, 0, 0, 1, 1)$
- $\tilde{\mathbf{q}}^{68} = (0, 1, 0, 0, 0, 1, 0, 0)$
- $\tilde{\mathbf{q}}^{69} = (0, 1, 0, 0, 0, 1, 0, 1)$
- $\tilde{\mathbf{q}}^{70} = (0, 1, 0, 0, 0, 1, 1, 0)$
- $\tilde{\mathbf{q}}^{71} = (0, 1, 0, 0, 0, 1, 1, 1)$
- $\tilde{\mathbf{q}}^{72} = (0, 1, 0, 0, 1, 0, 0, 0)$
- $\tilde{\mathbf{q}}^{73} = (0, 1, 0, 0, 1, 0, 0, 1)$
- $\tilde{\mathbf{q}}^{74} = (0, 1, 0, 0, 1, 0, 1, 0)$
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- $\tilde{\mathbf{q}}^{78} = (0, 1, 0, 0, 1, 1, 1, 0)$
- $\tilde{\mathbf{q}}^{79} = (0, 1, 0, 0, 1, 1, 1, 1)$
- $\tilde{\mathbf{q}}^{80} = (0, 1, 0, 1, 0, 0, 0, 0)$
- $\tilde{\mathbf{q}}^{81} = (0, 1, 0, 1, 0, 0, 0, 1)$
- $\tilde{\mathbf{q}}^{82} = (0, 1, 0, 1, 0, 0, 1, 0)$
- $\tilde{\mathbf{q}}^{83} = (0, 1, 0, 1, 0, 0, 1, 1)$
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- $\tilde{\mathbf{q}}^{200} = (1, 1, 0, 0, 1, 0, 0, 0)$
- $\tilde{\mathbf{q}}^{201} = (1, 1, 0, 0, 1, 0, 0, 1)$
- $\tilde{\mathbf{q}}^{202} = (1, 1, 0, 0, 1, 0, 1, 0)$
- $\tilde{\mathbf{q}}^{203} = (1, 1, 0, 0, 1, 0, 1, 1)$
- $\tilde{\mathbf{q}}^{204} = (1, 1, 0, 0, 1, 1, 0, 0)$
- $\tilde{\mathbf{q}}^{205} = (1, 1, 0, 0, 1, 1, 0, 1)$
- $\tilde{\mathbf{q}}^{206} = (1, 1, 0, 0, 1, 1, 1, 0)$
- $\tilde{\mathbf{q}}^{207} = (1, 1, 0, 0, 1, 1, 1, 1)$
- $\tilde{\mathbf{q}}^{208} = (1, 1, 0, 1, 0, 0, 0, 0)$
- $\tilde{\mathbf{q}}^{209} = (1, 1, 0, 1, 0, 0, 0, 1)$
- $\tilde{\mathbf{q}}^{210} = (1, 1, 0, 1, 0, 0, 1, 0)$
- $\tilde{\mathbf{q}}^{211} = (1, 1, 0, 1, 0, 0, 1, 1)$
- $\tilde{\mathbf{q}}^{212} = (1, 1, 0, 1, 0, 1, 0, 0)$
- $\tilde{\mathbf{q}}^{213} = (1, 1, 0, 1, 0, 1, 0, 1)$
- $\tilde{\mathbf{q}}^{214} = (1, 1, 0, 1, 0, 1, 1, 0)$
- $\tilde{\mathbf{q}}^{215} = (1, 1, 0, 1, 0, 1, 1, 1)$
- $\tilde{\mathbf{q}}^{216} = (1, 1, 0, 1, 1, 0, 0, 0)$
- $\tilde{\mathbf{q}}^{217} = (1, 1, 0, 1, 1, 0, 0, 1)$
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- $\tilde{\mathbf{q}}^{252} = (1, 1, 1, 1, 1, 1, 0, 0)$
- $\tilde{\mathbf{q}}^{253} = (1, 1, 1, 1, 1, 1, 0, 1)$
- $\tilde{\mathbf{q}}^{254} = (1, 1, 1, 1, 1, 1, 1, 0)$
- $\tilde{\mathbf{q}}^{255} = (1, 1, 1, 1, 1, 1, 1, 1)$

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