۱)

$$A=\begin{bmatrix}2 & 4 & 6\\ 4 & 11 & 15\\ 6 & 15 & 23\end{bmatrix}$$

الف) ماتریس‌های پایین‌مثلثی L و بالامثلثی U را به گونه‌ای بیابید که $A = LU$.

ب) با استفاده از پاسخ قسمت (الف) ماتریس را به فرم $A = \mathrm{LDL^T}$ (که در آن ماتریس D قطری است) در آورید.

ج) با استفاده از پاسخ قسمت (ب) تجزیه Cholesky ماتریس A را به فرم $A = R^TR$ (که در آن R یک ماتریس بالامثلثی است) بیابید.

الف)

$$A=\begin{bmatrix}2 & 4 & 6\\ 4 & 11 & 15\\ 6 & 15 & 23\end{bmatrix}\begin{array}{l}\to \text{pivot}=2\\ \to \text{mult}=2\\ \to \text{mult}=3\end{array} \Rightarrow E_1=\begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ -3 & 0 & 1\end{bmatrix}$$

$$E_1A\Rightarrow\begin{bmatrix}2 & 4 & 6\\ 0 & 3 & 3\\ 0 & 3 & 5\end{bmatrix}\begin{array}{l}\\ \to \text{pivot}=3\\ \to \text{mult}=+1\end{array} \Rightarrow E_2=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & -1 & 1\end{bmatrix} \Rightarrow E_2E_1A=\begin{bmatrix}2 & 4 & 6\\ 0 & 3 & 3\\ 0 & 0 & 2\end{bmatrix}$$

$$U=\begin{bmatrix}2 & 4 & 6\\ 0 & 3 & 3\\ 0 & 0 & 2\end{bmatrix} \qquad L=(E_2E_1)^{-1}=\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 3 & 1 & 1\end{bmatrix} \Rightarrow A=LU$$

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۱ ب)

$$\left.\begin{array}{l}A=LDL^T\\ A=LU\end{array}\right\} \Rightarrow U=DL^T \Rightarrow D=U(L^T)^{-1}$$

$$D=\begin{bmatrix}2 & 4 & 6\\ 0 & 3 & 3\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}^{-1}=\begin{bmatrix}2 & 4 & 6\\ 0 & 3 & 3\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}1 & -2 & -3\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}2 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 2\end{bmatrix}$$

$$\Rightarrow D=\begin{bmatrix}2 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 2\end{bmatrix},\ L=\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 3 & 1 & 1\end{bmatrix} \Rightarrow A=LDL^T$$

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۱ ج)

$$\left.\begin{array}{l}A=R^TR\\ A=LDL^T\end{array}\right\} \Rightarrow A=LD^{1/2}D^{1/2}L^T=(D^{1/2}L^T)^T(D^{1/2}L^T)=R^TR$$

$$\Rightarrow R=D^{1/2}L^T=\begin{bmatrix}\sqrt{2} & 0 & 0\\ 0 & \sqrt{3} & 0\\ 0 & 0 & \sqrt{2}\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}\sqrt{2} & 2 & 3\\ 0 & \sqrt{3} & 1\\ 0 & 0 & \sqrt{2}\end{bmatrix} \Rightarrow R=\begin{bmatrix}\sqrt{2} & 2\sqrt{2} & 3\sqrt{2}\\ 0 & \sqrt{3} & \sqrt{3}\\ 0 & 0 & \sqrt{2}\end{bmatrix},\ A=R^TR$$

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$$3x + y + 6z = 0$$
$$-6x - 16z = 4$$
$$8y - 17z = 17$$

$A = \begin{bmatrix} 3 & 1 & 6 \\ -6 & 0 & -16 \\ 0 & 8 & -17 \end{bmatrix}$ → pivot = 3, → mult = -2 $\Rightarrow E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow E_1 A = \begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & -4 \\ 0 & 8 & -17 \end{bmatrix}$ → pivot = +2, → mult = +4 $\Rightarrow E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \Rightarrow$

$E_2 E_1 A = \begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & -4 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow U = \begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & -4 \\ 0 & 0 & -1 \end{bmatrix}$, $L = (E_2 E_1)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$

$LUx = b \Rightarrow Lc = b \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 17 \end{bmatrix} \Rightarrow \begin{cases} a = 0 \\ -2a + b = 4 \Rightarrow b = 4 \\ 4b + c = 17 \Rightarrow c = 1 \end{cases}$

$Ux = c \Rightarrow \begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & -4 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} \Rightarrow \begin{cases} z = -1 \\ 2y - 4z = 4 \Rightarrow y = 0 \\ 3x + y + 6z = 0 \Rightarrow x = 2 \end{cases} \Rightarrow \boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}}$

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$A = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 3 & 3 \\ -1 & -1 & 5 \\ 1 & 3 & 7 \end{bmatrix}$

$\tilde{q}_1 = a_1 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \Rightarrow q_1 = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$

$\tilde{q}_2 = a_2 - \frac{\tilde{q}_1 \cdot a_2}{\tilde{q}_1 \cdot \tilde{q}_1} \tilde{q}_1 = \begin{bmatrix} -1 \\ 3 \\ -1 \\ 3 \end{bmatrix} - 2 \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow q_2 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$\tilde{q}_3 = a_3 - \frac{\tilde{q}_1 \cdot a_3}{\tilde{q}_1 \cdot \tilde{q}_1} \tilde{q}_1 - \frac{\tilde{q}_2 \cdot a_3}{\tilde{q}_2 \cdot \tilde{q}_2} \tilde{q}_2 = \begin{bmatrix} 1 \\ 3 \\ 5 \\ 7 \end{bmatrix} - \frac{4}{4} \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} - \frac{16}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 2 \\ 2 \end{bmatrix} \Rightarrow q_3 = \frac{1}{4} \begin{bmatrix} -2 \\ -2 \\ 2 \\ 2 \end{bmatrix}$

$\Rightarrow Q = [q_1 \ q_2 \ q_3] = \begin{bmatrix} -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 \end{bmatrix}$

$R = \begin{bmatrix} q_1^T a_1 & q_1^T a_2 & q_1^T a_3 \\ 0 & q_2^T a_2 & q_2^T a_3 \\ 0 & 0 & q_3^T a_3 \end{bmatrix} \Rightarrow R = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 2 & 8 \\ 0 & 0 & 4 \end{bmatrix}$

$\Rightarrow A = QR$

$$\left.\begin{array}{l}\hat{y} = XB, \; X \in R^{n\times k} \\ B \in R^{k}, \; \hat{y} \in R^{n}\end{array}\right\} \Rightarrow B = (X^T X)^{-1} X^T y$$ (4

suppose we have QR decomposition of X : $X = QR$ $\Rightarrow$ we can simplify

B : $B = (X^T X)^{-1} X^T y = (R^T \underbrace{Q^T Q}_{I} R)^{-1} R^T Q^T y = (R^T R)^{-1} R^T Q^T y \xlongequal{R \text{ is non-singular}} R^{-1} \underbrace{(R^T)^{-1} R^T}_{I} Q^T y$

$\Rightarrow B = R^{-1} Q^T y$

As we can see here we calculate $R^{-1} Q^T$ and calculating inverse of R: (R^{-1}) is computationally easier than inverse of $X^T X$: $(X^T X)^{-1}$ because R is upper triangular. So we conclude that QR decomposition can reduce our computational complexity in linear regression.

$Q_i \rightarrow$ orthogonal $\Rightarrow A = A_p \cdots A_2 A_1 = Q_p (Q_p^T A_p Q_{p-1}) \cdots (Q_2^T A_2 Q_1)(Q_1^T A_1 \underbrace{Q_0}_{I})$ (5

$\hookrightarrow$ Result of QR decomposition of $A_i Q_{i-1} \Rightarrow A_i Q_{i-1} = Q_i R_i \Rightarrow Q_i^T A_i Q_{i-1} = R_i$

$\Rightarrow A = A_p \cdots A_2 A_1 = Q_p (Q_p^T A_p Q_{p-1}) \cdots \underbrace{(Q_2^T A_2 Q_1)}_{R_2} \underbrace{(Q_1^T A_1 Q_0)}_{R_1} = \underbrace{Q_p}_{\text{orthogonal}} \underbrace{R_p R_{p-1} \cdots R_2 R_1}_{R \rightarrow \text{Upper triangular}}$

$\Rightarrow A = Q_p R$

$A \in R^{n\times n}$ $\quad a_i = A(:, i)$

$|\det(A)| \leqslant \|a_1\|_2 \cdots \|a_n\|_2$

$A = QR \Rightarrow \det(A) = \det(QR) = \det(Q)\det(R) \xrightarrow[\det(Q) = \pm 1]{Q \text{ is orthogonal}} |\det(A)| = |\det(R)|$

R is upper triangular $\Rightarrow |\det(A)| = |\det(R)| = \left|\prod_{i=1}^{n} r_{ii}\right| = \prod_{i=1}^{n} |r_{ii}|$ (A)

نظیر : $a_i = Q r_i \xrightarrow{Q \text{ is orthogonal}} \|a_i\|_2 = \|r_i\|_2 = \sqrt{\sum_{j=1}^{n} r_{ij}^2}$

$r_{ii}^2 \leqslant r_{i1}^2 + r_{i2}^2 + \dots + r_{in}^2 \Rightarrow \sqrt{r_{ii}^2} \leqslant \sqrt{r_{i1}^2 + \dots + r_{in}^2} \Rightarrow |r_{ii}| \leqslant \sqrt{\sum_{j=1}^{n} r_{ij}^2}$

$\Rightarrow |r_{ii}| \leqslant \|a_i\|_2$ (B)

$A, B \Rightarrow |\det(A)| = \prod_{i=1}^{n} |r_{ii}| \leqslant \prod_{i=1}^{n} \|a_i\|_2 \Rightarrow |\det(A)| \leqslant \|a_1\|_2 \cdots \|a_n\|_2$

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7) الف) $A = \mathrm{diag}(A_1, \dots, A_k)$ $\quad A \in R^{m_i \times m_i}$

1- استقرا : $A^1 = [A_1]$ $\quad A^2 = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & A_2 \end{bmatrix}$

$A^3 = \begin{bmatrix} A_1 & & 0 \\ & A_2 & \\ 0 & & A_3 \end{bmatrix} = \begin{bmatrix} A^2 & 0 \\ 0 & A_3 \end{bmatrix} = \begin{bmatrix} A^2 & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & A_3 \end{bmatrix} =$

$\rightarrow$ we assume the decomposition is available for A^k :

$$A^k = \begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_k \end{bmatrix} = \begin{bmatrix} A_1 & & & 0 \\ & I & & \\ & & \ddots & \\ 0 & & & I \end{bmatrix}\begin{bmatrix} I & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & I \end{bmatrix} \cdots \begin{bmatrix} I & & & 0 \\ & \ddots & & \\ & & I & \\ 0 & & & A_k \end{bmatrix}$$

$\Rightarrow$ for A^{k+1} we have :

$$A^{k+1} = \begin{bmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_{k+1} \end{bmatrix} = \begin{bmatrix} A^{(k)} & 0 \\ 0 & A_{k+1} \end{bmatrix} = \begin{bmatrix} A^{(k)} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & A_{k+1} \end{bmatrix} = \begin{bmatrix} A_1 & & & 0 \\ & I & & \\ & & \ddots & \\ 0 & & & I \end{bmatrix}\begin{bmatrix} I & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & I \end{bmatrix} \cdots \begin{bmatrix} I & & & 0 \\ & \ddots & & \\ & & I & \\ 0 & & & A_{k+1} \end{bmatrix}$$

2- ★ det of a 2x2 block matrix $\Rightarrow \det\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \det(A) \times \det(D - CA^{-1}B)$

$= \det(D) \times \det(A - BD^{-1}C)$

$$\det\left(\begin{bmatrix} I & & & 0 \\ & \ddots & & \\ & & A_i & \\ 0 & & & \ddots \; I \end{bmatrix}\right) = \det\left(\begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix}\right) \overset{\star}{=} \det(I) \times \det(X - 0 I 0) = \det(X)$$

$$\det(X) = \det\left(\begin{bmatrix} I & 0 \\ 0 & A_i \end{bmatrix}\right) = \det\left(\begin{bmatrix} I & 0 \\ 0 & A_i \end{bmatrix}\right) \overset{\star}{=} \det(I) \times \det(A_i - 0 I 0) = \det(A_i)$$

3-

$$A=\begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_n \end{bmatrix} \overset{\text{Part 1}}{=} \overbrace{\begin{bmatrix} A_1 & & 0 \\ & I & \\ 0 & & I \end{bmatrix}}^{B_1} \overbrace{\begin{bmatrix} I & & \\ & A_2 & 0 \\ & & I \\ 0 & & & I \end{bmatrix}}^{B_2} \cdots \overbrace{\begin{bmatrix} I & & \\ & I & \\ & & A_n \end{bmatrix}}^{B_3} \Rightarrow (A):\ \det(A)=\prod_{i=1}^{n}\det(B_i)$$

Part 2 $\Rightarrow$ (B): $\det(B_i)=\det(A_i)$

A, B $\Rightarrow$ $\det(A)=\prod_{i=1}^{n}\det(B_i)=\prod_{i=1}^{n}\det(A_i)$

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7) ب)

1- A is non-singular: Gaussian elimination $\Rightarrow$

$$M=\begin{bmatrix} A & B \\ C & D \end{bmatrix} \rightsquigarrow E_1=\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} \Rightarrow E_1M=\begin{bmatrix} A & B \\ 0 & D-CA^{-1} \end{bmatrix} \rightsquigarrow E_2=\begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix}$$

$$\Rightarrow E_1ME_2=\Lambda=\begin{bmatrix} A & 0 \\ 0 & D-CA^{-1}B \end{bmatrix} \Rightarrow M=E_1^{-1}\Lambda E_2^{-1}$$

$$\Rightarrow \begin{bmatrix} A & B \\ C & D \end{bmatrix}=\begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & D-CA^{-1}B \end{bmatrix}\begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix}$$

B is non-singular: Gaussian elimination $\Rightarrow$

$$M=\begin{bmatrix} A & B \\ C & D \end{bmatrix} \rightsquigarrow E_1=\begin{bmatrix} I & -BD^{-1} \\ 0 & I \end{bmatrix} \Rightarrow ME_1=\begin{bmatrix} A-BD^{-1}C & B \\ 0 & D \end{bmatrix} \rightsquigarrow E_2=\begin{bmatrix} I & 0 \\ -D^{-1}C & I \end{bmatrix}$$

$$\Rightarrow E_1ME_2=\Lambda=\begin{bmatrix} A-BD^{-1}C & 0 \\ 0 & D \end{bmatrix} \Rightarrow M=E_1^{-1}\Lambda E_2^{-1}$$

$$\Rightarrow \begin{bmatrix} A & B \\ C & D \end{bmatrix}=\begin{bmatrix} I & BD^{-1} \\ 0 & I \end{bmatrix}\begin{bmatrix} A-BD^{-1}C & 0 \\ 0 & D \end{bmatrix}\begin{bmatrix} I & 0 \\ D^{-1}C & I \end{bmatrix}$$

2- part 1: A & D non-singular $\Rightarrow$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}=\begin{bmatrix} I & BD^{-1} \\ 0 & I \end{bmatrix}\begin{bmatrix} A-BD^{-1}C & 0 \\ 0 & D \end{bmatrix}\begin{bmatrix} I & 0 \\ D^{-1}C & I \end{bmatrix}=\begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & D-CA^{-1}B \end{bmatrix}\begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix}$$

$$\Rightarrow \det\begin{bmatrix} A & B \\ C & D \end{bmatrix}=\overbrace{\det\begin{bmatrix} I & BD^{-1} \\ 0 & I \end{bmatrix}}^{1}\det\begin{bmatrix} A-BD^{-1}C & 0 \\ 0 & D \end{bmatrix}\overbrace{\det\begin{bmatrix} I & 0 \\ D^{-1}C & I \end{bmatrix}}^{1}=\det(D)\times\det(A-BD^{-1}C)$$

$$\det\begin{bmatrix} A & B \\ C & D \end{bmatrix}=\overbrace{\det\begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix}}^{1}\det\begin{bmatrix} A & 0 \\ 0 & D-CA^{-1}B \end{bmatrix}\overbrace{\det\begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix}}^{1}=\det(A)\times\det(D-CA^{-1}B)$$

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7) ج) 1-

$$\det(I_m+AB) = \det(A^{-1})\det(I_m+AB)\det(A) = \det\left(A^{-1}(I_m+AB)A\right) = \det(I_m+BA)$$

2- $\det(X+AB) = \det\left(X(I+X^{-1}AB)\right) = \det(X)\det(B^{-1})\det(I+X^{-1}AB)\det(B)$

$= \det(X)\det\left(B(I+X^{-1}AB)B^{-1}\right) = \det(X)\det(I+BX^{-1}A)$

3- we showed $\Rightarrow$ $\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(A)\det(D-CA^{-1}B) = \det(D)\det(A-BD^{-1}C)$

we define $\Rightarrow$ $M = \begin{bmatrix} X_{m\times m} & u_{m\times m} \\ -v^T_{1\times m} & 1 \end{bmatrix} \Rightarrow \det(M) = \det(X)\det\overbrace{(1+v^TX^{-1}u)}^{num} = \overbrace{\det(1)}^{1}\det(X+uv^T)$

$\Rightarrow (1+v^TX^{-1}u)\det(X) = \det(X+uv^T)$
