Αριθμητική Γραμμική Άλγεβρα
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ΘΕΜΑ 1ο

$$A=\begin{bmatrix} a-b-\gamma & 2a & 2a \\ 2b & b-\gamma-a & 2b \\ 2\gamma & 2\gamma & \gamma-a-b \end{bmatrix}$$

$$\det(A)=(a-b-\gamma)\det\begin{pmatrix} b-\gamma-a & 2b \\ 2\gamma & \gamma-a-b \end{pmatrix}-2a\det\begin{pmatrix} 2b & 2b \\ 2\gamma & \gamma-a-b \end{pmatrix}+2a\det\begin{pmatrix} 2b & b-\gamma-a \\ 2\gamma & 2\gamma \end{pmatrix}$$

$$\left.\begin{aligned} \det\begin{pmatrix} b-\gamma-a & 2b \\ 2\gamma & \gamma-a-b \end{pmatrix} &= -b^2-2b\gamma+a^2-\gamma^2 \\ \det\begin{pmatrix} 2b & 2b \\ 2\gamma & \gamma-a-b \end{pmatrix} &= -2b^2-2ab-2b\gamma \\ \det\begin{pmatrix} 2b & b-\gamma-a \\ 2\gamma & 2\gamma \end{pmatrix} &= 2b\gamma+2\gamma^2+2a\gamma \end{aligned}\right\}$$

$$=(a-b-\gamma)(-b^2-2b\gamma+a^2-\gamma^2)-2a(-2b^2-2ab-2b\gamma)+2a(2b\gamma+2\gamma^2+2a\gamma)$$

$$=a^3+3a^2b+3a^2\gamma+3ab^2+3a\gamma^2+6ab\gamma+b^3+\gamma^3+3b\gamma^2+3b^2\gamma$$

$$=a^3+3a^2b+3a^2\gamma+3a\gamma^2+6ab\gamma+b^3+\gamma^3+3b\gamma^2+3b^2\gamma.$$

ΘΕΜΑ 2ο

$$\underbrace{\begin{bmatrix} a & 2 & 1 \\ 1 & 1 & a \\ 1 & 2 & 1 \end{bmatrix}}_{A_{3\times3}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X_{3\times1}} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{B_{3\times1}}$$

$$[A|B] \longrightarrow \left[\begin{array}{ccc|c} a & 2 & 1 & 1 \\ 1 & 1 & a & 0 \\ 1 & 2 & 1 & 1 \end{array}\right] \xrightarrow{r_1 = r_1/a} \left[\begin{array}{ccc|c} 1 & 2/a & 1/a & 1/a \\ 1 & 1 & a & 0 \\ 1 & 2 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{r_2 = r_2 + r_1 \cdot 1} \left[\begin{array}{ccc|c} 1 & 2/a & 1/a & 1/a \\ 0 & \frac{a-2}{a} & a-1/a & -1/a \\ 1 & 2 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{r_3 = r_3 - r_1} \left[\begin{array}{ccc|c} 1 & 2/a & 1/a & 1/a \\ 0 & \frac{a-2}{a} & a-1/a & -1/a \\ 0 & 2-\frac{2}{a} & \frac{a-1}{a} & \frac{a-1}{a} \end{array}\right] \xrightarrow{r_2 = \frac{a}{a-2} r_2} \left[\begin{array}{ccc|c} 1 & 2/a & \frac{1}{a} & \frac{1}{a} \\ 0 & 1 & \frac{a^2-1}{a-2} & -\frac{1}{a-2} \\ 0 & 2-\frac{2}{a} & \frac{a-1}{a} & \frac{a-1}{a} \end{array}\right]$$

$$\xrightarrow{r_1 = r_1 - \frac{2}{a} r_2} \left[\begin{array}{ccc|c} 1 & 0 & \frac{1-2a}{a-2} & \frac{1}{a-2} \\ 0 & 1 & \frac{a^2-1}{a-2} & -\frac{1}{a-2} \\ 0 & 2-\frac{2}{a} & \frac{a-1}{a} & \frac{a-1}{a} \end{array}\right]$$

$$\xrightarrow{r_3 = r_3 - \left(2-\frac{2}{a}\right) r_3} \left[\begin{array}{ccc|c} 1 & 0 & \frac{1-2a}{a-2} & \frac{1}{a-2} \\ 0 & 1 & \frac{a^2-1}{a-2} & -\frac{1}{a-2} \\ 0 & 0 & -\frac{(a-1)(2a-1)}{a-2} & \frac{a-1}{a-2} \end{array}\right]$$

$$\xrightarrow{r_3 = r_3 - \left(2 - \frac{2}{a}\right) r_2} \left[\begin{array}{ccc|c} 1 & 0 & \frac{1-2a}{a-2} & \frac{1}{a-2} \\ 0 & 1 & \frac{a^2-1}{a-2} & -\frac{1}{a-2} \\ 0 & 0 & -\frac{(a-1)(2a-1)}{a-2} & \frac{a-1}{a-2} \end{array}\right]$$

$$\xrightarrow{r_3 = -\frac{a-2}{(a-1)(2a-1)} r_3} \left[\begin{array}{ccc|c} 1 & 0 & \frac{1-2a}{a-2} & \frac{1}{a-2} \\ 0 & 1 & \frac{a^2-1}{a-2} & -\frac{1}{a-2} \\ 0 & 0 & 1 & -\frac{1}{2a-1} \end{array}\right]$$

$$\xrightarrow{r_1 = r_1 - \frac{1-2a}{a-2} r_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{a^2-1}{a-2} & \frac{a}{2a-1} \\ 0 & 0 & 1 & -\frac{1}{2a-1} \end{array}\right]$$

$$\xrightarrow{r_2 = r_2 - \frac{a^2-1}{a-2} R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{a}{2a-1} \\ 0 & 0 & 1 & -\frac{1}{2a-1} \end{array}\right]$$

$x_1 = 0$

$x_2 = \frac{a}{2a-1}$ $\qquad$ x_1, x_2, x_3 IR

$x_3 = -\frac{1}{2a-1}$

ΘΕΜΑ 4ο

$$A = \begin{bmatrix} 5 & -4 \\ 6 & -6 \end{bmatrix}$$

$$\begin{bmatrix} 5-\lambda & -4 \\ 6 & -\lambda-6 \end{bmatrix} = \lambda - 2 \quad \lambda + 3$$

$\boxed{\lambda_1 = 2}$ $\boxed{\lambda_2 = -3}$

$\lambda_1 = 2$

$$\begin{bmatrix} 5-\lambda & -4 \\ 6 & -\lambda-6 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 6 & -8 \end{bmatrix} \quad \left\{ \begin{bmatrix} 4/3 \\ 1 \end{bmatrix} \right\}$$

$\lambda_2 = -3$

$$\begin{bmatrix} 5-\lambda & -4 \\ 6 & -\lambda-6 \end{bmatrix} = \begin{bmatrix} 8 & -4 \\ 6 & -3 \end{bmatrix} \quad \left\{ \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \right\}$$

$$P = \begin{bmatrix} \frac{4}{3} & \frac{1}{2} \\ 1 & 1 \end{bmatrix}$$ Απο τη διαγωνοποιηση $\Delta = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$

Οι πινακες P και Δ έχουν τετοια σχέση ώστε να προκυπτει ο πινακας A

$$P \cdot \Delta \cdot P^{-1} = \begin{bmatrix} 5 & -4 \\ 6 & -6 \end{bmatrix}$$

$$A^{k} = A^{4726} = (A^{2363})^{2} = (A^{2})^{2363}$$

e)

$$\det(A) = 5 \cdot (-6) - 4 \cdot 6 = -54$$

ΘΕΜΑ 5ο

β) $P = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$

$\det(P) = -1 \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} - 1 \begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix} = 4$

$P^T = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 3 & 1 \end{pmatrix}$

$adj(P) = \begin{pmatrix} 1(2-3) & -(1-0) & +(3-0) \\ -(0-3) & +(-1-0) & -(-3-0) \\ -(0-2) & -(-1-1) & +(-2-0) \end{pmatrix}$

$= \begin{bmatrix} -1 & -1 & 3 \\ 3 & -1 & 3 \\ -2 & 2 & -2 \end{bmatrix}$

Άρα $P^{-1} = \frac{1}{4} \begin{pmatrix} -1 & -1 & 3 \\ 3 & -1 & 3 \\ -2 & 2 & -2 \end{pmatrix}$

Συνεπώς

$\Gamma = P \cdot \Delta \cdot P^{-1} = \frac{1}{4} \begin{pmatrix} -1 & 1 & 0 \\ 0 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} -1 & -1 & 3 \\ 3 & -1 & 3 \\ -2 & 2 & -2 \end{pmatrix}$

$= \frac{1}{4} \begin{pmatrix} 4 & 0 & 0 \\ -6 & 10 & -6 \\ 0 & 0 & 4 \end{pmatrix} = ($