Reactive strategies with longer memory

ICSD 2024

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MAX-PLANCK-GESELLSCHAFT

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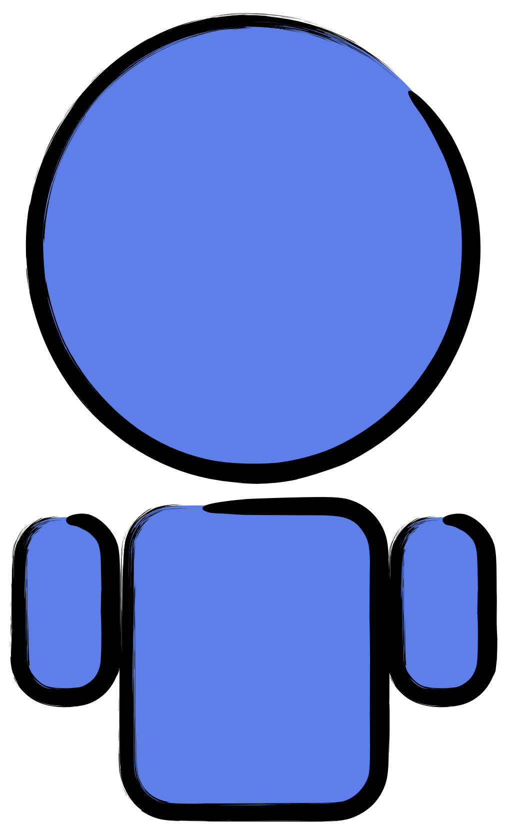
R-CCS

$$\begin{array}{c c} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} r & s \\ t & p \end{pmatrix} \end{array}$$

$$\begin{array}{c|cc} & C & D \\ \hline C & r & s \\ D & t & p \end{array}$$



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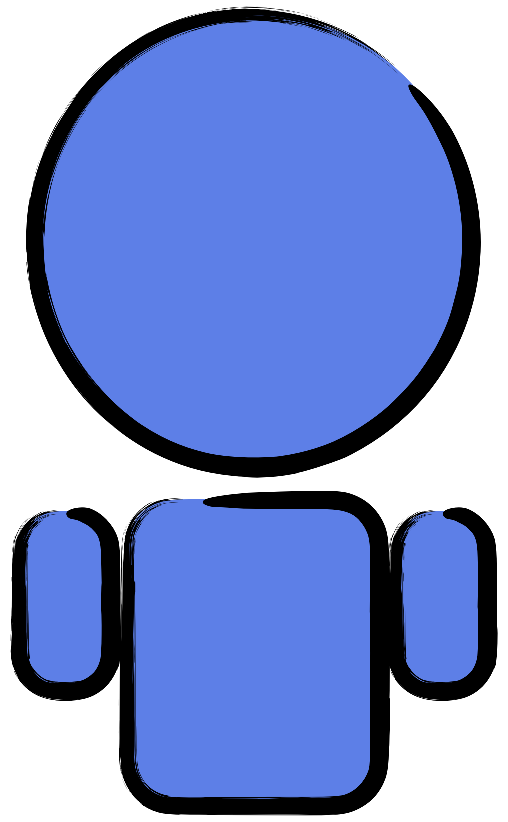
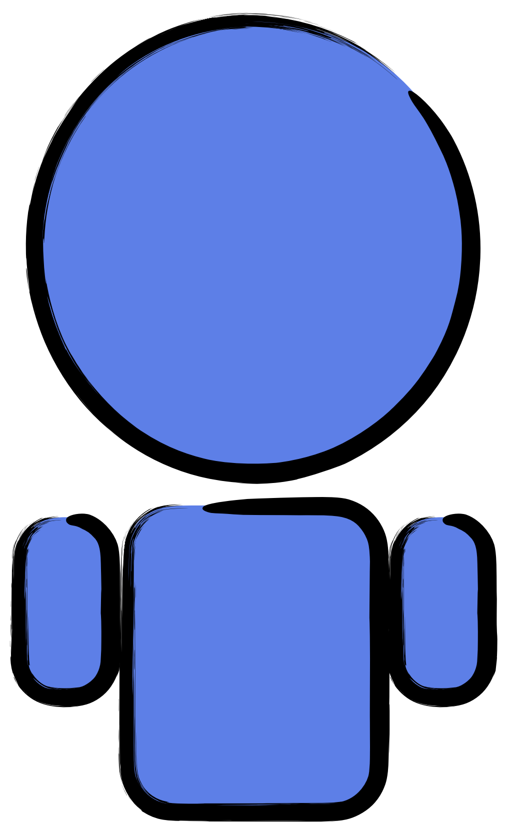


$$\begin{array}{cc} & \begin{matrix} C & D \end{matrix} \\ \begin{matrix} C \\ D \end{matrix} & \begin{pmatrix} r & s \\ t & p \end{pmatrix} \end{array}$$



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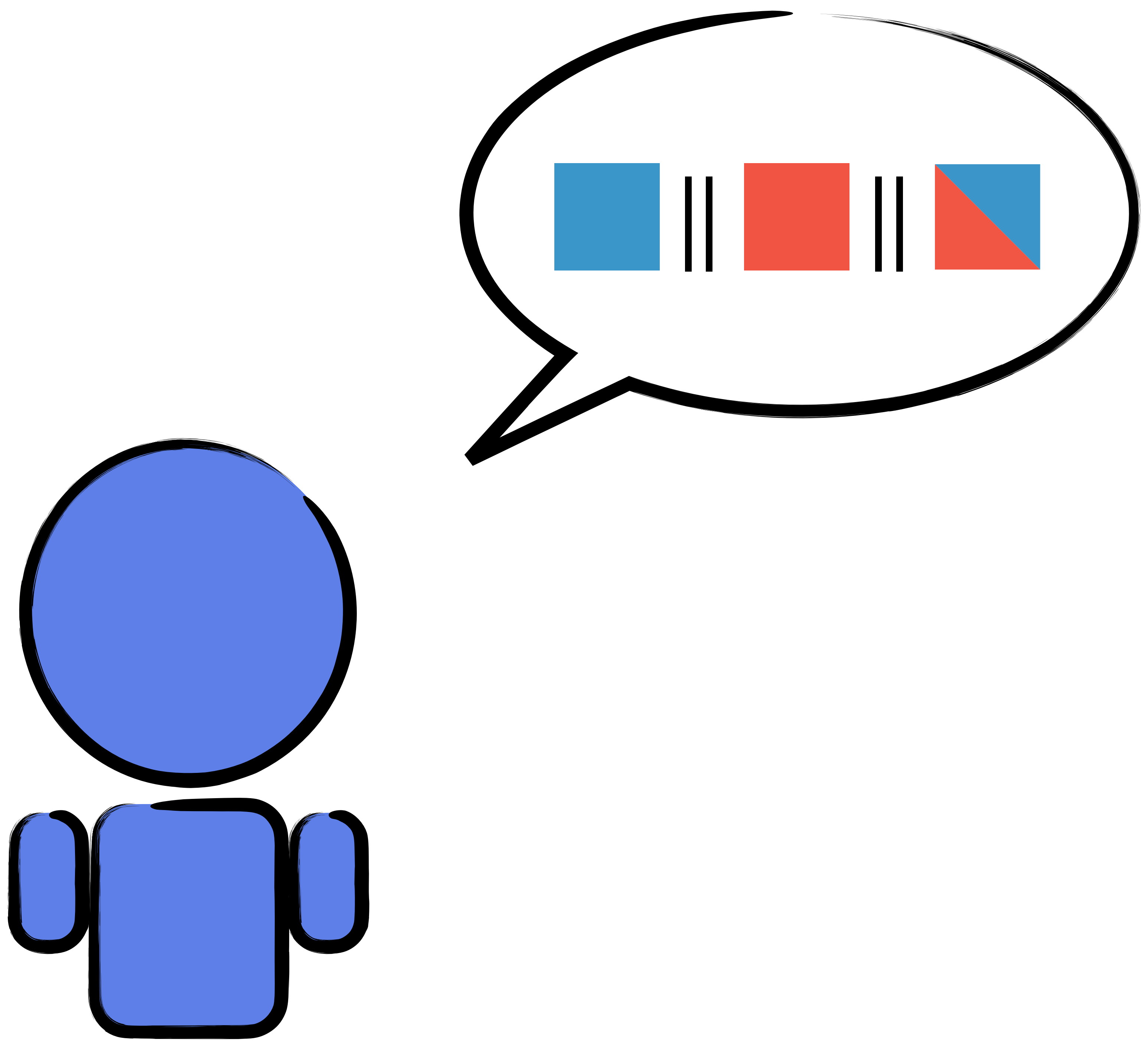
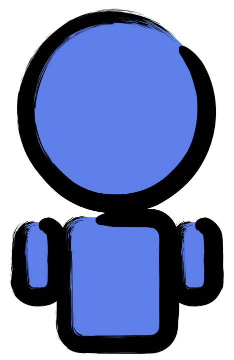
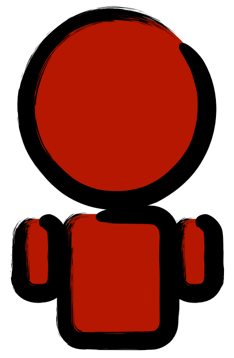


$$\overset{1}{\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} r & s \\ t & p \end{pmatrix} \end{array}} \quad \overset{2}{\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} r & s \\ t & p \end{pmatrix} \end{array}} \quad \overset{3}{\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} r & s \\ t & p \end{pmatrix} \end{array}} \cdots \overset{n-1}{\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} r & s \\ t & p \end{pmatrix} \end{array}} \quad \overset{n}{\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} r & s \\ t & p \end{pmatrix} \end{array}}$$





$$
\begin{array}{cccccc}
1 & 2 & 3 & & n-1 & n \\
\begin{array}{c|cc} & C & D \\ \hline C & r & s \\ D & t & p \end{array} &
\begin{array}{c|cc} & C & D \\ \hline C & r & s \\ D & t & p \end{array} &
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\begin{array}{c|cc} & C & D \\ \hline C & r & s \\ D & t & p \end{array}
\end{array}
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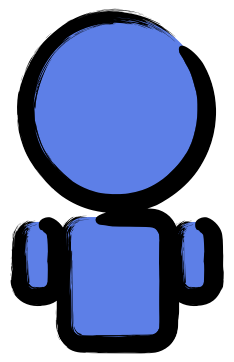
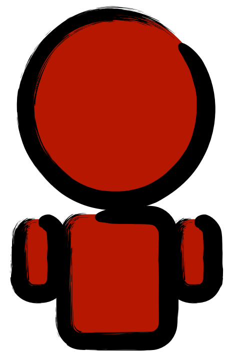
	C	D	C	C	C
	D	C	C	D	C

	1	2	3		$n-1$	n
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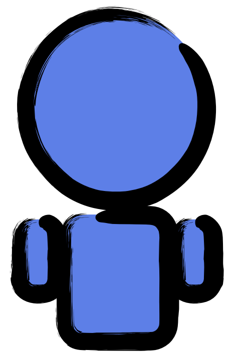
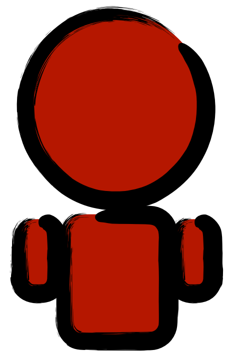
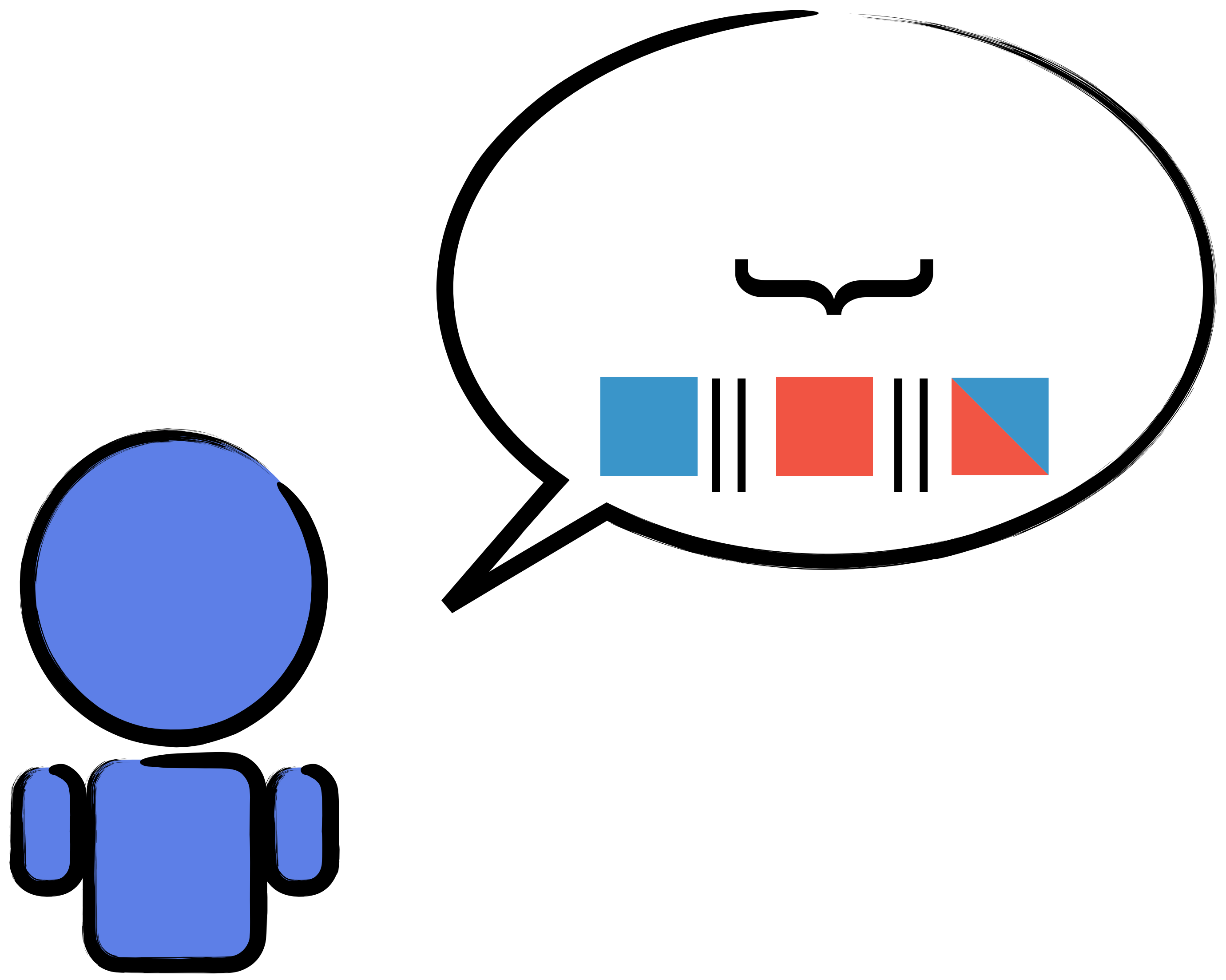
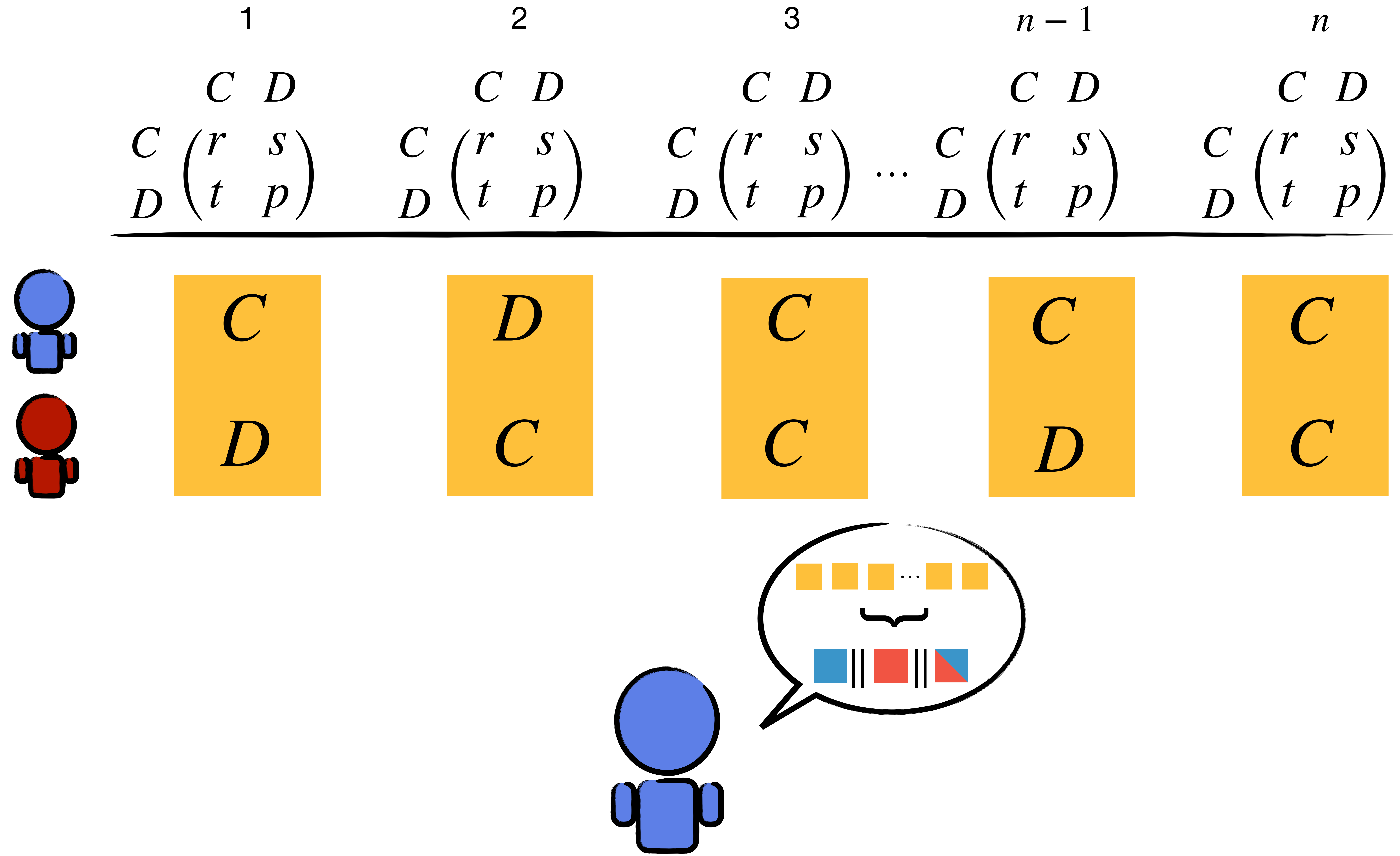
	C	D	C		C	C
	D	C	C		D	C





1
2
3
$n-1$
n
$\begin{array}{c|cc} & C & D \\ \hline C & r & s \\ D & t & p \end{array}$
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D
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C
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C
C
D
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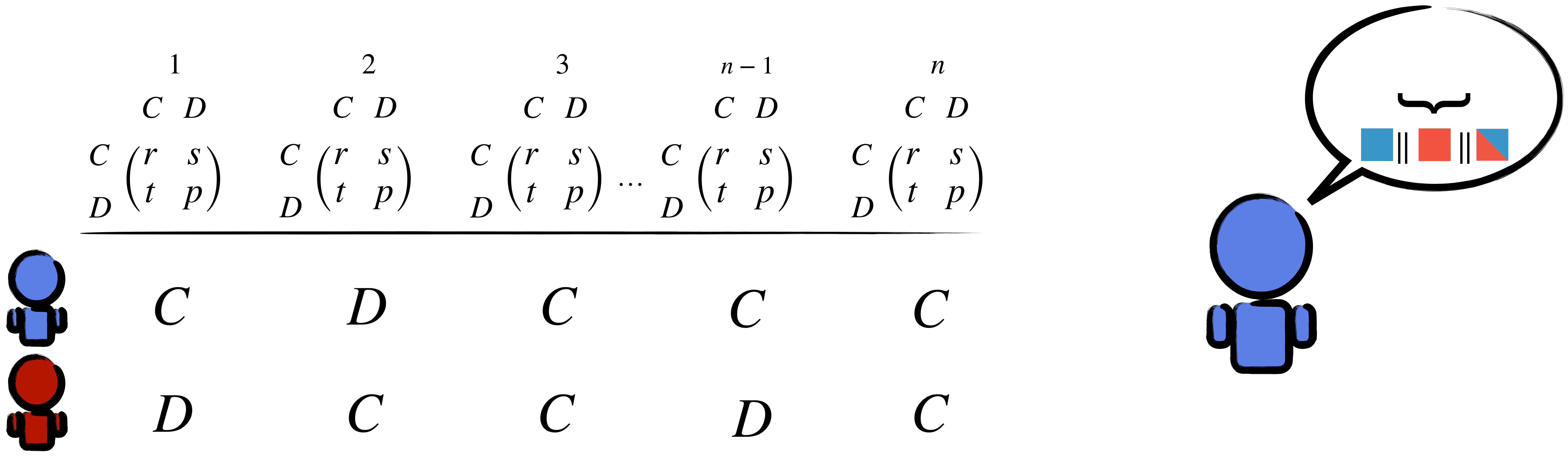
C



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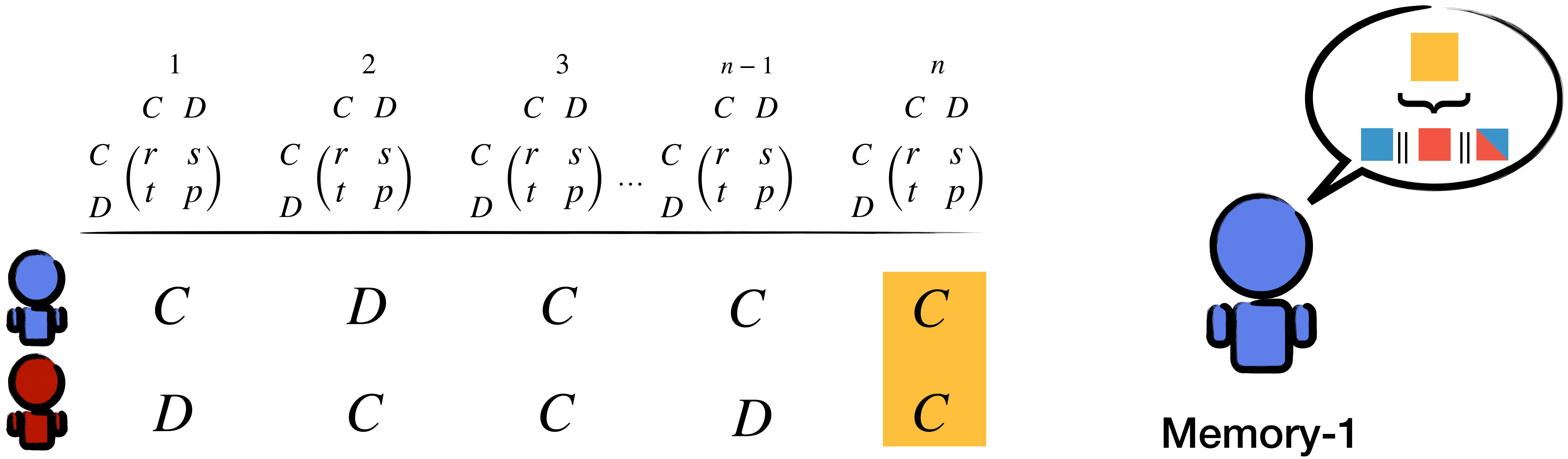
C D C C C
D C C D C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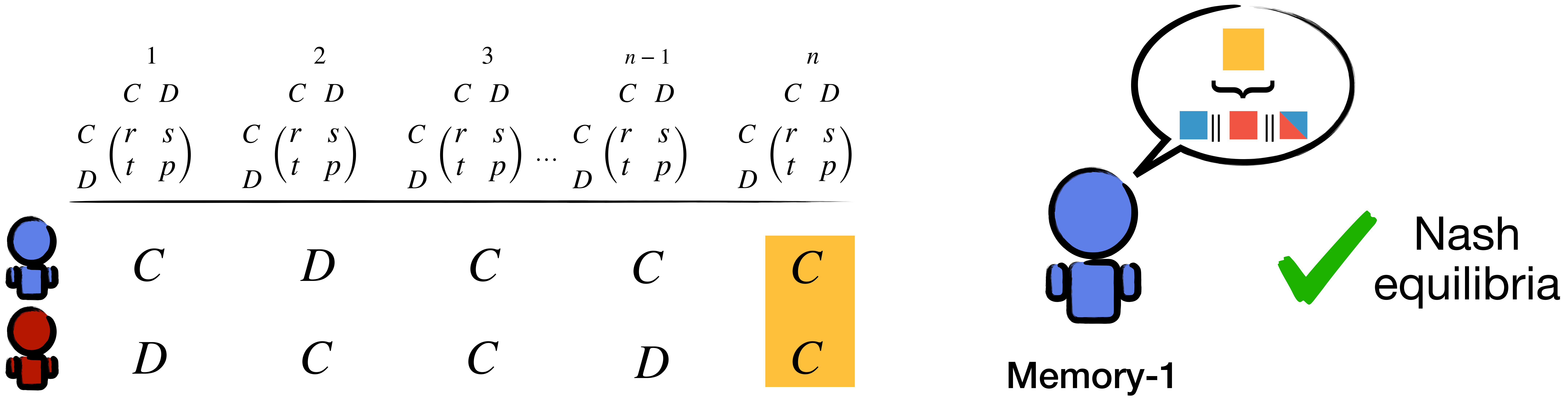
D C C D C
Memory-1


1
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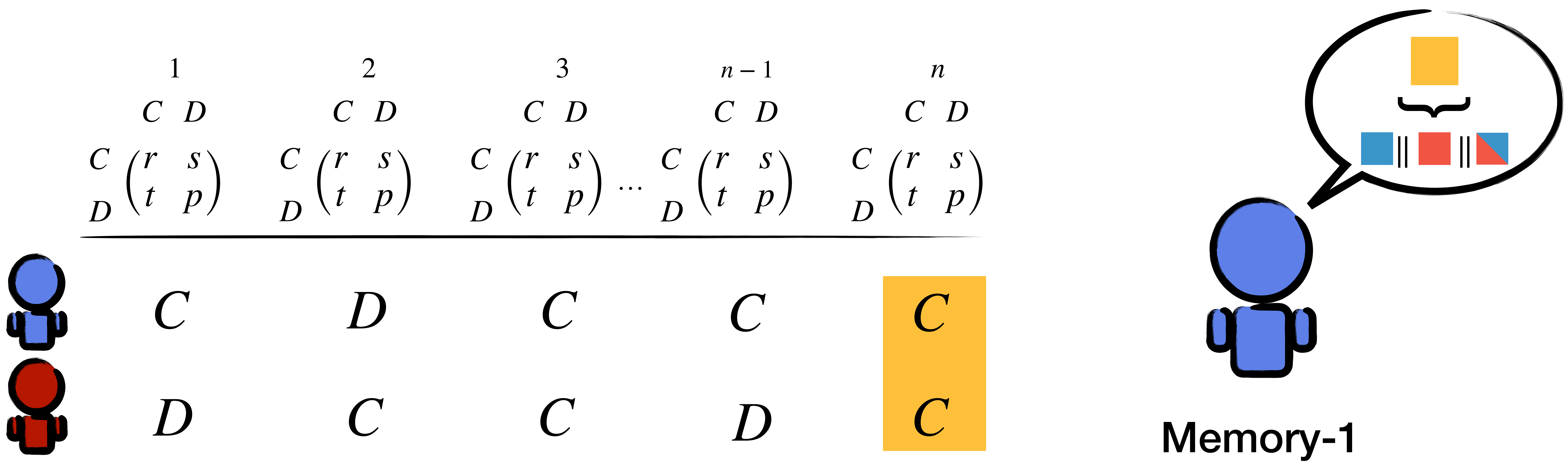
D
r s
t p
C D C C C
D C C D C
Memory-1
Nash
equilibria



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C D C D C C
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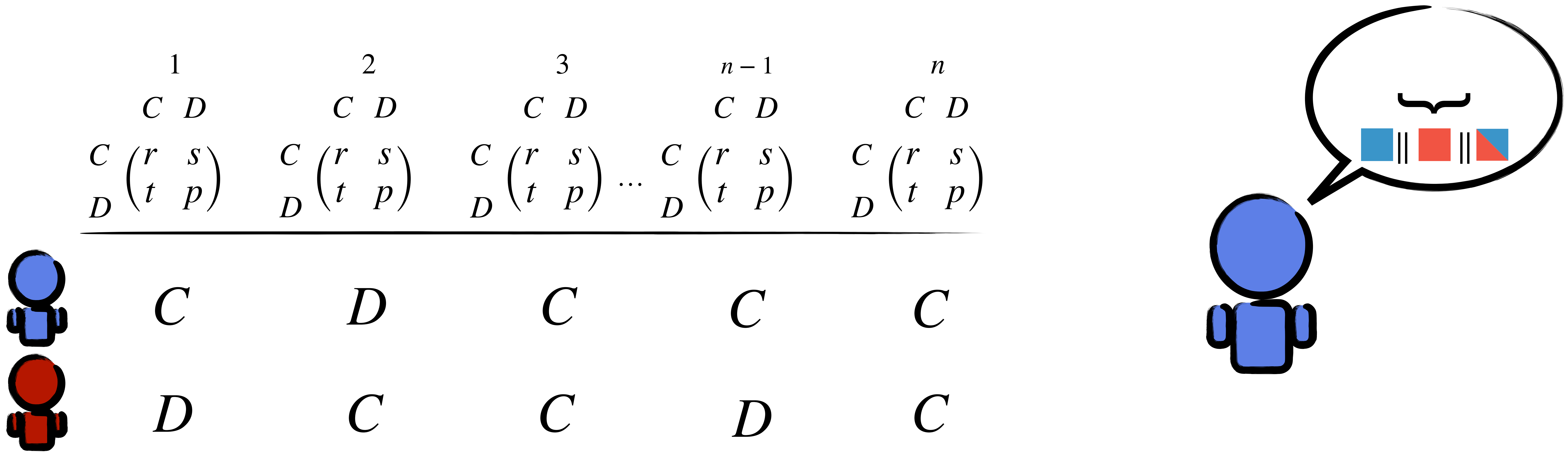
Memory-1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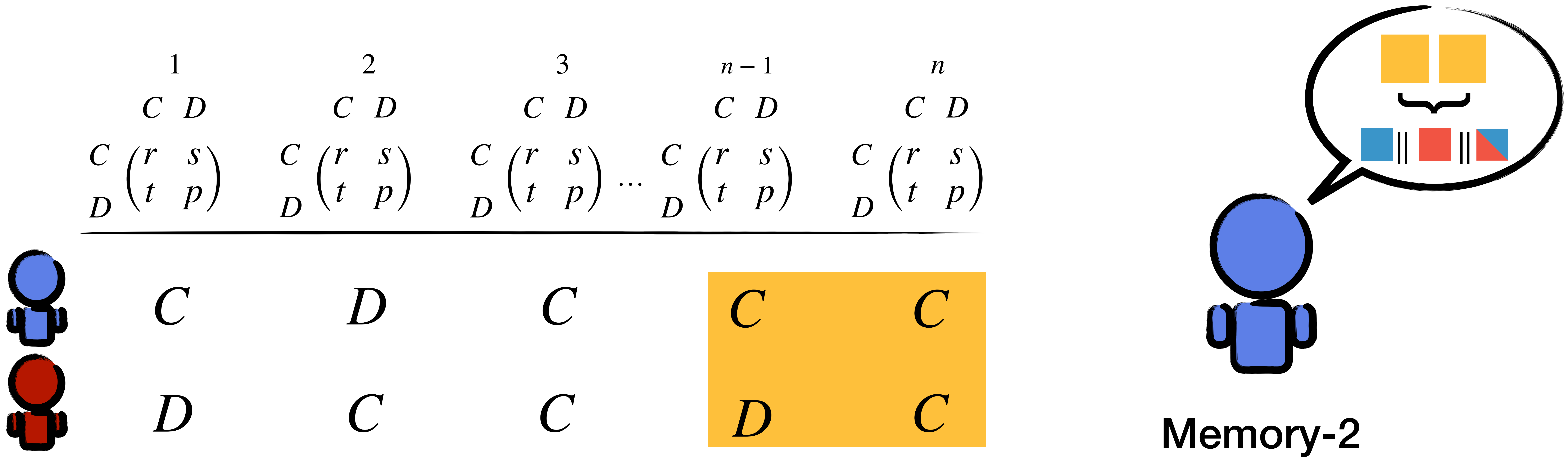
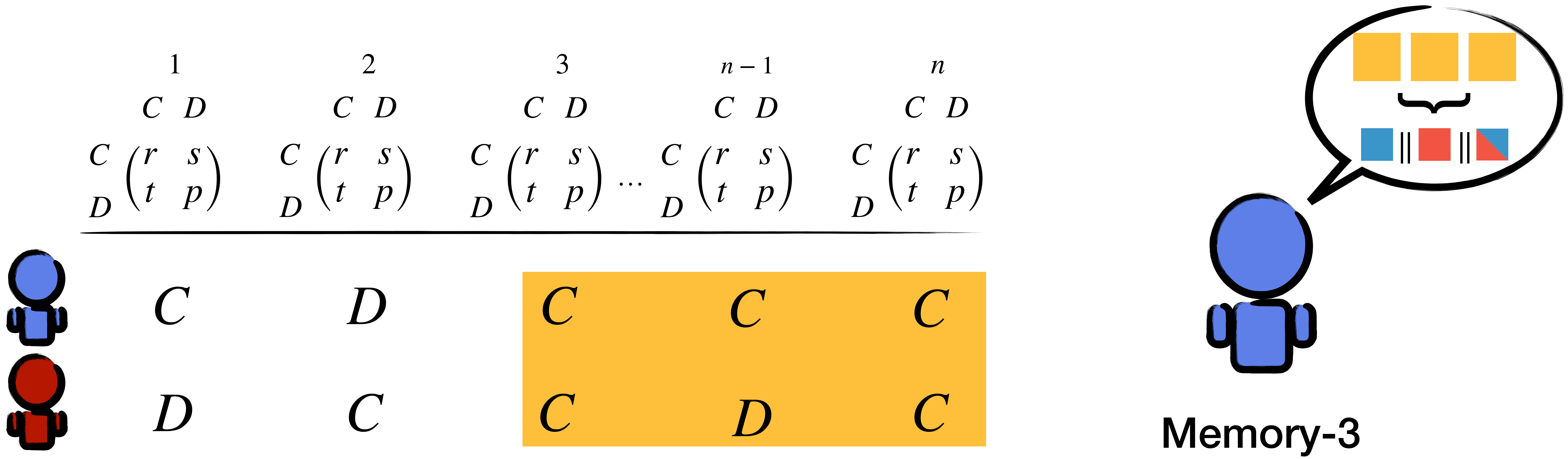
n − 1
n
C D
C r s
D t p
C D C C C
D C C D C



Memory-2



1
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n − 1
n
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D
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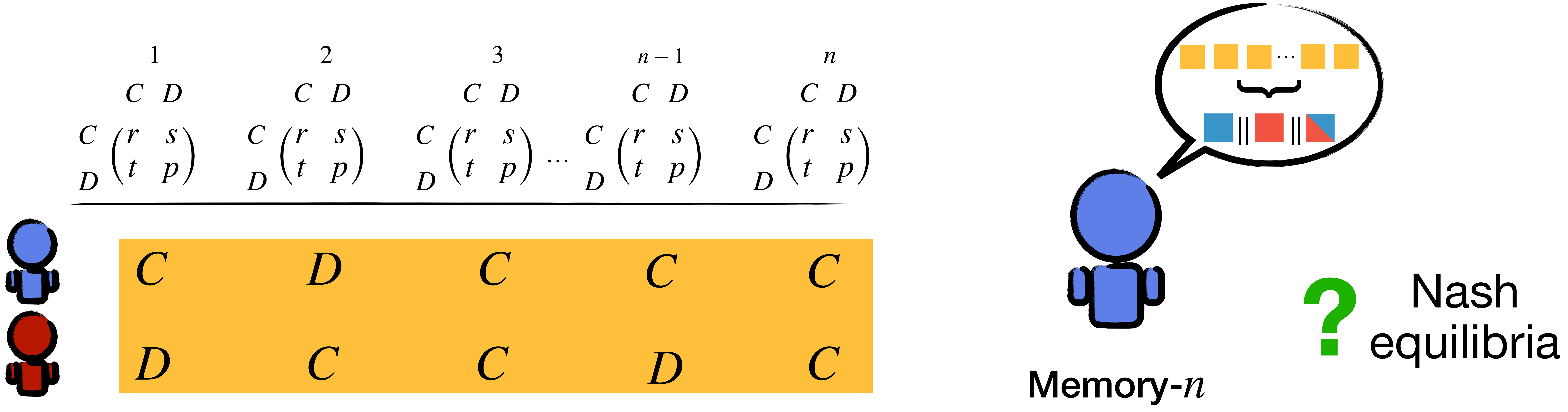
D
C
C
C
D
C
C
D
C
Memory-3

Can we say anything about Nash equilibria in repeated games for any n?



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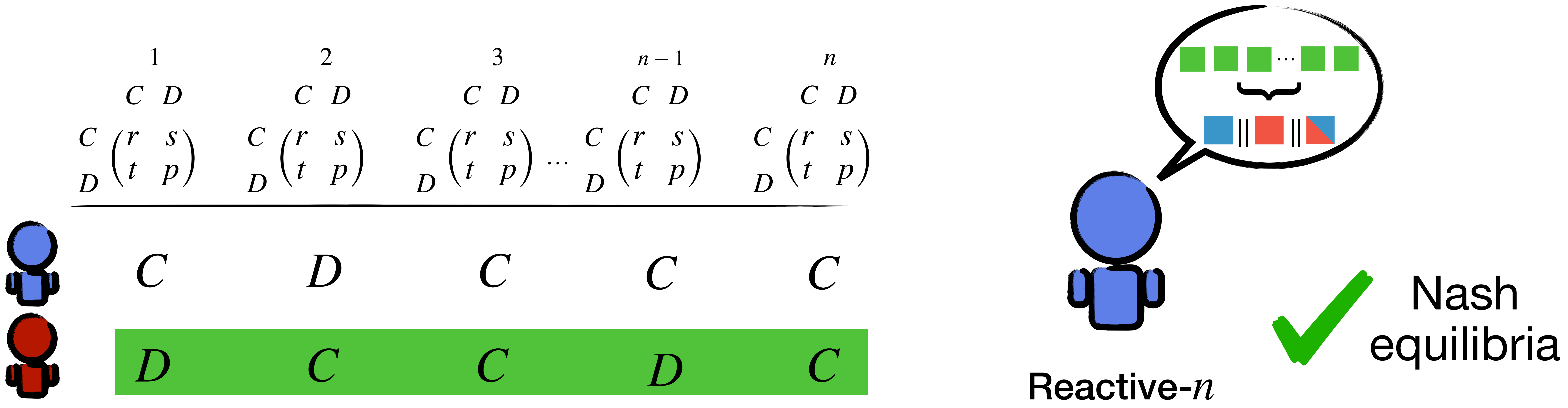

Definitions

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Definition 1.

A reactive-n strategy can be defined as 2^n-dimensional vector $\mathbf{p} = (p_{\mathbf{h^{-i}}})_{\mathbf{h^{-i}} \in H^{-i}}$ with $0 \leq p_{\mathbf{h^{-i}}} \leq 1$ where $\mathbf{h^{-i}}$ refers to an n-history of the co-player from the space of all possible co-player histories.

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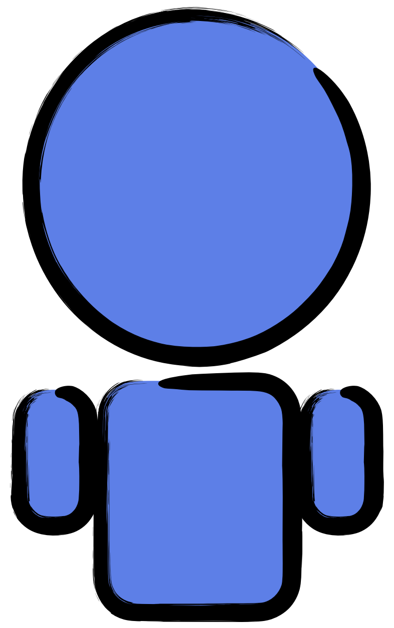
Definition 2.

A strategy $\mathbf{p}$ for a repeated game is a Nash equilibrium if it is a best response to itself.

That is $\pi(\mathbf{p}, \mathbf{p}) \geq \pi(\sigma, \mathbf{p})$ for all other strategies σ.



p


p

Nash equilibria in higher n

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Theorem. A reactive strategy $\mathbf{p} \in \mathscr{R}_n$ is a Nash equilibrium if and only if $\pi(\mathbf{p}, \mathbf{p}) \geq \pi(\tilde{\mathbf{p}}, \mathbf{p})$ for all pure self-reactive strategies $\tilde{\mathbf{p}}$.

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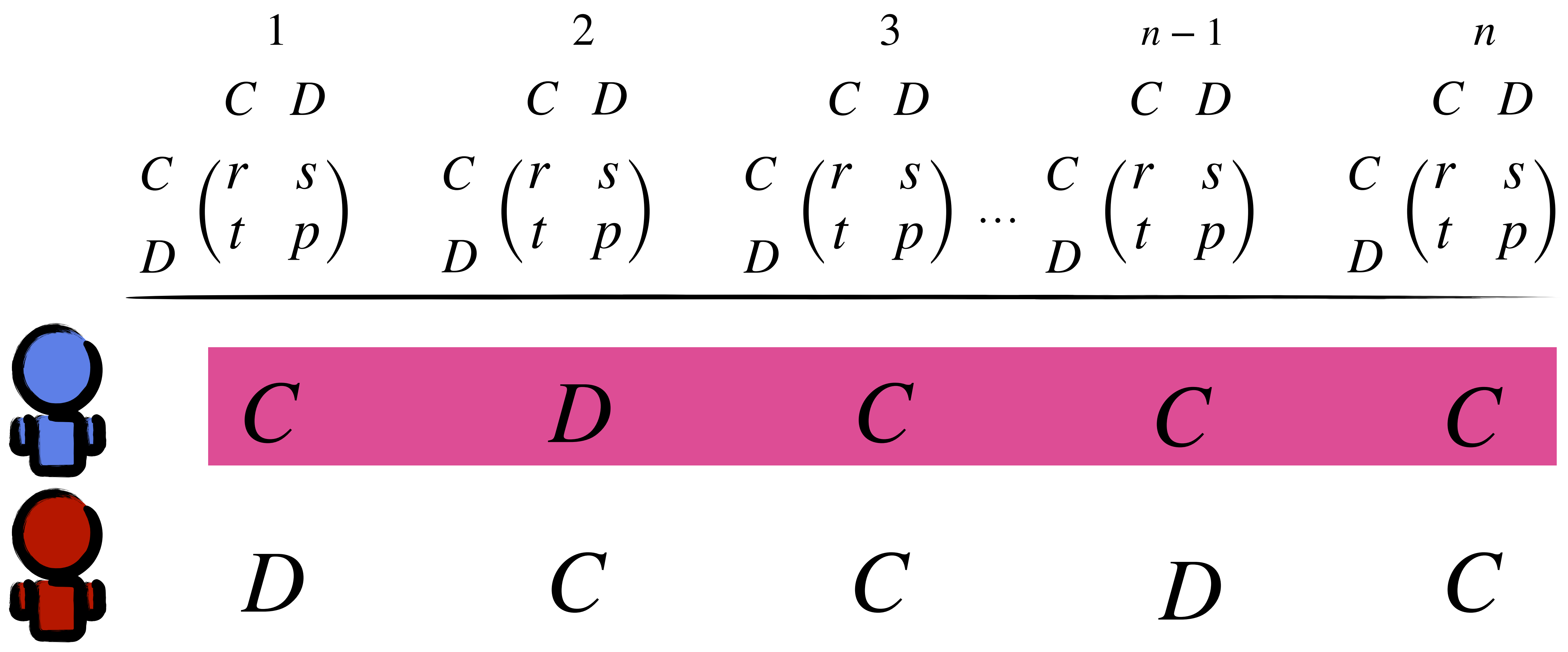

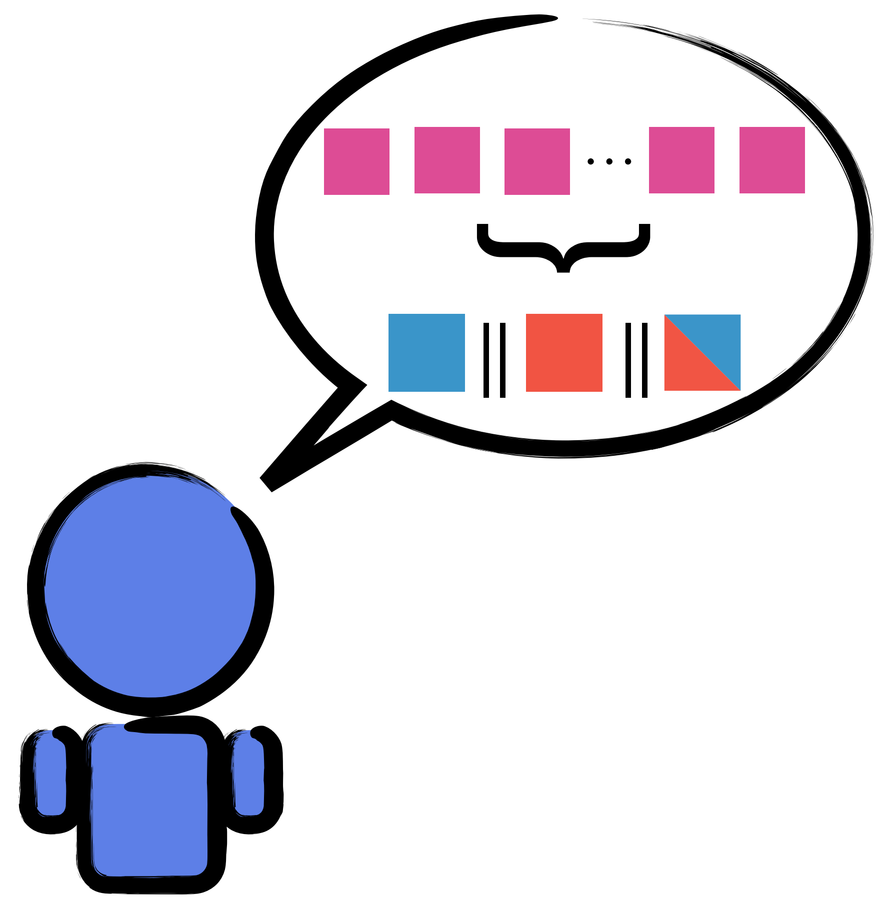

Reactive-n

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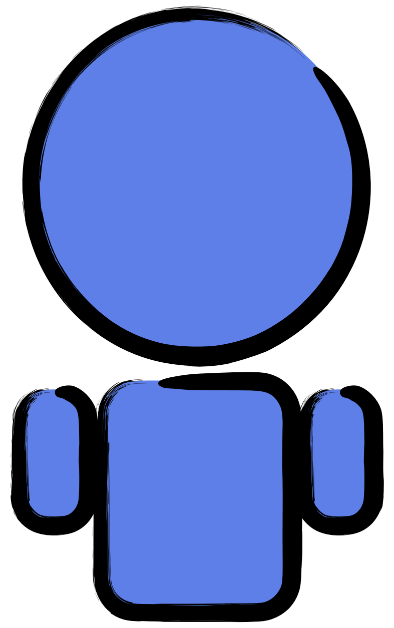

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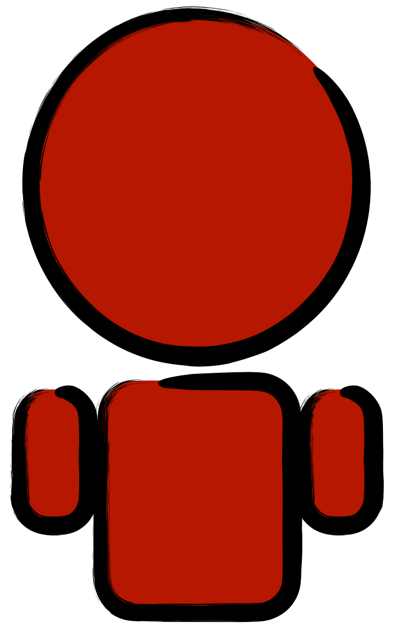

4

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$\mathbf{p} = (p_{CC}, p_{CD}, p_{DC}, p_{DD})$ 16

Donation game for $n = 2$

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} b-c & -c \\ b & 0 \end{pmatrix} \end{array}$$

$$b > c > 0$$

Donation game for $n = 2$

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Theorem. A reactive-2 strategy $\mathbf{p} = (p_{CC}, p_{CD}, p_{DC}, p_{DD})$ is a cooperative Nash equilibrium if and only if its entries satisfy the conditions,

$$p_{CC} = 1, \quad \frac{p_{CD} + p_{DC}}{2} < 1 - \frac{1}{2} \cdot \frac{c}{b}, \quad p_{DD} \leq 1 - \frac{c}{b}\,.$$

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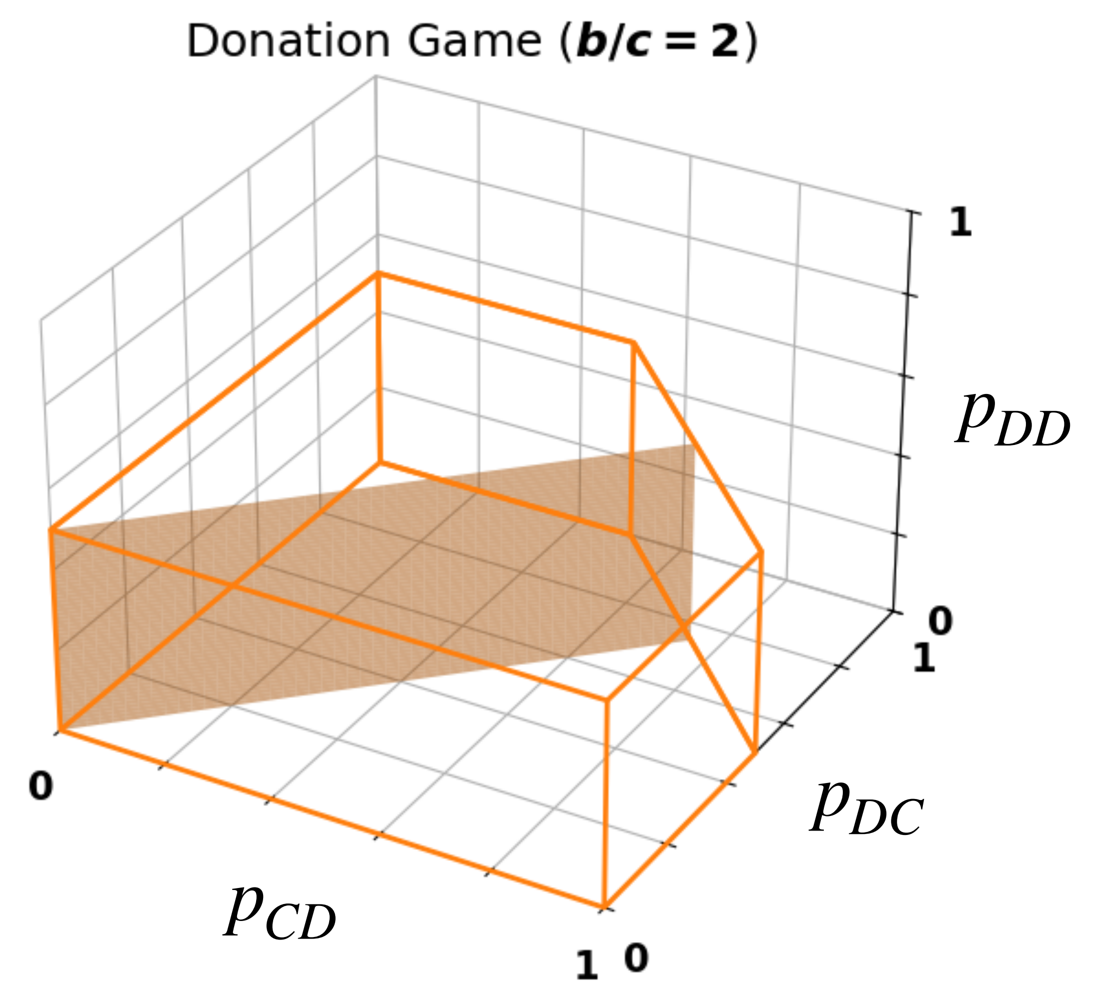

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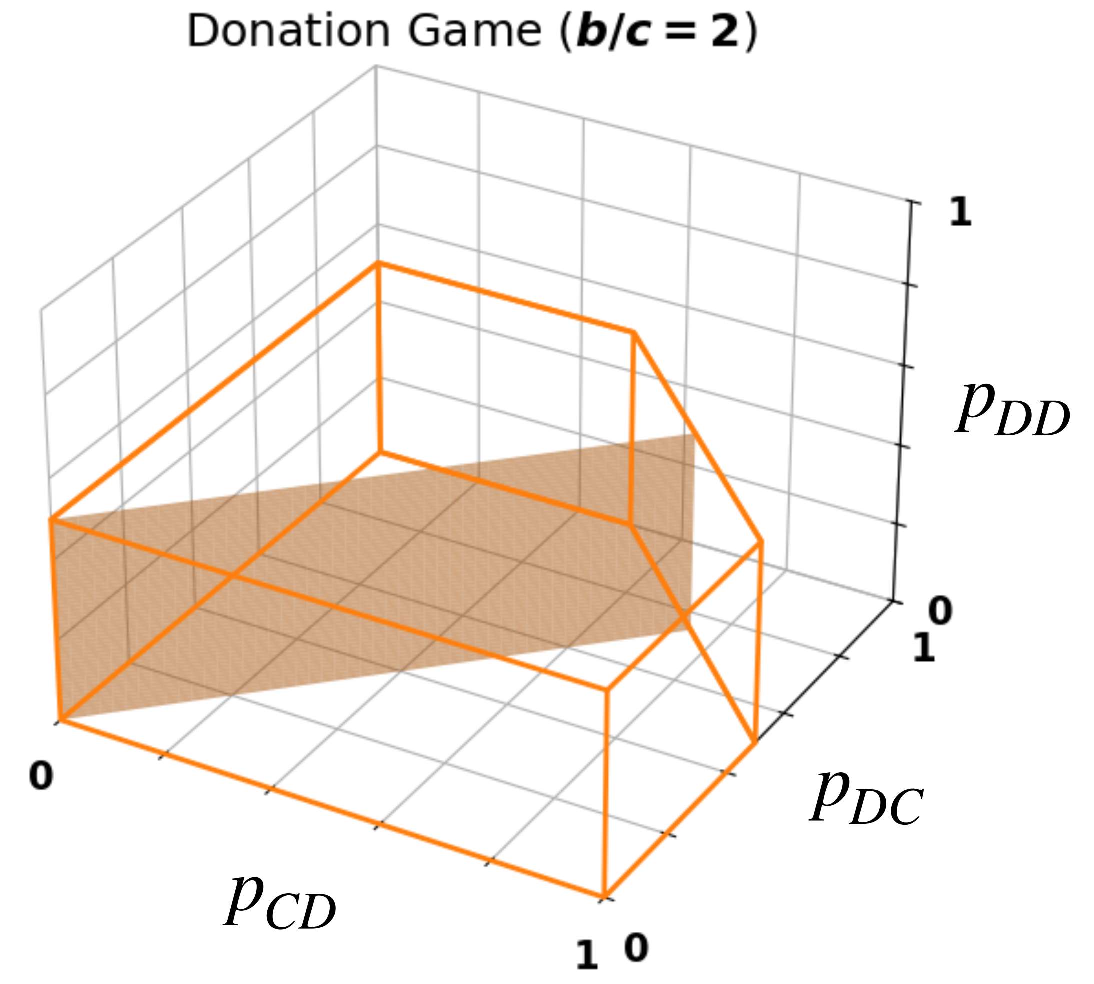

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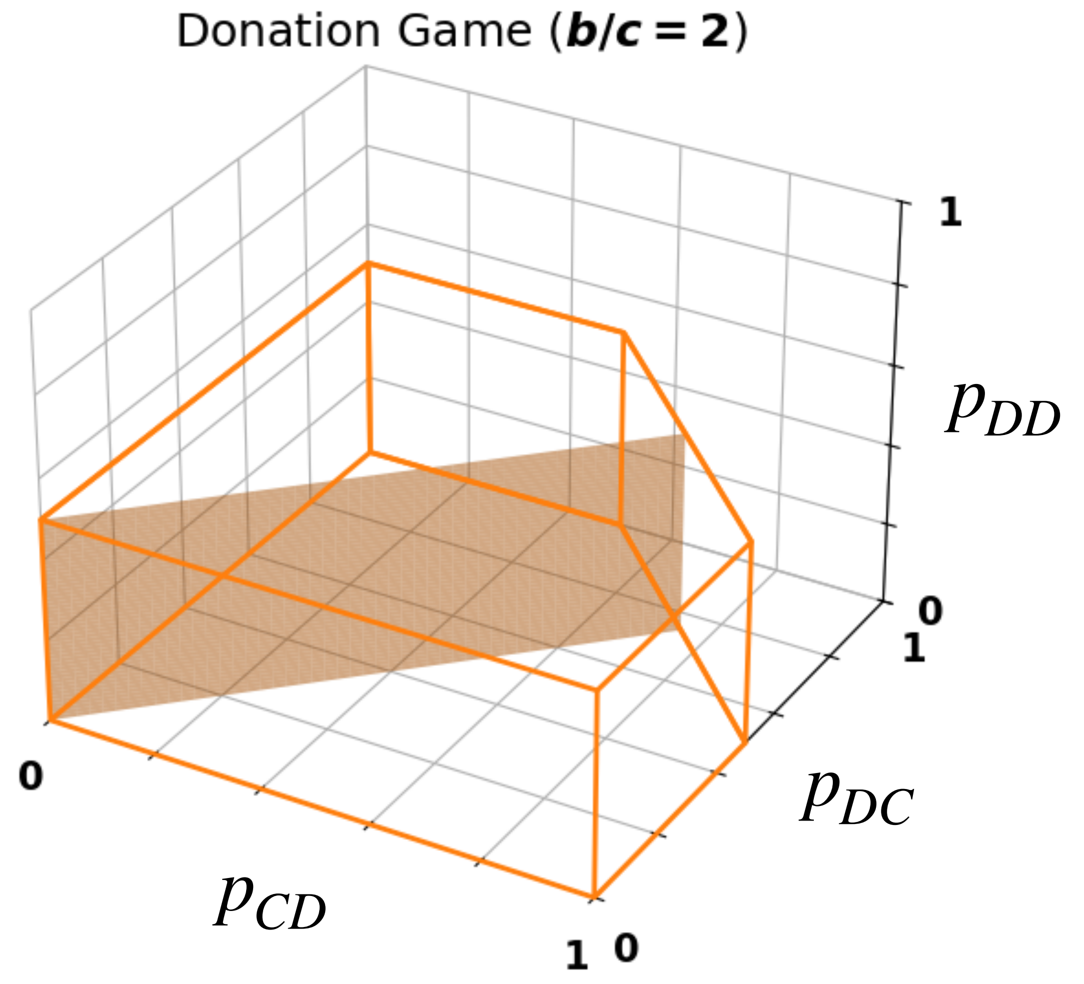

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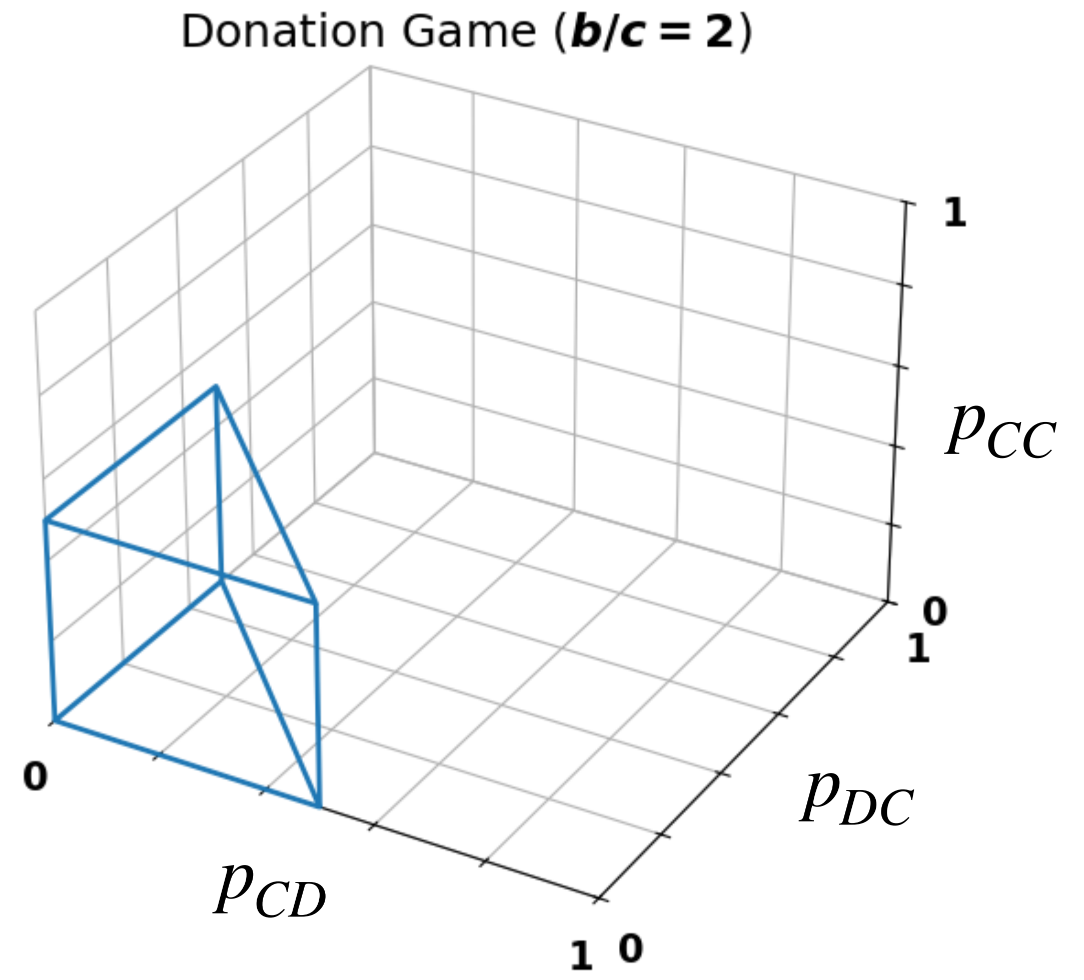
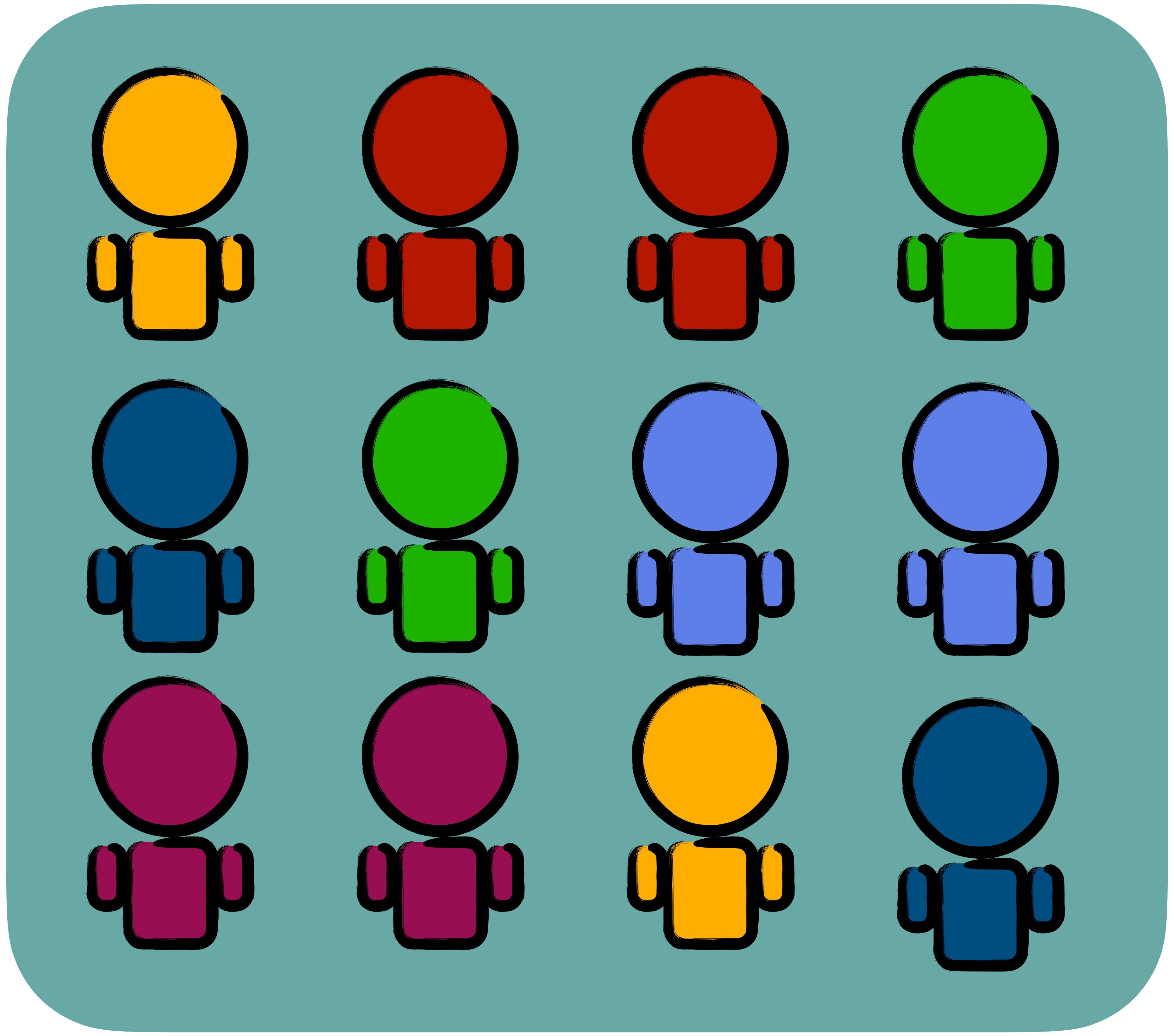


Evolutionary simulations



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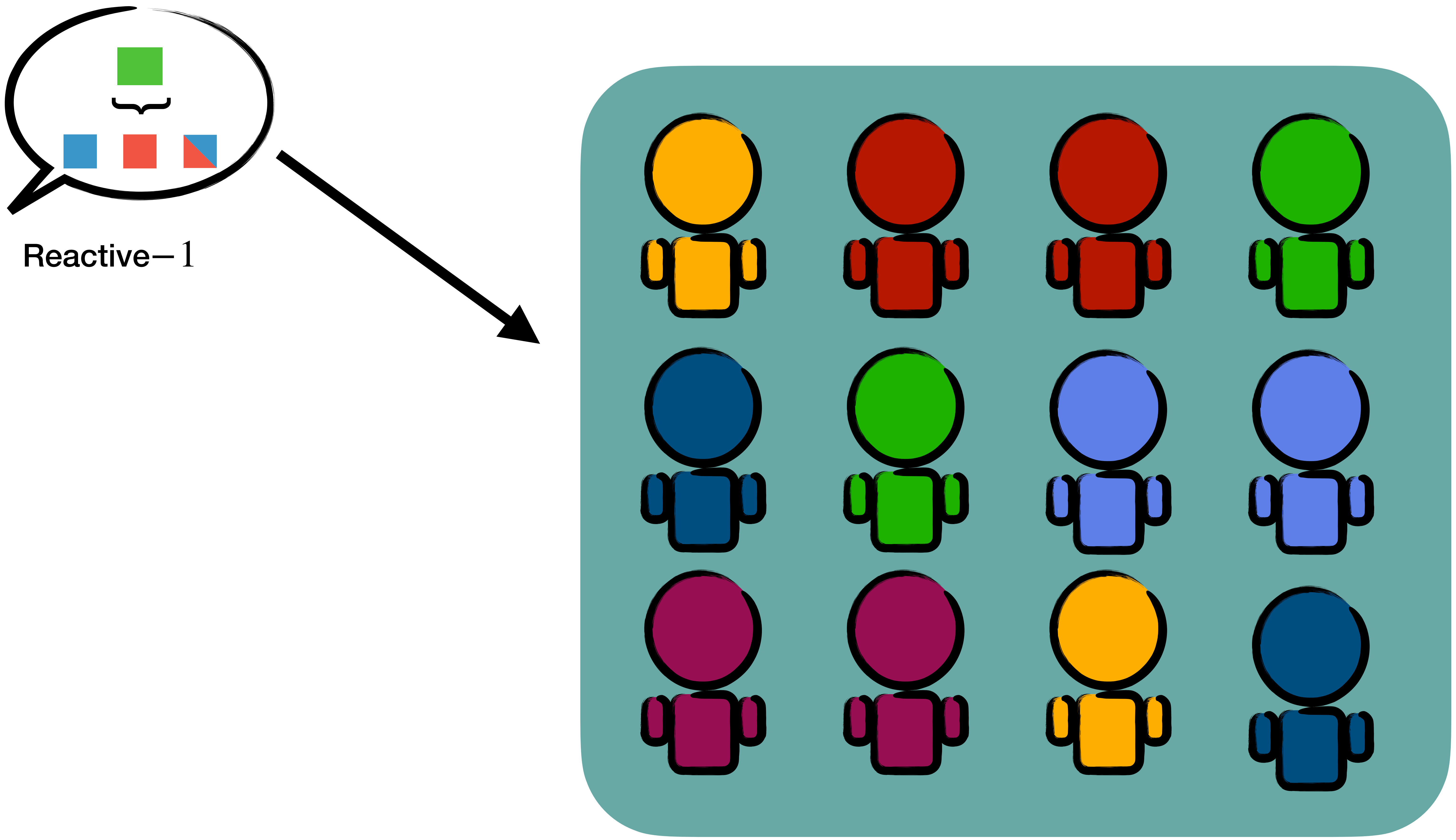

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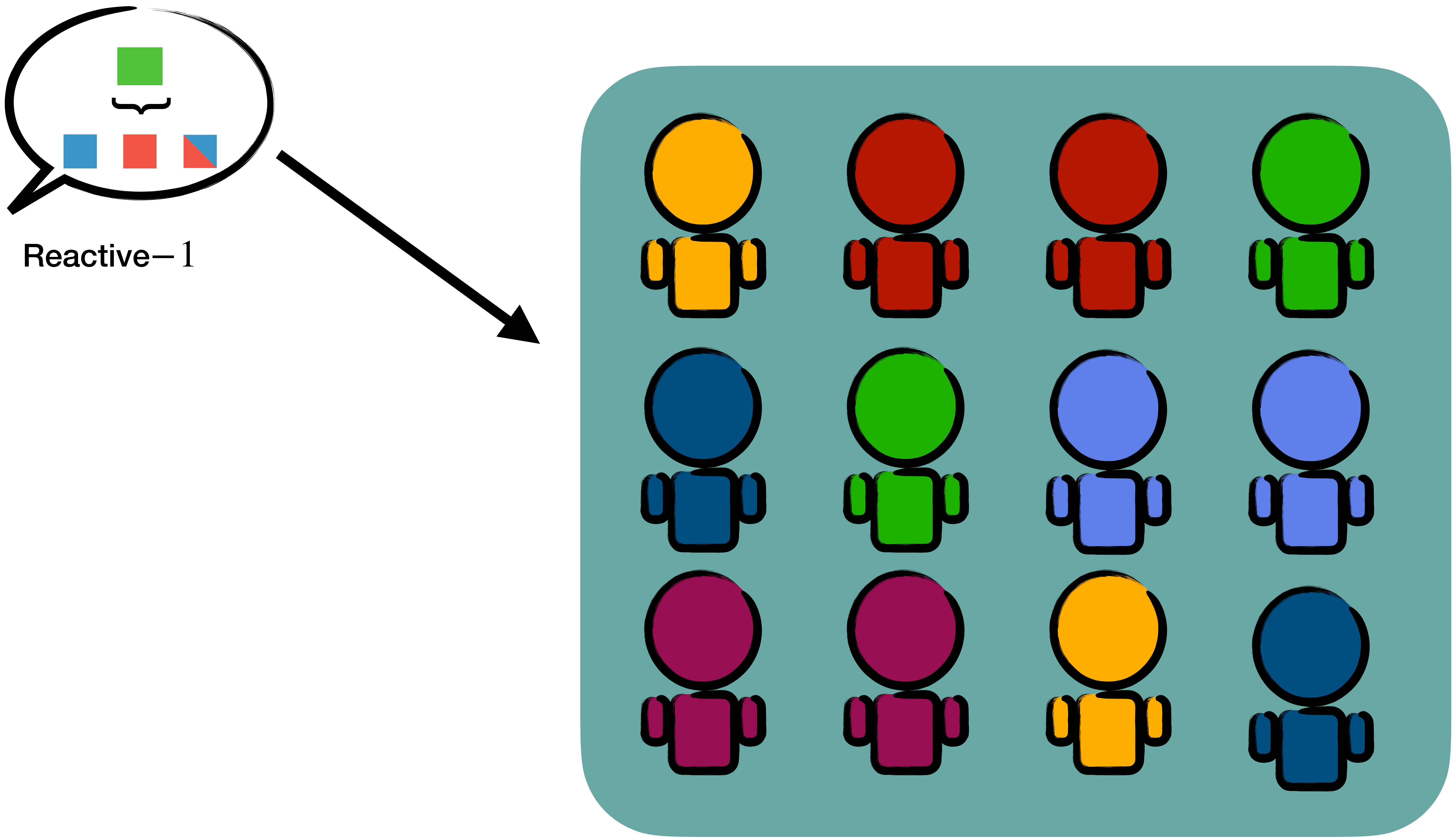
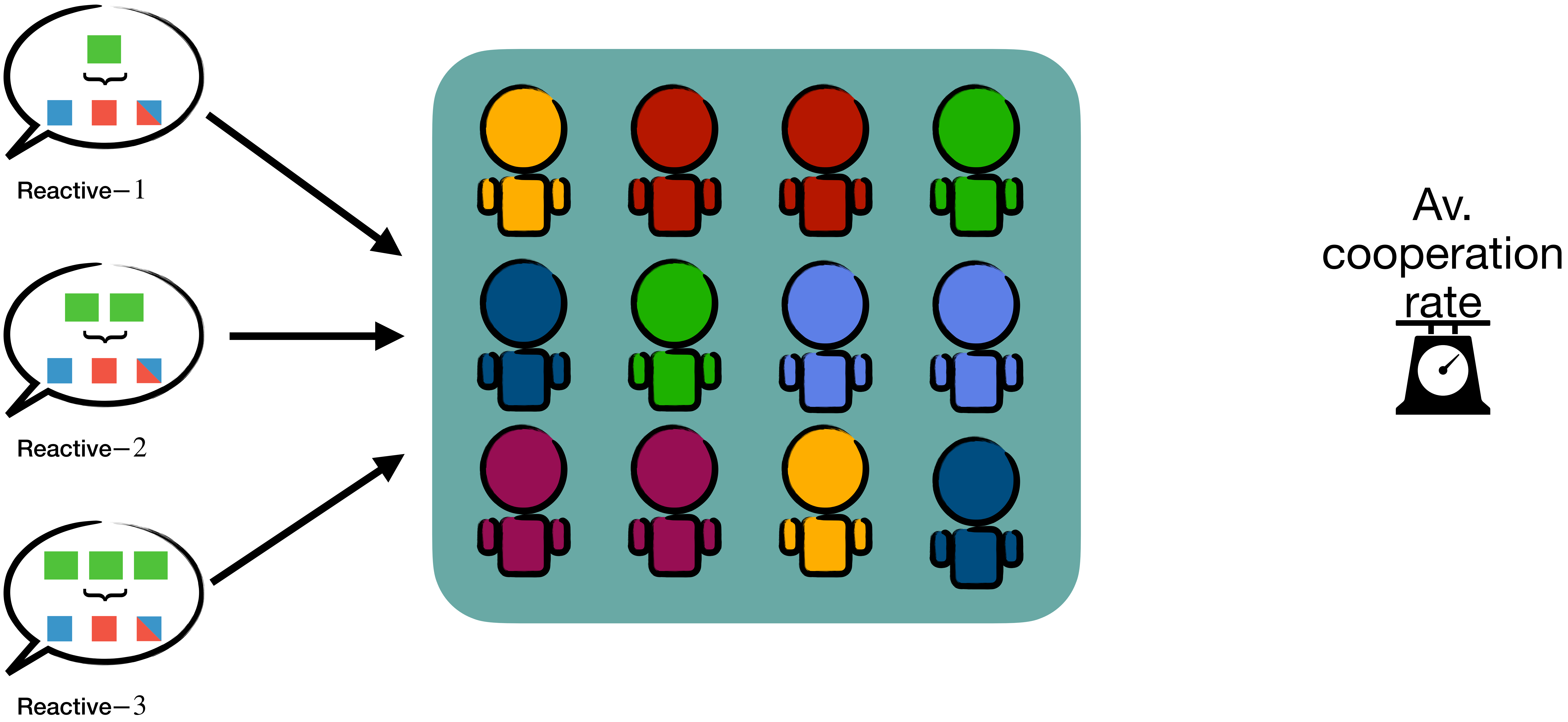
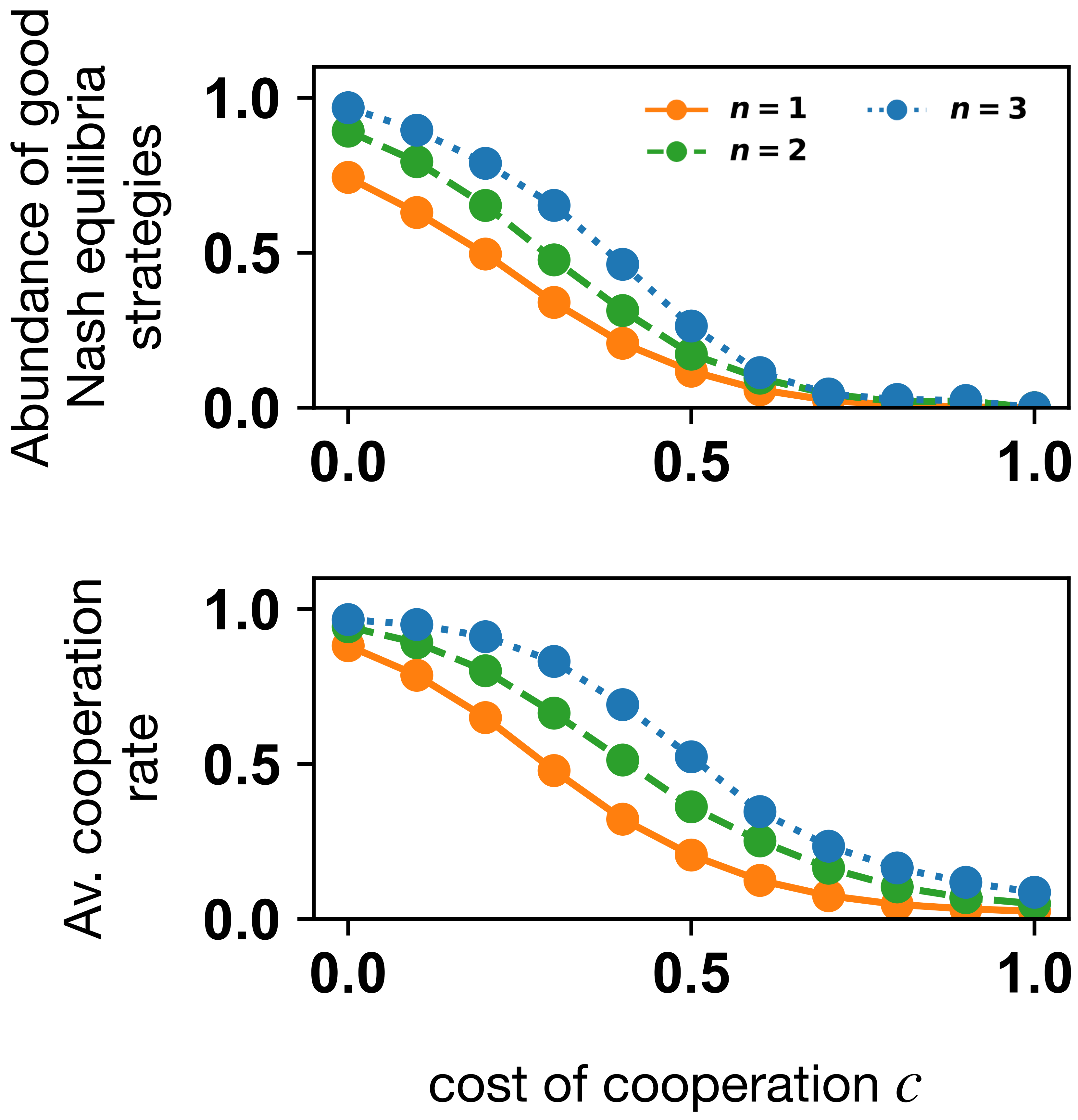


Av.
cooperation
rate

Evolutionary simulations





Abundance of good Nash equilibria strategies
1.0
0.5
0.0
0.0
0.5
1.0
n = 1
n = 2
n = 3
Av. cooperation rate
1.0
0.5
0.0
0.0
0.5
1.0
cost of cooperation c

1. Algorithm to verify whether a given reactive-n strategy is an equilibrium.

input: p, n

pure_self_reactive_strategies $\leftarrow \left\{ \tilde{\mathbf{p}} \;\middle|\; \tilde{\mathbf{p}} \in \{0,1\}^{2^n} \right\}$;

isNash ←True;

for $\tilde{\mathbf{p}} \in$ *pure_self_reactive_strategies* **do**

  if $\mathbf{p}$ is not a best response $\tilde{\mathbf{p}}$ to **then**

    isNash ←False;

return ($\mathbf{p}$, isNash);

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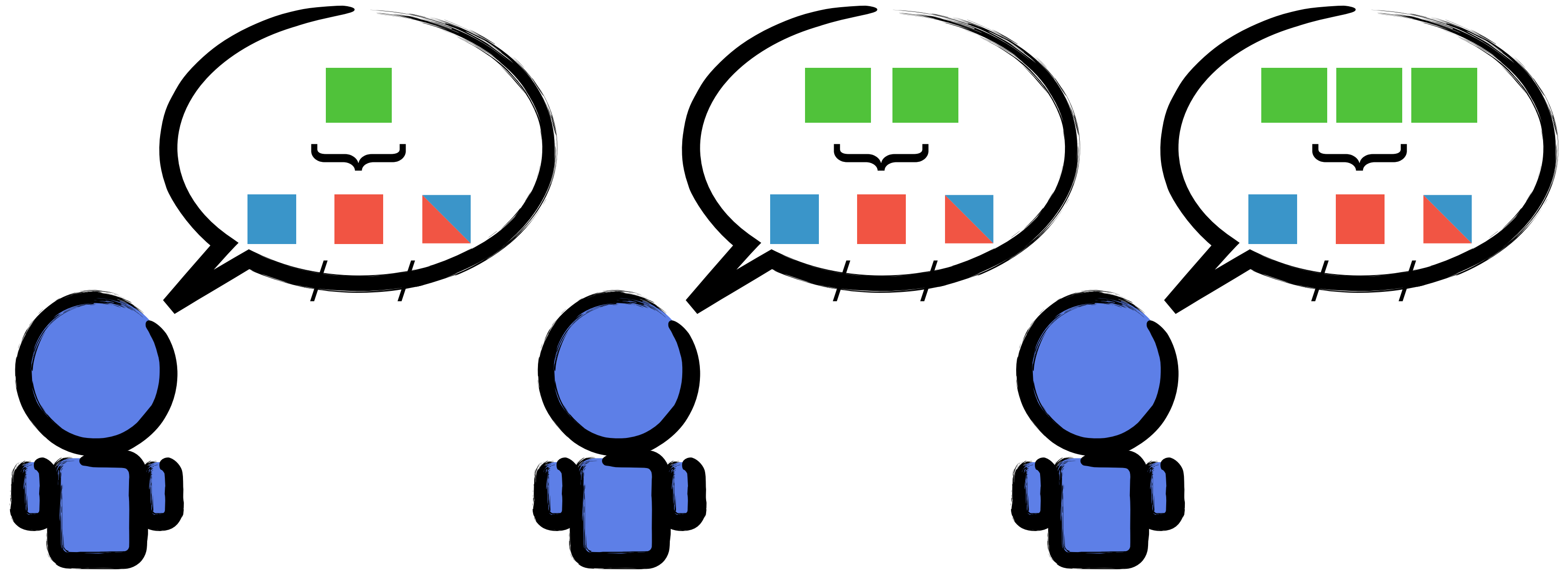
2. Fully characterize cooperative & defective equilibria for $n = 2$ and $n = 3$.



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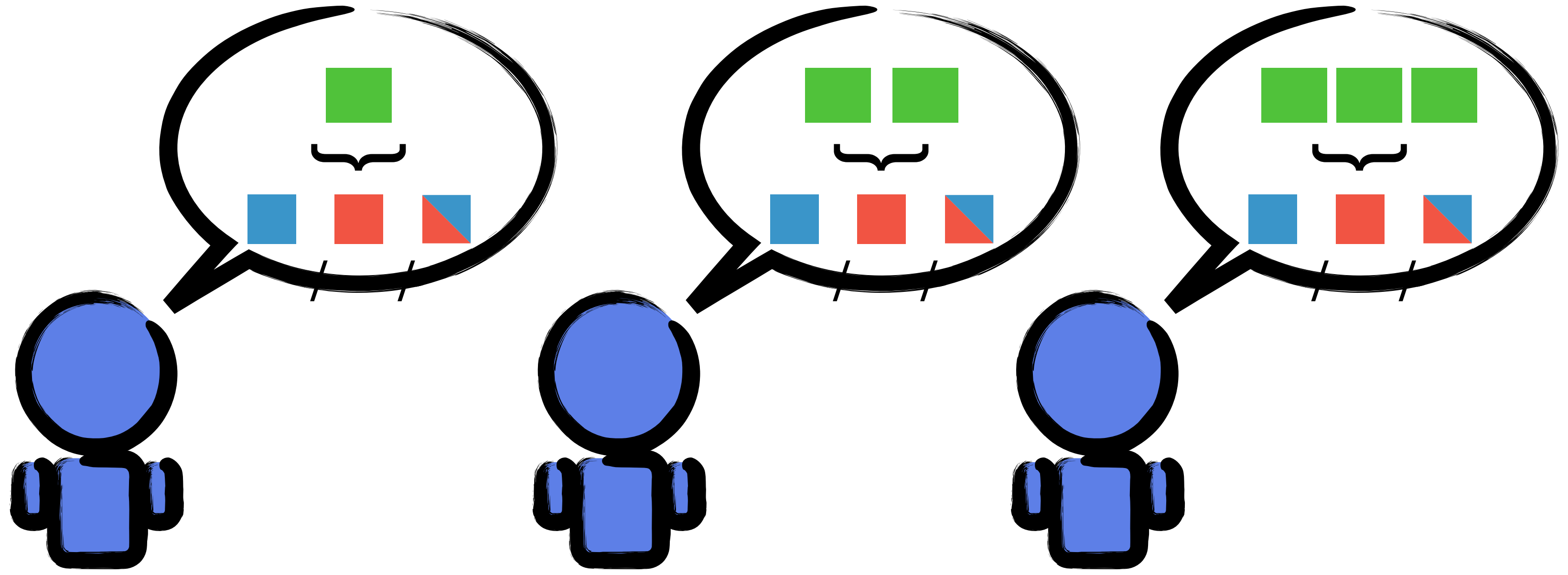
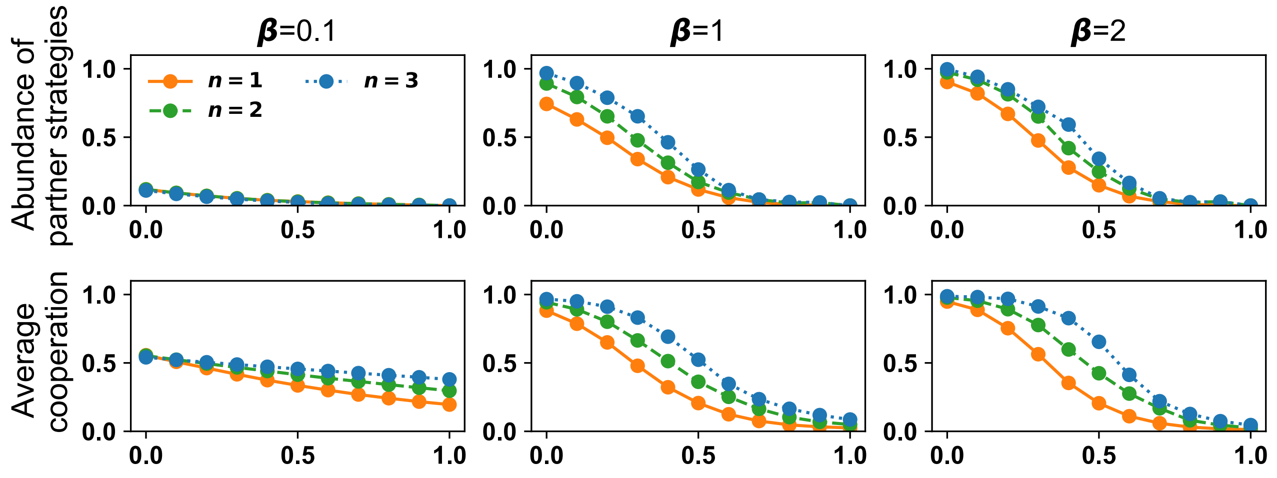


3. Longer memory helps sustain cooperation.



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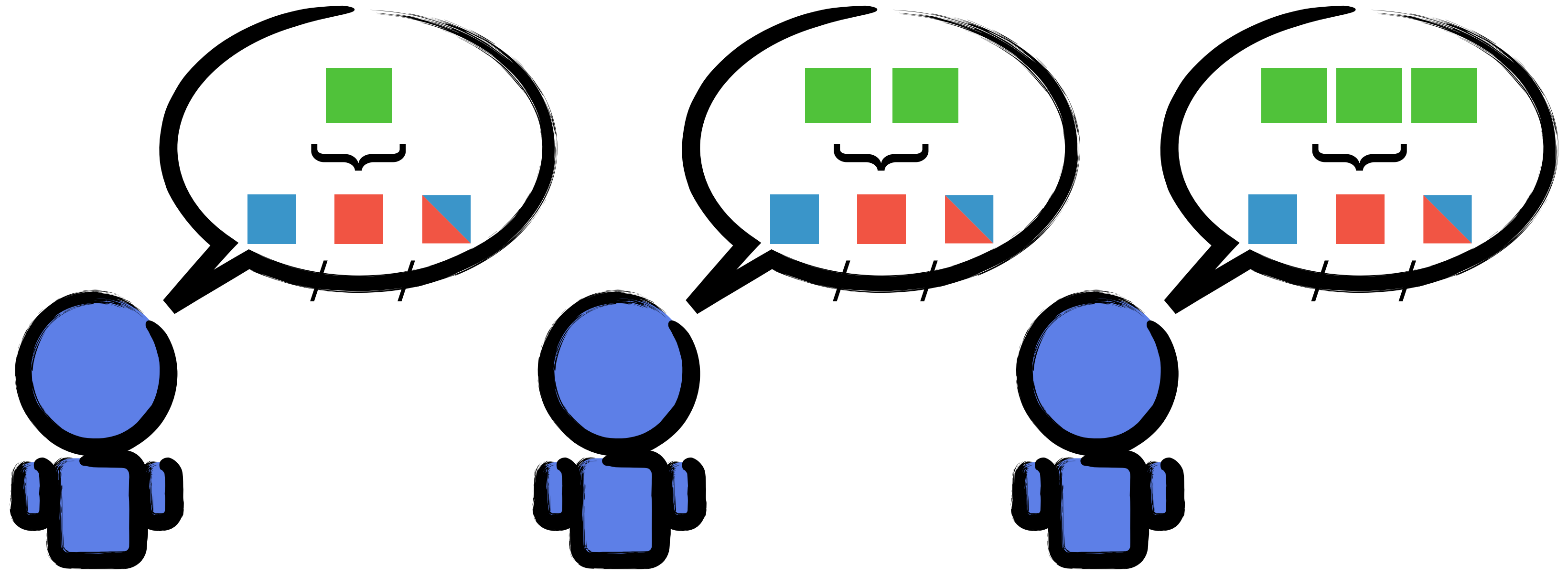

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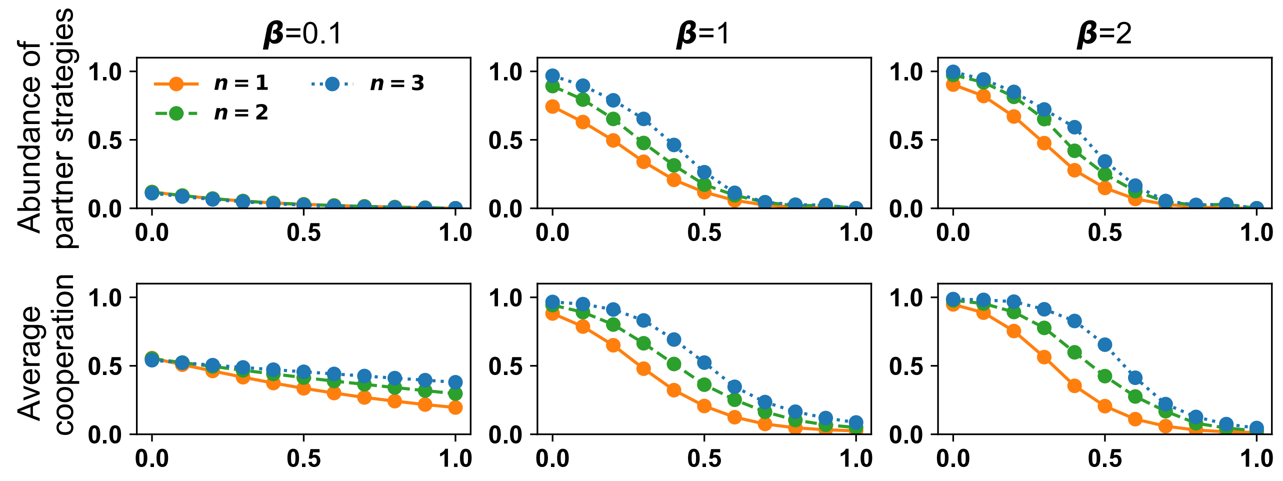

More information

Conditional cooperation with longer memory

https://arxiv.org/abs/2402.02437

@NikoletaGlyn

https://nikoleta-v3.github.io/

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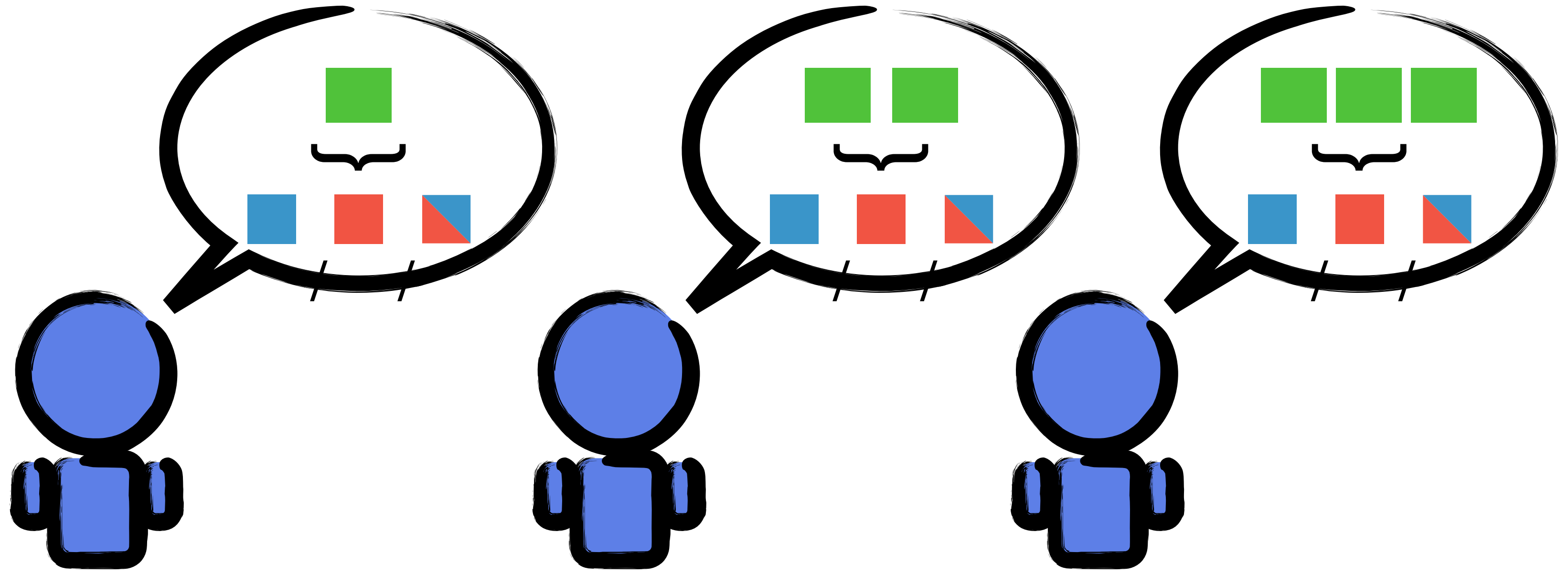

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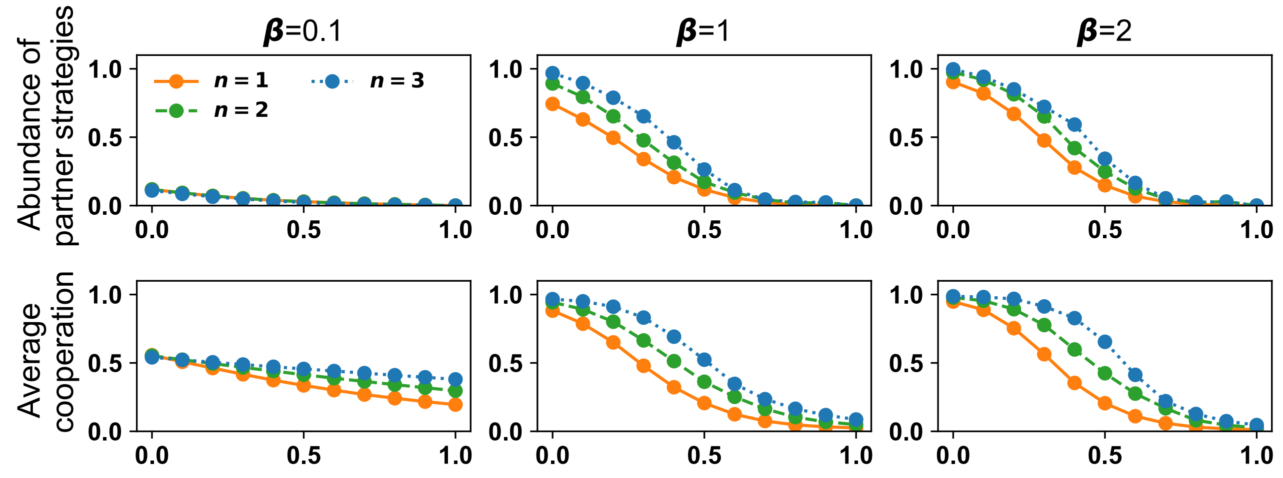

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