The Curry–Howard Isomorphism

Introduction

- An example

$$\frac{A \to B \quad A}{B}$$

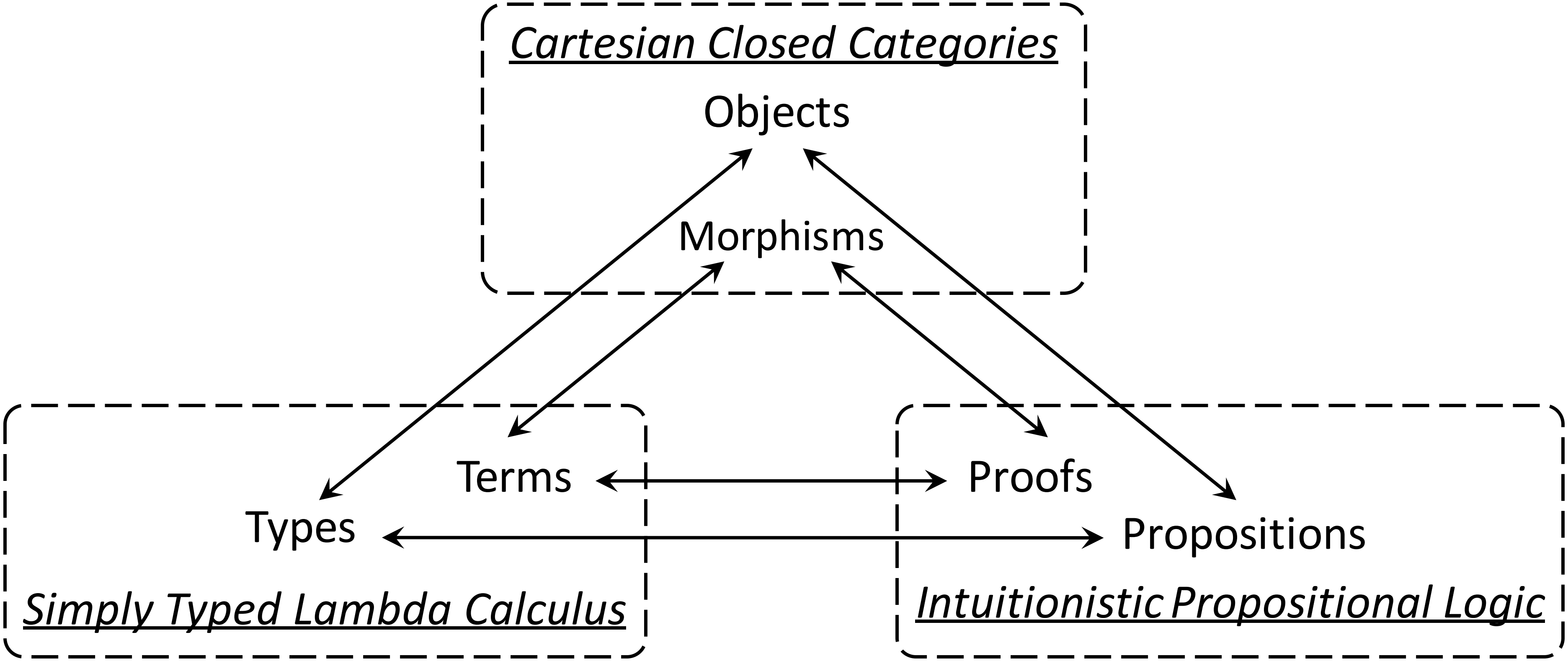
Modus ponens

$$\frac{f : A \to B \quad x : A}{f\,x : B}$$

Function application

- The Curry-Howard Isomorphism

A Three-way Correspondence



Intuitionistic Logic

- Syntactically, *reductio ad absurdum* (*RAA*) is not a rule in the natural deduction system of IL.
- The law of excluded middle $\varphi \vee \neg\varphi$ and double negation elimination $\neg\neg\varphi \to \varphi$ are no longer axioms.
- Semantically, the judgements about statements are based on the existence of a proof (or construction) of that statement

Simply Typed Lambda Calculus

- A family of prototype programming languages
- Simply typed lambda calculus with function types $\lambda^{\rightarrow}$
  - Types $\sigma \ ::= \ \iota \ | \ \sigma \rightarrow \sigma$
  - Terms $M \ ::= \ x \ | \ \lambda x.M \ | \ MM$
- Three kinds of equivalence
  - α-equivalence $\lambda x.M =_{\alpha} \lambda y.[y/x]M$
  - β-equivalence $(\lambda x.M)N =_{\beta} [N/x]M$
  - η-equivalence $\lambda x.Mx =_{\eta} M$

Cartesian Closed Categories

A *CCC* is a category with the following extra structure:

- A terminal object *unit* with unique arrow $\mathbf{One}^A$ for every A
- Object map $\times$, function $\langle\cdot,\cdot\rangle$, and arrows $\mathbf{Proj_1}$ and $\mathbf{Proj_2}$ satisfying $\mathbf{Proj}_i \circ \langle f_1, f_2 \rangle = f_i$ and $\langle \mathbf{Proj_1} \circ g, \mathbf{Proj_2} \circ g \rangle = g$
- Object map $\rightarrow$, function **Curry**, and arrow **App** satisfying $\mathbf{App} \circ \langle \mathbf{Curry}(h) \circ \mathbf{Proj_1}, \mathbf{Proj_2} \rangle = h$ and $\mathbf{Curry}(\mathbf{App} \circ \langle k \circ \mathbf{Proj_1}, \mathbf{Proj_2} \rangle) = k$

Encoding Proofs in Typed Lambda Terms

- Every proof leads to a type derivation
- Every type-derivation brings a well-typed lambda term
- Therefore, every proof can be encoded in a well-typed lambda term
  - assumptions are encoded in type context
  - proofs of theorems are encoded in closed terms

Encoding Proofs in Typed Lambda Terms

- Example – Theorem $\varphi \to (\psi \to \varphi)$

$$\dfrac{\dfrac{\dfrac{\overline{\varphi \vdash \varphi}}{\varphi,\ \psi \vdash \varphi}\,(add)}{\varphi \vdash \psi \to \varphi}\,(\to \mathrm{I})}{\vdash \varphi \to (\psi \to \varphi)}\,(\to \mathrm{I})$$

- The proof of theorem $\varphi \to (\psi \to \varphi)$ is encoded in combinator $\mathbf{K} \equiv \lambda x{:}\varphi.\lambda y{:}\psi.x : \varphi \to (\psi \to \varphi)$

Decoding Well-typed Lambda Terms

- Every well-typed lambda term uniquely determines a type derivation
- Every type derivation leads to a proof (by erasing terms)
- By decoding a well-typed lambda term, we obtain a proof
  - Rebuild the type derivation of the term
  - Erase all the terms in the derivation

Decoding Well-typed Lambda Terms

- Example – Combinator $\mathbf{K} \equiv \lambda x{:}\varphi.\,\lambda y{:}\psi.\,x : \varphi \to (\psi \to \varphi)$

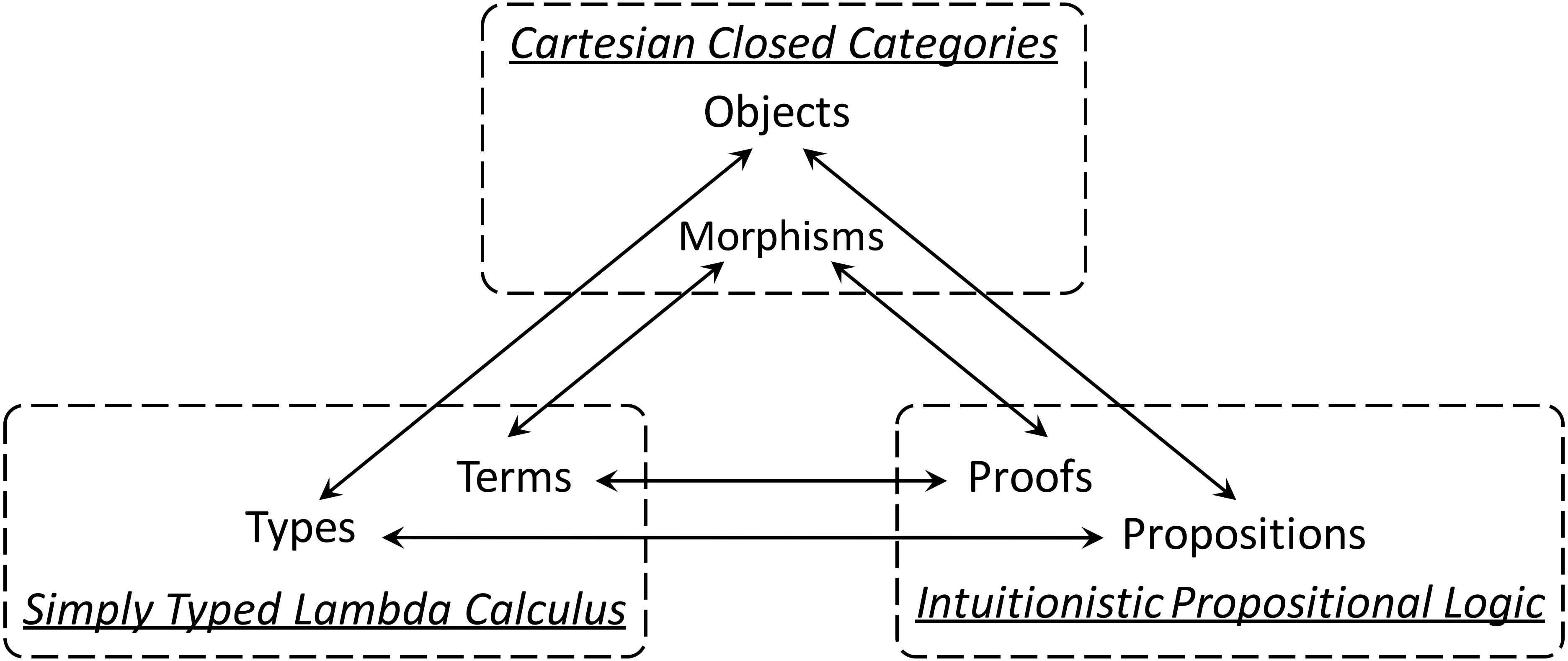
$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \vdash \varphi}}{\varphi,\ \psi \vdash \varphi}\,(add)}{\varphi \vdash \psi \to \varphi}\,(\to \mathrm{I})}{\vdash \varphi \to (\psi \to \varphi)}\,(\to \mathrm{I})
$$

- The proof of theorem $\varphi \to (\psi \to \varphi)$ is obtained by decoding the combinator $\mathbf{K} \equiv \lambda x{:}\varphi.\,\lambda y{:}\psi.\,x : \varphi \to (\psi \to \varphi)$

Some Correspondence

- Propositions correspond to types
  - propositional connectives ≌ type constructors
  - set of propositions ≌ set of simple types
- Proofs correspond to terms
  - natural deduction system ≌ typing rules
  - normalization ≌ reduction
- Some typical questions
  - proof checking ≌ type checking
  - provability ≌ inhabitation

A Three-way Correspondence



$\lambda^{unit,\times,\rightarrow}$ or $\lambda^{\rightarrow}$

- $\lambda^{unit,\times,\rightarrow}$ — CCCs

$\lambda^{unit,\times,\rightarrow}$	CCCs
terminal type $unit$	terminal object $unit$
product types $\sigma \times \tau$	products $A \times B$
function types $\sigma \rightarrow \tau$	exponentials $A \rightarrow B$

- $\lambda^{\rightarrow}$ is as expressive as $\lambda^{unit,\times,\rightarrow}$
  - product $\sigma_1 \times \ldots \times \sigma_n \cong$ sequence $\sigma_1, \ldots, \sigma_n$
  - function $f: \sigma \times \tau \rightarrow \rho \cong \bar{f}: \sigma \rightarrow (\tau \rightarrow \rho)$
  - function $g: \sigma \rightarrow (\tau \times \rho) \cong g_1: \sigma \rightarrow \tau$ and $g_2: \sigma \rightarrow \rho$

Interpreting Well-typed Terms as Morphisms

Given a cartesian closed category $\mathcal{C}$ with some object constants

- the interpretation of type expressions
  - $\mathcal{C}[\![\iota]\!] = \hat{b}$
  - $\mathcal{C}[\![\sigma \to \tau]\!] = \mathcal{C}[\![\sigma]\!] \to \mathcal{C}[\![\tau]\!]$
- the interpretation of type contexts
  - $\mathcal{C}[\![\emptyset]\!] = unit$
  - $\mathcal{C}[\![\Gamma, x{:}\,\sigma]\!] = \mathcal{C}[\![\Gamma]\!] \times \mathcal{C}[\![\sigma]\!]$

Interpreting Well-typed Terms as Morphisms

- the interpretation of well-typed terms

Given $\Gamma \rhd M : \sigma$, $\mathcal{C}[\![\Gamma \rhd M : \sigma]\!] : \mathcal{C}[\![\Gamma]\!] \to \mathcal{C}[\![\sigma]\!]$

- $\mathcal{C}[\![\Gamma, x : \sigma \rhd x : \sigma]\!] = \mathrm{Proj}_2^{\mathcal{C}[\![\Gamma]\!], \mathcal{C}[\![\sigma]\!]}$
- $\mathcal{C}[\![\Gamma \rhd MN : \tau]\!] =$
$$\mathrm{App}^{\mathcal{C}[\![\sigma]\!], \mathcal{C}[\![\tau]\!]} \circ \langle \mathcal{C}[\![\Gamma \rhd M : \sigma \to \tau]\!], \mathcal{C}[\![\Gamma \rhd N : \sigma]\!] \rangle$$
- $\mathcal{C}[\![\Gamma \rhd \lambda x : \sigma . M : \sigma \to \tau]\!] = \mathrm{Curry}(\mathcal{C}[\![\Gamma, x : \sigma \rhd M : \tau]\!])$
- $\mathcal{C}[\![\Gamma_1, x : \sigma, \Gamma_2 \rhd M : \tau]\!] = \mathcal{C}[\![\Gamma_1, \Gamma_2 \rhd M : \tau]\!] \circ \chi_f^{[\![\Gamma_1, x : \sigma, \Gamma_2]\!]}$

Soundness

Theorem (Soundness) Given any well typed terms M and N,

if $\Gamma \rhd M \equiv_{\alpha\beta\eta} N : \sigma$, then $[\![\Gamma \rhd M : \sigma]\!] = [\![\Gamma \rhd N : \sigma]\!]$ in every *CCC*.

Soundness (cont.)

Proof

- α-equivalence:

  no term-variable name appears in the calculations

- β-equivalence: $[\![\Gamma \rhd (\lambda x{:}\sigma.\, M)N : \tau]\!] = [\![\Gamma \rhd [N/x]M : \tau]\!]$
  - $\mathbf{App} \circ \langle \mathbf{Curry}(h) \circ \mathbf{Proj_1}, \mathbf{Proj_2} \rangle = h$
  - Substitution Lemma

- η-equivalence: $[\![\Gamma \rhd \lambda x{:}\sigma.\, Mx : \tau]\!] = [\![\Gamma \rhd M : \sigma \to \tau]\!]$
  - $\mathbf{Curry}(\mathbf{App} \circ \langle k \circ \mathbf{Proj_1}, \mathbf{Proj_2} \rangle) = k$

Completeness

Theorem (Completeness) Given any $\Gamma \rhd M : \sigma$ and $\Gamma \rhd N : \sigma$, there exists a CCC $\mathcal{C}$ such that if $\mathcal{C}[\![\Gamma \rhd M : \sigma]\!] = \mathcal{C}[\![\Gamma \rhd N : \sigma]\!]$, then $\Gamma \rhd M \equiv_{\alpha\beta\eta} N : \sigma$.

Completeness (cont.)

Proof

- Construct one category $\mathcal{C}$ from $\lambda^{\rightarrow}$
  - Objects – sequences of types
    - [] – terminal object
    - $[\sigma_1, \ldots, \sigma_m]$ – products of $[\sigma_i]$ – $\vec{\sigma}_m$
  - Morphisms – tuples of equivalence classes of terms

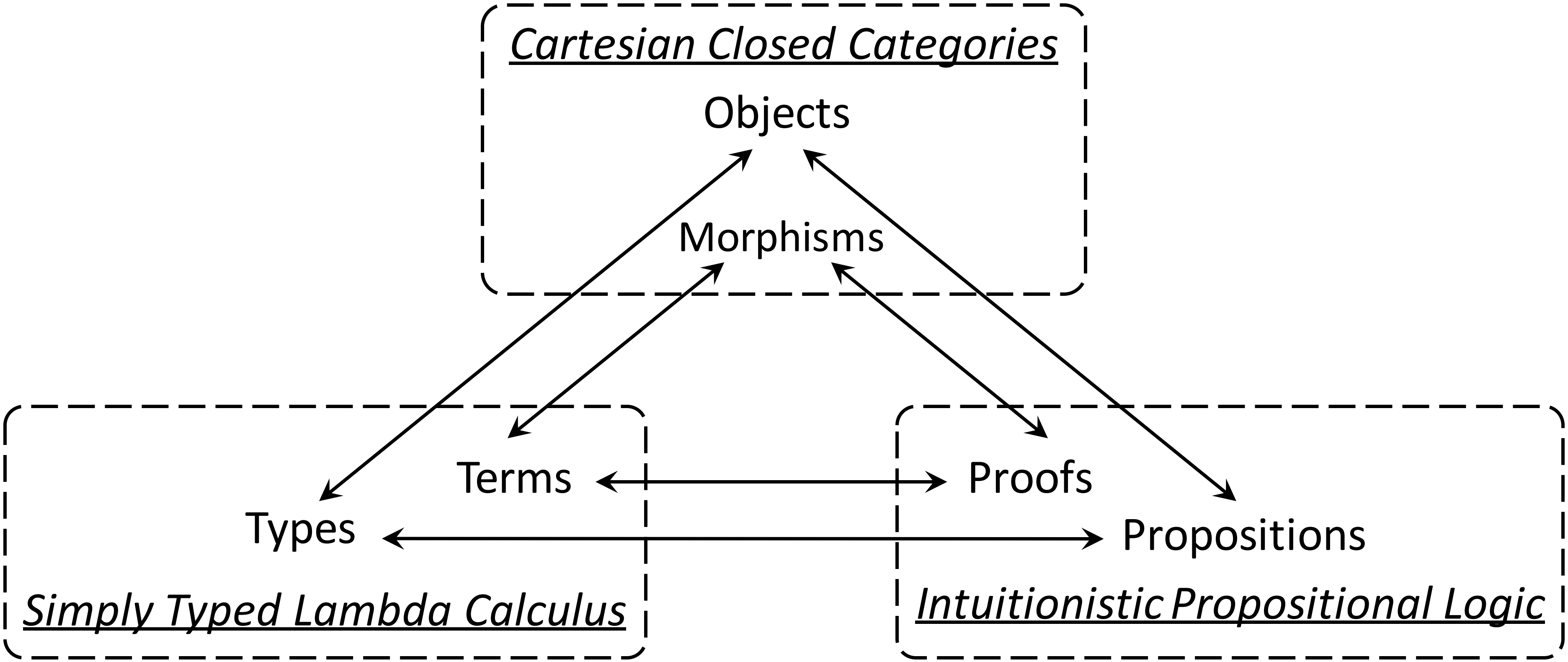
    $\langle \{\vec{x}_m : \vec{\sigma}_m \rhd N_{\boldsymbol{i}} : \tau_{\boldsymbol{i}} | \Gamma \rhd N_{\boldsymbol{i}} = M_{\boldsymbol{i}} : \tau_{\boldsymbol{i}}\} \mid \boldsymbol{i} = 1, \ldots, n \rangle : \vec{\sigma}_m \rightarrow \vec{\tau}_n$
- $\mathcal{C}$ is cartesian closed (the most difficult part)
- $\mathcal{C}[\![\Gamma \rhd M : \sigma]\!] = \langle \{\Gamma \rhd N : \sigma | \Gamma \rhd N = M : \sigma\} \rangle$

  If $\mathcal{C}$ satisfies $[\![\Gamma \rhd M : \sigma]\!] = [\![\Gamma \rhd N : \sigma]\!]$,

  then $\Gamma \rhd M \equiv_{\alpha\beta\eta} N : \sigma$.

Reflections

- A connection between very diverse mathematical subjects



Reflections

- The C.H. isomorphism can be extended to stronger logics, more expressive languages and richer categories
  - First order logic ≌ Dependent types
  - PCF ≌ Category of domains
  - …

Reflections

- The C.H. isomorphism helps to prove theorems in each field

  Q: Is the inhabitation problem in Dependent Types decidable?

  A: No, it is undecidable.

  - Inhabitation ≌ Provability
  - Dependent Types ≌ First Order Logic
  - Provability in First Order Logic is undecidable

Thank you for your listening!

Question?

Appendix

- Connectives VS. Type Constructors

Formulas in intuitionist propositional logic	Types in full simply-typed lambda calculus
False formula $\bot$	Initial type $null$
True formula $\top$ $(\neg\bot)$	Terminal type $unit$
Implication $\varphi_0 \to \varphi_1$	Function type $\sigma_0 \to \sigma_1$
Conjunction $\varphi_0 \wedge \varphi_1$	Product type $\sigma_0 \times \sigma_1$
Disjunction $\varphi_0 \vee \varphi_1$	Sum type $\sigma_0 + \sigma_1$

Appendix

- Natural Deduction System VS. Typing Rules

Natural deduction in intuitionist propositional logic	Typing rules in full simply-typed lambda calculus
$\dfrac{}{\Gamma_1, \varphi, \Gamma_2 \vdash \varphi}\, Axiom$	$\dfrac{}{\Gamma_1, x{:}\sigma, \Gamma_2 \vdash x{:}\sigma}\, Var$
$\dfrac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \to \psi} \to I$	$\dfrac{\Gamma, x{:}\sigma \vdash M{:}\tau}{\Gamma \vdash \lambda x. M{:}\sigma \to \tau} \to I$
$\dfrac{\Gamma \vdash \varphi \to \psi,\ \Gamma \vdash \varphi}{\Gamma \vdash \psi} \to E$	$\dfrac{\Gamma \vdash M{:}\sigma \to \tau,\ \Gamma \vdash N{:}\sigma}{\Gamma \vdash MN{:}\tau} \to E$
$\dfrac{\Gamma \vdash \varphi,\ \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi} \wedge I$	$\dfrac{\Gamma \vdash M{:}\sigma,\ \Gamma \vdash N{:}\tau}{\Gamma \vdash \langle M, N\rangle{:}\sigma \times \tau} \times I$

Appendix

◆ Natural Deduction System VS. Typing Rules (cont.)

$\dfrac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \varphi} \wedge E_1$	$\dfrac{\Gamma \vdash M : \sigma \times \tau}{\Gamma \vdash \mathrm{Proj}_1^{\sigma,\tau} M : \sigma} \times E_1$
$\dfrac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \psi} \wedge E_2$	$\dfrac{\Gamma \vdash M : \sigma \times \tau}{\Gamma \vdash \mathrm{Proj}_2^{\sigma,\tau} M : \tau} \times E_2$
$\dfrac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi} \vee I_1$	$\dfrac{\Gamma \vdash M : \sigma}{\Gamma \vdash \mathrm{Inleft}^{\sigma,\tau} M : \sigma + \tau} + I_1$
$\dfrac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi} \vee I_2$	$\dfrac{\Gamma \vdash M : \tau}{\Gamma \vdash \mathrm{Inright}^{\sigma,\tau} M : \sigma + \tau} + I_1$
$\dfrac{\Gamma \vdash \varphi \vee \psi, \Gamma \vdash \varphi \to \chi, \Gamma \vdash \psi \to \chi}{\Gamma \vdash \chi} \vee E$	$\dfrac{\Gamma \vdash M : \sigma + \tau, \Gamma \vdash N : \sigma \to \rho, \Gamma \vdash P : \tau \to \rho}{\Gamma \vdash \mathrm{Case}^{\sigma,\tau,\rho} MNP : \rho} + E$

Appendix

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