The Curry–Howard Isomorphism

Project Aims

Intuitionistic Logic & λ-Calculus

Cartesian Closed Categories

Correspondence between *CCC* & $\lambda^{unit,\to,\times}$

Project Aims

- Study the foundations of the Curry-Howard Isomorphism
- Probe into one of its extensions:

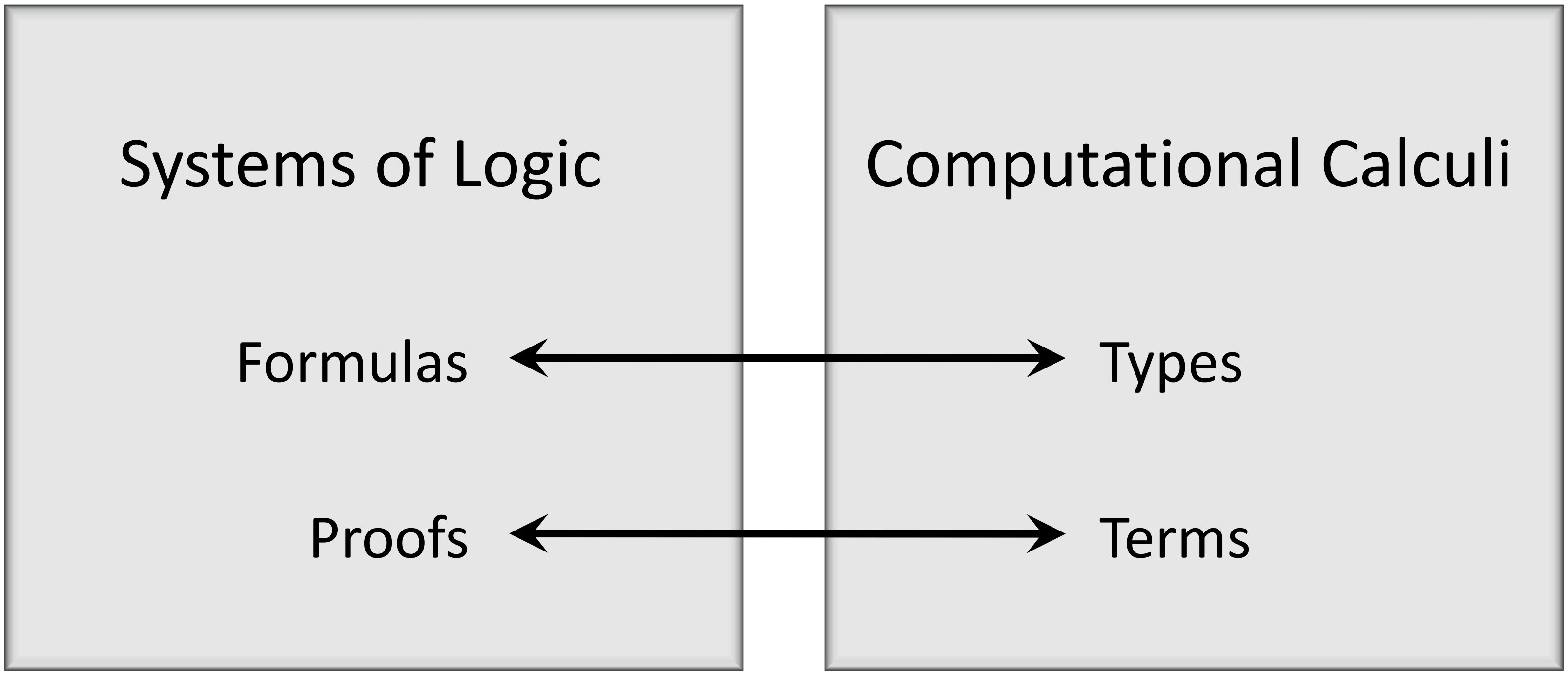
  The three-way-correspondence between intuitionistic logic,

  typed lambda calculus and cartesian closed categories

Intuitionistic Logic & Lambda Calculus

- Intuitionistic Logic
  - The judgments about statements are based on the existence of a proof or construction of that statement
  - The law of excluded middle ($\vdash \varphi \vee \neg\varphi$) is not an axiom
- Lambda Calculus
  - A family of prototype programming languages
  - Three types of equivalence: α-, β-, and η-equivalence

The Curry-Howard Isomorphism



Extension of the Isomorphism

$$
\begin{tikzcd}
 & \text{Cartesian Closed Categories} \arrow[dl, shift left] \arrow[dr, shift left] & \\
\text{Intuitionistic Propositional Logic} \arrow[ur, shift left] \arrow[rr, shift left] & & \text{Simply Typed Lambda Calculus} \arrow[ul, shift left] \arrow[ll, shift left]
\end{tikzcd}
$$

Cartesian Closed Categories

A *CCC* is a category with the following extra structure:

- A terminal object *unit* with unique arrow $\mathbf{One}^{\sigma}$ for every σ
- Object map $\times$, function $\langle\cdot,\cdot\rangle$, and arrows $\mathbf{Proj_1}$ and $\mathbf{Proj_2}$ satisfying $\mathbf{Proj}_{\boldsymbol{i}}\circ\langle f_1, f_2\rangle = f_i$ and $\langle\mathbf{Proj_1}\circ g, \mathbf{Proj_2}\circ g\rangle = g$
- Object map $\rightarrow$, function **Curry**, and arrow **App** satisfying $\mathbf{App}\circ\langle\mathbf{Curry}(h)\circ\mathbf{Proj_1}, \mathbf{Proj_2}\rangle = h$ and $\mathbf{Curry}(\mathbf{App}\circ\langle k\circ\mathbf{Proj_1}, \mathbf{Proj_2}\rangle) = k$

The Interpretation of λ-terms in *CCCs*

- The interpretation of type expressions:
  - $[\![unit]\!] = unit$
  - $[\![\sigma \times \tau]\!] = [\![\sigma]\!] \times [\![\tau]\!]$
  - $[\![\sigma \to \tau]\!] = [\![\sigma]\!] \to [\![\tau]\!]$
- The interpretation of typing contexts:
  - $[\![\emptyset]\!] = unit$
  - $[\![\Gamma, x\!: \sigma]\!] = [\![\Gamma]\!] \times [\![\sigma]\!]$

The Interpretation of λ-terms in *CCCs* (cont.)

- The interpretation of well-typed terms
  - $[\![\Gamma, x{:}\sigma \rhd x{:}\sigma]\!] = \mathrm{Proj}_2^{[\![\Gamma]\!],\sigma}$
  - $[\![\emptyset \rhd *{:}unit]\!] = \mathrm{One}^{unit}$
  - $[\![\Gamma \rhd \mathrm{Proj}_1^{\sigma,\tau} M{:}\sigma]\!] = \mathrm{Proj}_1^{\sigma,\tau} \circ [\![\Gamma \rhd M{:}\sigma \times \tau]\!]$
  - $[\![\Gamma \rhd \mathrm{Proj}_2^{\sigma,\tau} M{:}\tau]\!] = \mathrm{Proj}_2^{\sigma,\tau} \circ [\![\Gamma \rhd M{:}\sigma \times \tau]\!]$
  - $[\![\Gamma \rhd \langle M, N\rangle{:}\sigma \times \tau]\!] = \langle [\![\Gamma \rhd M{:}\sigma]\!], [\![\Gamma \rhd N{:}\tau]\!] \rangle$
  - $[\![\Gamma \rhd MN{:}\tau]\!] = \mathrm{App}^{\sigma,\tau} \circ \langle [\![\Gamma \rhd M{:}\sigma \to \tau]\!], [\![\Gamma \rhd N{:}\sigma]\!] \rangle$
  - $[\![\Gamma \rhd \lambda x{:}\sigma. M{:}\sigma \to \tau]\!] = \mathrm{Curry}([\![\Gamma, x{:}\sigma \rhd M{:}\tau]\!])$
  - $[\![\Gamma_1, x{:}\sigma, \Gamma_2 \rhd M{:}\tau]\!] = [\![\Gamma_1, \Gamma_2 \rhd M{:}\tau]\!] \circ \chi_f^{[\![\Gamma_1, x:\sigma, \Gamma_2]\!]}$

Soundness & Completeness

Theorem (Soundness) Given any well typed terms M and N,

if $\Gamma \rhd M \equiv_{\alpha\beta\eta} N : \sigma$, then $[\![\Gamma \rhd M : \sigma]\!] = [\![\Gamma \rhd N : \sigma]\!]$ in very *CCC*.

Proof

- α-equivalence:
  no term-variable name appears in the calculations
- β-equivalence:
  $[\![\Gamma \rhd (\lambda x{:}\sigma.M)N : \tau]\!] = [\![\Gamma \rhd [N/x]M : \tau]\!]$
- η-equivalence:
  $[\![\Gamma \rhd \lambda x{:}\sigma.Mx : \tau]\!] = [\![\Gamma \rhd M : \sigma \to \tau]\!]$

Soundness & Completeness (cont.)

Theorem (Completeness) if $[\![\Gamma \rhd M : \sigma]\!] = [\![\Gamma \rhd N : \sigma]\!]$ in all CCCs, then $\Gamma \rhd M \equiv_{\alpha\beta\eta} N : \sigma$.

Proof

- Construct one category $\mathcal{C}$ by $\lambda^{unit, \to, \times}$
  - Objects – types
  - Morphisms – equivalence classes of terms
- $\mathcal{C}$ is cartesian closed
- If $\mathcal{C}$ satisfies $[\![\Gamma \rhd M : \sigma]\!] = [\![\Gamma \rhd N : \sigma]\!]$, then $\Gamma \rhd M \equiv_{\alpha\beta\eta} N : \sigma$.

Internal Language of *CCCs*

Let $\mathcal{C}$ be a CCC. The internal language $\mathrm{L}(\mathcal{C})$ of $\mathcal{C}$ is a typed lambda calculus $\lambda^{unit, \rightarrow, \times}$.

- Types of $\mathrm{L}(\mathcal{C})$ – objects of $\mathcal{C}$
  terminal type, function types and product types
- Terms of $\mathrm{L}(\mathcal{C})$ – morphisms of $\mathcal{C}$
  atoms (term-variables), term of terminal type, applications, abstractions, product terms, and terms with projections
- The equations between terms in $\mathrm{L}(\mathcal{C})$:
  α-, β-, and η-equivalence

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