نیکو مرادی - 400101934 - تمرین سری 1 TD

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1)

$$\left.\begin{matrix} A\in R^{m\times p} \\ B\in R^{p\times n}\end{matrix}\right\} AB = C\in R^{m\times n} \rightarrow c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

نامساوی کوشی شوارتز

$$* \|AB\|_F^2 = \|C\|_F^2 = \sum_{i=1}^{m}\sum_{j=1}^{n} |c_{ij}|^2 = \sum_{i=1}^{m}\sum_{j=1}^{n} \left|\sum_{k=1}^{p} a_{ik} b_{kj}\right|^2 \leq \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} |a_{ik}|^2 |b_{kj}|^2$$

$$= \left(\sum_{i=1}^{m}\sum_{k=1}^{p} |a_{ik}|^2\right)\left(\sum_{j=1}^{n}\sum_{k=1}^{p} |b_{kj}|^2\right) = \|A\|_F^2 \|B\|_F^2 \Rightarrow \|AB\|_F^2 \leq \|A\|_F^2 \|B\|_F^2$$

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2)

$$A = [a_1 | a_2 | \dots | a_n] \qquad \|A\|_1 = \max_{1\leq j\leq n} \|a\|_1$$

$$\left.\begin{matrix} \|A\|_1 = \sup \dfrac{\|Ax\|_1}{\|x\|_1} = \max\limits_{\|x\|_1=1} \|Ax\|_1 \\ \|a_j\|_1 = \sum\limits_{i=1}^{m} |a_{ij}| \\ \|Ax\|_1 = \left\|\sum\limits_{j=1}^{n} a_j x_j\right\|_1 = \sum\limits_{i=1}^{m}\left|\sum\limits_{j=1}^{n} a_{ij} x_j\right| \end{matrix}\right\} \rightarrow \max \sum_{i=1}^{m}\left|\sum_{j=1}^{n} a_{ij} x_j\right| = \max_{1\leq j\leq n} \sum_{i=1}^{m} |a_{ij}|$$

$$\Rightarrow \sum_{i=1}^{m}\left|\sum_{j=1}^{n} a_{ij} x_j\right| < \sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}||x_j| = \sum_{j=1}^{n} |x_j|\left(\sum_{i=1}^{m} |a_{ij}|\right) \leq \underbrace{\sum_{j=1}^{n} |x_j|}_{\|x\|_1=1} \max_j \underbrace{\left(\sum_{i=1}^{m} |a_{ij}|\right)}_{\|a\|_1}$$

$$\Rightarrow \max \sum_{i=1}^{m}\left|\sum_{j=1}^{m} a_{ij} x_j\right| = \max_j \|a\|_1 \Rightarrow \|A\|_1 = \max_j \|a\|_1$$

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3)

$A^2 = A \rightarrow$

الف) $|A| = |A^2| = |A|^2 \Rightarrow |A|^2 - |A| = 0 \Rightarrow |A|(|A|-1) = 0$

$\Rightarrow |A| = 0$ یا 1

ب) $B = I - A \rightarrow B^2 = I + A^2 - 2A \xrightarrow{A = A^2} B^2 = I + A - 2A$

$\rightarrow B^2 = B = I - A \Rightarrow$ ماتریس $B = I - A$ هم توان است.

$A \in R^{n\times n}$, $B \in R^{m\times m}$ $\qquad A \otimes B = ?$ 14

$A\underline{x} = \lambda \underline{x}$, $B\underline{y} = \mu \underline{y}$

$$(A\otimes B)(\underline{x}\otimes \underline{y}) = (A\underline{x})\otimes(B\underline{y}) = (\lambda \underline{x})\otimes(\mu \underline{y}) = \lambda\mu(\underline{x}\otimes \underline{y})$$

$X\in R^{m\times n}$ $\qquad P(m,n)\in R^{mn\times mn}$ $\qquad P(m,n)=P(n,m)^T$ 15

$$P(m,n)=\sum_{i=1}^{m}\sum_{j=1}^{n} (\overbrace{E_{ij}}^{m\times n}\otimes \underbrace{E_{ij}^T}_{n\times m})$$

$$n\times m \leftarrow E'_{ij} = E_{ji}^T \rightarrow m\times n$$

$$P(n,m)=\sum_{i=1}^{n}\sum_{j=1}^{m} (\underbrace{E'_{ij}}_{n\times m}\otimes \underbrace{E'^T_{ij}}_{m\times n}) \xrightarrow{T} P(n,m)^T=\sum_{i=1}^{n}\sum_{j=1}^{m} (\underbrace{E'^T_{ij}}_{m\times n}\otimes \underbrace{E'_{ij}}_{n\times m})$$

$$\xrightarrow{i=j',\ j=i'} P(n,m)^T=\sum_{i'=1}^{m}\sum_{j'=1}^{n} (\overbrace{E'^T_{j'i'}}^{m\times n}\otimes \overbrace{E'_{j'i'}}^{n\times m}) = \sum_{i'=1}^{m}\sum_{j'=1}^{n} (\underbrace{E_{i'j'}}_{m\times n}\otimes \underbrace{E^T_{i'j'}}_{n\times m}) = P(m,n)$$

$$P(m,n)=P(n,m)^{-1} \Leftrightarrow P(m,n)P(n,m)=I$$

$$\underbrace{E_{ij}}_{m\times n}\underbrace{E'_{lm}}_{n\times m} = \begin{cases} 0 & j\neq l \\ E_{ij}E'_{jm} & j=l\end{cases} \Rightarrow \sum_{l} E_{ij}E'_{lm} = E_{ij}E'_{jm} \Rightarrow \sum_{l} E'^T_{lm}E^T_{ij} = E'^T_{jm}E^T_{ij}$$

$$\Rightarrow P(m,n)P(n,m)=\left[\sum_{i=1}^{m}\sum_{j=1}^{n}(E_{ij}\otimes E_{ij}^T)\right]\left[\sum_{a=1}^{n}\sum_{b=1}^{m}(E'_{ab}\otimes E'^T_{ab})\right]$$

$$=\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=1}^{n}\sum_{b=1}^{m}(E_{ij}\otimes E_{ij}^T)(E'_{ab}\otimes E'^T_{ab}) = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=1}^{n}\sum_{b=1}^{m}(E_{ij}E'_{ab}\otimes E_{ij}^T E'^T_{ab}) \xrightarrow{j=a}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{b=1}^{m}(E_{ij}E'_{jb}\otimes E_{ij}^T E'^T_{jb}) \xrightarrow{b=i} \sum_{i=1}^{m}\sum_{j=1}^{n}(E_{ij}E'_{ji}\otimes E_{ij}^T E'^T_{ji}) = \sum_{i=1}^{m}\sum_{j=1}^{n} 1_{ii}\otimes 1_{jj}$$

$$=\left(\sum_{i=1}^{m} 1_{ii}\right)\otimes\left(\sum_{j=1}^{n} 1_{jj}\right) = I_{m\times m}\otimes I_{n\times n} = I_{mn\times mn} \Rightarrow P(m,n)=P(n,m)^{-1}$$

پس می‌توانیم نتیجه بگیریم P ماتریس جایگشت است.

$$\frac{1}{p}+\frac{1}{q}=1 \qquad p,q\in[1,\infty) \qquad u,v\in\mathbb{R}^n$$

$$|u^Tv|\leqslant \|u\|_p\,\|v\|_q \qquad \|u\|_p=\left(|u_1|^p+\dots+|u_n|^p\right)^{1/p}$$

$$\|v\|_q=\left(|v_1|^q+\dots+|v_n|^q\right)^{1/q}$$

نامساوی یانگ: $\forall x,y\in\mathbb{R} \quad \frac{1}{q}+\frac{1}{p}=1 \Rightarrow |xy|\leqslant \frac{x^p}{p}+\frac{y^q}{q}$

$$\rightarrow x=\frac{u_i}{\|u\|_p}\,,\quad y=\frac{v_i}{\|v\|_q} \Rightarrow$$

$$\left|\frac{u_i\cdot v_i}{\|u\|_p\,\|v\|_q}\right|\leqslant \frac{1}{p}\,\frac{u_i^p}{\|u\|_p^p}+\frac{1}{q}\,\frac{v_i^q}{\|v\|_q^q} \xrightarrow{\sum_{i=1}^{n}}$$

$$\sum_{i=1}^{n}\left|\frac{u_i\cdot v_i}{\|u\|_p\,\|v\|_q}\right|\leqslant \frac{1}{p}\underbrace{\frac{\sum u_i^p}{\|u\|_p^p}}_{1}+\frac{1}{q}\underbrace{\frac{\sum v_i^q}{\|v\|_q^q}}_{1}=\frac{1}{p}+\frac{1}{q}=1 \Rightarrow$$

$$\frac{\sum_{i=1}^{n}|u_i\cdot v_i|}{\|u\|_p\,\|v\|_q}\leqslant 1 \Rightarrow \underbrace{\left|\sum_{i=1}^{n}u_i\cdot v_i\right|}_{|u^Tv|}\leqslant \sum_{i=1}^{n}|u_i\cdot v_i|\leqslant \|u\|_p\,\|v\|_q$$

$$\Rightarrow \boxed{|u^Tv|\leqslant \|u\|_p\,\|v\|_q}$$

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17 الف)

$$\left.\begin{array}{l} PD:\ \forall x\in\mathbb{R}^n,\ x\neq 0 \quad x^TAx>0 \\ Ax=\lambda x \end{array}\right\} \Rightarrow x^T\lambda x>0 \Rightarrow \lambda|x|^2>0 \Rightarrow \boxed{\lambda>0}$$

مقادیر ویژه A

ب)

$$\left.\begin{array}{l} A\to PD \Rightarrow \lambda_i>0 \quad (\lambda_i:\ \text{مقادیر ویژه } A) \\ B\to PD \Rightarrow \mu_j>0 \quad (\mu_j:\ \text{مقادیر ویژه } B) \\ C=A\otimes B \to \lambda_i\mu_j:\ \text{مقادیر ویژه } C \end{array}\right\} \lambda_i\mu_j>0 \Rightarrow \boxed{C \text{ is PD}}$$

$$\Uparrow$$

$$0<\min_{i,j}\lambda_i\mu_j\leqslant\frac{x^TCx}{x^Tx}\Rightarrow x^TCx\geqslant 0:\ \forall x\in\mathbb{R}^{nm}$$

الف)

$A \in R^{n\times n} \quad , B \in R^{m\times m}$

$A \otimes B = AI \otimes IB = (A\otimes I)(I \otimes B) \Rightarrow$

$\det(A\otimes B) = \det(A\otimes I)\det(I\otimes B)$

$$\left.\begin{array}{l} I \otimes B = \underbrace{\begin{bmatrix} B_{m\times m} & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & & \ddots \end{bmatrix}}_{mn \times mn} \Big\}_{m} \Rightarrow \det(I\otimes B) = \det(B)^m \\ \\ A \otimes I = I \otimes A \Rightarrow \det(A\otimes I) = \det(A)^n \end{array}\right\} \det(A\otimes B) = \det(A)^n \det(B)^m$$

راه دوم : $A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & & \vdots \\ a_{m1}B & \cdots & a_{mm}B \end{bmatrix} \Rightarrow \det(A\otimes B) = \prod_{i=1}^{m} (a_{ii}^n \det(B))$

$= \det(B)^m \prod_{i=1}^{m} a_{ii}^n = \det(B)^m \det(a)^n$

ب) $P \in R^{n\times n}$ ماتریس جایگشت ← $\det(P\otimes P) \overset{\text{الف}}{=} \det(P)^{2n}$

می‌دانیم : $\det(P) = \pm 1$ ← بستگی به تعداد جایگشت‌ها دارد

$\Rightarrow \det(P \times P) = \det(P)^{2n} = (\pm 1)^{2n} = 1^n = 1$

ج) $A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & & \vdots \\ a_{m1}B & \cdots & a_{mm}B \end{bmatrix} \qquad A \in R^{m\times m} \quad , B \in R^{n\times n}$

element (p,q) in $A \otimes B$ $a_{ij}b_{kl}$: $\begin{cases} p : (i-1)n+k \\ q : (j-1)m+l \end{cases} \Rightarrow$

diagonal elements of $A\otimes B$ $p=q \Rightarrow i=j, k=l \Rightarrow a_{ii}b_{kk}$

$tr(A\otimes B) = \sum_{i=1}^{m}\sum_{k=1}^{n} a_{ii}b_{kk} = (\sum_{i=1}^{m} a_{ii})(\sum_{k=1}^{n} b_{kk}) = tr(A).tr(b)$

$A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times n}$ (د

$$\underbrace{(vec(A))^T}_{1\times mn}\underbrace{(vec(B))}_{mn\times 1} = (a_{11}, \dots, a_{m1}, \dots, a_{mn})\begin{bmatrix} b_{11} \\ \vdots \\ b_{m1} \\ \vdots \\ b_{mn}\end{bmatrix} = \boxed{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\, b_{ij}} \; : (1)$$

element (i,j) of $\underset{n\times m}{A^T}\,\underset{m\times n}{B}$: (ith row of A^T).(jth column of B)

$$= (\text{ith column of } A).(\text{jth column of } B) = \begin{bmatrix} a_{1i} \\ \vdots \\ a_{mi}\end{bmatrix} \cdot \begin{bmatrix} b_{1j} \\ \vdots \\ b_{mj}\end{bmatrix} = \sum_{k=1}^{m} a_{ki}\, b_{kj}$$

diagonal elements of $A^T B$: $i=j$: $\sum_{k=1}^{m} a_{ki}\, b_{ki}$

$$\underset{n\times m}{A^T}\,\underset{m\times n}{B} = C \in \mathbb{R}^{n\times n} \; : \; tr(A^T B) = \boxed{\sum_{i=1}^{n}\sum_{k=1}^{m} a_{ki}\, b_{ki}} \; : (2)$$

(1), (2) $\Rightarrow$ $\boxed{tr(A^T B) = vec(A)^T vec(B)}$

$A \in \mathbb{R}^{n\times n}$, $A^a = 0$ (الف 19

$B \in \mathbb{R}^{n\times n}$, $B^b = 0$

$$(A\otimes B)^2 = (A\otimes B)(A\otimes B) = A^2 \otimes B^2 \rightarrow (A\otimes B)^3 = (A^2\otimes B^2)(A\times B) = A^3 \otimes B^3$$

$$\Rightarrow (A\otimes B)^n = A^n \otimes B^n \xrightarrow{n=\min(a,b)} 0 \otimes B^n \text{ or } A^n \otimes 0 = 0$$

$\Rightarrow \exists k \in \mathbb{N}$, $k = \min(a,b) \Rightarrow (A\otimes B)^k = 0 \Rightarrow$ $\boxed{A\otimes B \text{ is nilpotent}}$

$A^n = A$, $B^n = B$ (ب

$(A\otimes B)^n = A^n \otimes B^n = A\otimes B \Rightarrow$ $\boxed{A\otimes B \text{ is idempotent}}$

$A \in R^{m\times n}$ $B \in R^{p\times n}$ $*(A \odot B)^T(A \odot B) =$

10 ام سوال

$$A \odot B = (a_1 \otimes b_1, \dots, a_n \otimes b_n) = \begin{bmatrix} a_{11}\underline{b}_1 & a_{12}\underline{b}_2 & \dots & a_{1n}\underline{b}_n \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}\underline{b}_1 & a_{m2}\underline{b}_2 & \dots & a_{mn}\underline{b}_n \end{bmatrix}$$

$$\Rightarrow (A \odot B)^T (A \odot B) = \begin{bmatrix} a_{11}\underline{b}_1^T & \dots & a_{m1}\underline{b}_1^T \\ a_{12}\underline{b}_2^T & \ddots & \vdots \\ \vdots & & \\ a_{1n}\underline{b}_n^T & \dots & a_{mn}\underline{b}_n^T \end{bmatrix} \begin{bmatrix} a_{11}\underline{b}_1 & a_{12}\underline{b}_2 & \dots & a_{1n}\underline{b}_n \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}\underline{b}_1 & a_{m2}\underline{b}_2 & \dots & a_{mn}\underline{b}_n \end{bmatrix} =$$

$$\begin{bmatrix} \sum_{i=1}^m a_{i1}^2 |\underline{b}_1|^2 & \sum_{i=1}^m a_{i1}a_{i2}\underline{b}_1^T\underline{b}_2 & \dots & \sum_{i=1}^m a_{i1}a_{in}\underline{b}_1^T\underline{b}_n \\ \vdots & & \ddots & \vdots \\ \sum_{i=1}^m a_{in}a_{i1}\underline{b}_n^T\underline{b}_1 & & \dots & \sum_{i=1}^m a_{in}^2 |\underline{b}_n|^2 \end{bmatrix}$$

the element in the (q,v) position: $\sum_{i=1}^m a_{iq}a_{iv}\underline{b}_q^T\underline{b}_v = \sum_{j=1}^p \sum_{i=1}^m a_{iq}a_{iv}b_{jq}b_{jv}$

$$* \; A^TA \,.*\, B^TB = \begin{bmatrix} a_{11} & \dots & a_{m1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} .* \begin{bmatrix} b_{11} & \dots & b_{p1} \\ \vdots & \ddots & \vdots \\ b_{1n} & \dots & b_{pn} \end{bmatrix} \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{p1} & \dots & b_{pn} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=1}^m a_{i1}^2 & \dots & \sum_{i=1}^m a_{i1}a_{in} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^m a_{in}a_{i1} & \dots & \sum_{i=1}^m a_{in}^2 \end{bmatrix} .* \begin{bmatrix} \sum_{j=1}^p b_{j1}^2 & \dots & \sum_{j=1}^p b_{j1}b_{jn} \\ \vdots & \ddots & \vdots \\ \sum_{j=1}^p b_{jn}b_{j1} & \dots & \sum_{j=1}^p b_{jn}^2 \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{j=1}^p\sum_{i=1}^m a_{i1}^2 b_{j1}^2 & \dots & \sum_{j=1}^p\sum_{i=1}^m a_{i1}a_{in}b_{j1}b_{jn} \\ \vdots & \ddots & \vdots \\ & \dots & \sum_{j=1}^p\sum_{i=1}^m a_{in}^2 b_{jn}^2 \end{bmatrix}$$

$\rightarrow$ The element in (q,v) position: $\sum_{i=1}^m \sum_{j=1}^p a_{iq}a_{iv}b_{jq}b_{jv}$

As we can see the element in the (q,v) position of $A^TA .* B^TB$ is the same as the element in the (q,v) position of $(A\odot B)^T(A \odot B)$ So we can conclude that: $(A\odot B)^T(A\odot B) = A^TA \,.*\, B^TB$

$(A\odot B)^+ = (A^TA .* B^TB)^{-1}(A\odot B)^T$ 16) ب)

$(X^TX)^{-1}X^T = X^+ \rightsquigarrow X = A\odot B \Rightarrow X^TX = (A\odot B)^T(A\odot B) \overset{\text{الف}}{=} A^TA .* B^TB$

$\Rightarrow (A\odot B)^+ = ((A\odot B)^T(A\odot B))^{-1}(A\odot B)^T = (A^TA .* B^TB)^{-1}(A\odot B)^T$

$A_k \in R^{N_k\times N_k}$ $\qquad \sum_{k=1}^{n} N_k = N$ 11)

الف) * $A = diag(A_1, \dots, A_n) \Rightarrow tr(A) = \sum_{i=1}^{N} a_{ii} = \sum_{i=1}^{N_1} a_{ii} + \sum_{i=N_1+1}^{N_2+N_1} a_{ii} + \dots + \sum_{i=\sum_{k=1}^{n-1} N_k+1}^{N} a_{ii}$

$= tr(A_1) + tr(A_2) + \dots + tr(A_n) = \sum_{k=1}^{n} tr(A_k)$

* $A = \begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_n \end{bmatrix} = \overbrace{\begin{bmatrix} A_1 & & 0 \\ & I & \\ & & \ddots \\ 0 & & I \end{bmatrix}}^{B_1} \overbrace{\begin{bmatrix} I & & & 0 \\ & A_2 & & \\ & & I & \\ 0 & & & \ddots I \end{bmatrix}}^{B_2} \cdots \overbrace{\begin{bmatrix} I & & \\ & I & \\ & & \ddots A_n \end{bmatrix}}^{B_3} \Rightarrow$ 1: $det(A) = \prod_{i=1}^{n} det(B_i)$

$B_1 = \begin{bmatrix} A_1 & \cdots & 0 \\ \vdots & I & 0 \\ 0 & 0 & \ddots I \end{bmatrix} \to B_1' = \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix}$

we can simplify B_1 into the form of B_1', Now we can calculate

$det(B_1') = det(B_1)$:

$det(B_1') = det(B_1) = det(A_1) \times det(I - 0\, A_1^{-1} 0) = det(A_1) \Rightarrow \quad det(A_1) = det(B_1)$

$det\left(\begin{bmatrix} I \ddots A_i & 0 \\ 0 & \ddots I \end{bmatrix}\right) = det\left(\begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix}\right) = det(X) = det\left(\begin{bmatrix} I & 0 \\ 0 & \ddots I\, A_i \end{bmatrix}\right) = det\left(\begin{bmatrix} I & 0 \\ 0 & A_i \end{bmatrix}\right) = det(A_i)$

$\Rightarrow$ 2: $det(B_i) = det(A_i)$

①, ② $\Rightarrow$ $det(A) = \prod_{i=1}^{n} det(B_i) = \prod_{i=1}^{n} det(A_i)$

(11

(ب

if A_k are non-singular $\Rightarrow A_k A_k^{-1} = I$: $A = \begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_n \end{bmatrix} \rightsquigarrow A^{-1} = \begin{bmatrix} A_1^{-1} & & 0 \\ & \ddots & \\ 0 & & A_n^{-1} \end{bmatrix}$

$$AA^{-1} = \begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_n \end{bmatrix} \begin{bmatrix} A_1^{-1} & & 0 \\ & \ddots & \\ 0 & & A_n^{-1} \end{bmatrix} = \begin{bmatrix} I & & 0 \\ & I & \\ 0 & & I \end{bmatrix} = I \Rightarrow$$ 1: A is non-singular

if A is non-singular $\Rightarrow AA^{-1} = I$ $\qquad A^{-1} = \begin{bmatrix} N_{11} & \cdots & N_{1n} \\ \vdots & \ddots & \vdots \\ N_{n1} & \cdots & N_{nn} \end{bmatrix}$

$$AA^{-1} = I \Rightarrow \begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_n \end{bmatrix} \begin{bmatrix} N_{11} & \cdots & N_{1n} \\ \vdots & & \vdots \\ N_{n1} & \cdots & N_{nn} \end{bmatrix} = \begin{cases} A_i N_{ii} = I \Rightarrow N_{ii} = A_i^{-1} \\ A_i N_{ij} = 0 \Rightarrow N_{ij} = 0 \end{cases}$$

$\Rightarrow A^{-1} = \begin{bmatrix} A_1^{-1} & & 0 \\ & \ddots & \\ 0 & & A_n^{-1} \end{bmatrix} \Rightarrow$ 2: A_k are also non-singular

1, 2 $\Rightarrow$ A is non-singular $\Leftrightarrow$ A_k are non-singular $(k = 1, n)$

$A \in \mathbb{R}^{m \times n}$ $\qquad A\underline{x} = 0$ $\qquad \forall \underline{x} \in null(A)$ $\qquad$ (12

$$\begin{bmatrix} - a_1^T - \\ \vdots \\ - a_n^T - \end{bmatrix} \underline{x} = \begin{bmatrix} - a_1^T \underline{x} - \\ \vdots \\ - a_n^T \underline{x} - \end{bmatrix} = 0 \Rightarrow \forall a_i : a_i^T \underline{x} = 0 \Rightarrow \forall a \in span(\{a_i\}) = ran(A^T)$$

$\Rightarrow \underline{a} \perp \underline{x} \Rightarrow null(A) \perp ran(A^T) \Rightarrow null(A) = ran(A^T)^{\perp}$

$$\left.\begin{aligned} dim(null(A)) &= dim(ran(A^T)^{\perp}) = n - dim(ran(A^T)) \\ rank(A) &= dim(ran(A)) = dim(ran(A^T)) \end{aligned}\right\} \boxed{dim(null(A)) + rank(A) = n}$$
