2.1 Intuitionistic Logic

Intuitionistic logic is also called constructive logic. As a formalization of intuitionism, it differs from classical logic not only in that syntactically some laws in classical logic are not axioms of the system but also in the meaning for statements to be true. The judgments about statements are based on the existence of a proof or a "construction" of that statement. Its existence property makes it practically useful, e.g. provided that a constructive proof that an object exists, one can turn it into an algorithm for generating an example of the object.

One vertex in the correspondence-triangle is the intuitionistic propositional logic. So the introduction to intuitionistic logic in this dissertation is up to the propositional one.

2.1.1 Syntax

The language of intuitionistic propositional logic is similar to the one of classical propositional logic. Customarily, people use $\bot$, $\to$, $\wedge$, $\vee$ as basic connectives and treat $\neg\varphi$ as an abbreviation for $\varphi \to \bot$.

2.1.1 Definition (Set of formulas) Given an infinite set of propositional variables, the *set* Φ *of formulas* in intuitionistic propositional logic is defined by induction, represented in the following grammar:

$$\Phi \ ::= \ P \ | \ \bot \ | \ \neg\Phi \ | \ (\Phi \to \Phi) \ | \ (\Phi \wedge \Phi) \ | \ (\Phi \vee \Phi)$$

where P is a *propositional variable*, $\bot$ is *contradiction*, $\neg$ is *negation*, $\to$ is *implication*, $\wedge$ is *conjunction*, and $\vee$ is *disjunction*.

Given a set Γ of propositions and a proposition φ, the relation $\Gamma \vdash \varphi$ says that there is a derivation with conclusion φ from hypotheses in Γ. Here Γ is also called *context*. If Γ is empty, we write $\vdash \varphi$ and say that φ is a *theorem*.

For notational convenience, we use the following abbreviations:

- $\varphi_1, \varphi_2, \ldots, \varphi_n$ for $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$;
- Γ, φ for $\Gamma \cup \{\varphi\}$;

The natural deduction system allows one to derive conclusions from premises. The axiom and inference rules in this deduction system show how the relation $\vdash$ is built.

2.1.2 Definition (Natural Deduction) Given a set of propositional variable, the relation $\Gamma \vdash \varphi$ is obtained by using the following axiom and inference rules

- *Axiom*

$$\frac{}{\varphi \vdash \varphi}\,(axiom)$$

Since proposition φ appears in the context, one can conclude it from the context.

- *Adding hypothesis into context*

$$\frac{\Gamma \vdash \varphi}{\Gamma, \psi \vdash \varphi}\,(add)$$

This rules allows one to add additional hypotheses into the context.

- →-*introduction*

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \to \psi}(\to \mathrm{I})$$

If one can derive ψ from φ as a hypothesis, then $\varphi \to \psi$ is derivable without hypothesis φ.

- →-*elimination*

$$\frac{\Gamma \vdash \varphi \to \psi, \quad \Gamma \vdash \varphi}{\Gamma \vdash \psi}(\to \mathrm{E})$$

Both the conditional claim "if φ then ψ" and φ are provided, one can conclude ψ. As mentioned in the beginning, this is a very common inference rule which is also called *modus ponens*.

- ∧-*introduction*

$$\frac{\Gamma \vdash \varphi, \quad \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi}(\wedge \mathrm{I})$$

If both φ and ψ are derivable from Γ, $\varphi \wedge \psi$ is also derivable.

- ∧-*elimination*

$$\frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \varphi}(\wedge \mathrm{E})_1 \qquad \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \psi}(\wedge \mathrm{E})_2$$

Provided that conjunction $\varphi \wedge \psi$ is derivable from Γ, both of its components are also derivable.

- ∨-*introduction*

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi}(\vee \mathrm{I})_1 \qquad \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi}(\vee \mathrm{I})_2$$

One can conclude disjunction $\varphi \vee \psi$ from either φ or ψ.

- ∨-*elimination*

$$\frac{\Gamma \vdash \varphi \to \rho, \quad \Gamma \vdash \psi \to \rho, \quad \Gamma \vdash \varphi \vee \psi}{\Gamma \vdash \rho}(\vee \mathrm{E})$$

If ρ follows φ, ρ follows ψ and $\varphi \vee \psi$ is given, one can conclude ρ.

- ⊥-*elimination*

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi}(\bot \mathrm{E})$$

From contradiction $\bot$, we can derive any propositions. This rule is also called *principle of explosion* or *ex falso quodlibet*.

Some examples are given here to show how the rules above are used to build a theorem:

(1) $\vdash \varphi \to (\psi \to \varphi)$

$$\cfrac{\cfrac{\cfrac{\overline{\varphi \vdash \varphi}}{\varphi, \psi \vdash \varphi}(add)}{\varphi \vdash \psi \to \varphi}(\to \mathrm{I})}{\vdash \varphi \to (\psi \to \varphi)}(\to \mathrm{I})$$

(2) $\vdash (\neg\varphi \vee \psi) \to (\varphi \to \psi)$

$$
\begin{prooftree}
\AxiomC{$\neg\varphi, \varphi \vdash \bot$}
\RightLabel{$(\bot\,\mathrm{E})$}
\UnaryInfC{$\neg\varphi, \varphi \vdash \psi$}
\RightLabel{$(\to \mathrm{I})$}
\UnaryInfC{$\neg\varphi \vdash \varphi \to \psi$}
\RightLabel{$(\to \mathrm{I})$}
\UnaryInfC{$\vdash \neg\varphi \to (\varphi \to \psi)$}
\RightLabel{(add)}
\UnaryInfC{$\neg\varphi \vee \psi \vdash \neg\varphi \to (\varphi \to \psi)$}
\AxiomC{}
\RightLabel{(1)}
\UnaryInfC{$\vdash \psi \to (\varphi \to \psi)$}
\RightLabel{(add)}
\UnaryInfC{$\neg\varphi \vee \psi \vdash \psi \to (\varphi \to \psi)$}
\AxiomC{}
\UnaryInfC{$\neg\varphi \vee \psi \vdash \neg\varphi \vee \psi$}
\RightLabel{$(\vee\,\mathrm{E})$}
\TrinaryInfC{$\neg\varphi \vee \psi \vdash \varphi \to \psi$}
\RightLabel{$(\to \mathrm{I})$}
\UnaryInfC{$\vdash (\neg\varphi \vee \psi) \to (\varphi \to \psi)$}
\end{prooftree}
$$

2.1.2 Constructive Semantics

In intuitionistic logic, each proposition is given an intuitive meaning that establishes the proposition, called a *proof* or *construction*. Then all the connectives in intuitionistic propositional logic are given another interpretation, sometimes called the *BHK-interpretation*.

2.1.3 Definition (Proofs) The expression $p : \varphi$ denotes that p is a construction that establishes proposition φ, we call this p a *proof of* φ. The following rules explain the constructive semantics of propositional connectives:

- $\bot$ — There is no proof of $\bot$.
- $p : \varphi \to \psi$ — The proof p of implication $\varphi \to \psi$ is a method converting every proof $a : \varphi$ into a proof $p(a) : \psi$.
- $p : \varphi \wedge \psi$ — The proof p of conjunction $\varphi \wedge \psi$ is a pair of proofs $\langle p_1, p_2 \rangle$ such that $p_1 : \varphi$ and $p_2 : \psi$, with projections π_1, π_2 that return the first and second proofs in a pair.
- $p : \varphi \vee \psi$ — The proof p of disjunction $\varphi \vee \psi$ is a pair $\langle i, a \rangle$ where $i \in \{0,1\}$ such that $i = 0$ and $a : \varphi$ or $i = 1$ and $a : \psi$. In words, the i indicates which disjunct is correct and a is the proof of that disjunct.
- $p : \neg\varphi$ — The proof p of negation of φ is a method that transforms every proof $a : \varphi$ into $p(a) : \bot$, i.e. p tells us that φ has no proofs.

The proof of $\bot \to \varphi$ is the proof of φ?
Or a particular unique proof (method) for every φ which is isomorphic to the proof of φ?

The following examples demonstrate the proof interpretation for some propositions:

(1) $\varphi \to (\psi \to \varphi)$

Let p is a proof of $\varphi \to (\psi \to \varphi)$. Then p is a method converting proofs of φ into a proof of $\psi \to \varphi$. If we already have $p_1 : \varphi$ and $p_2 : \psi$, then the proof of $\psi \to \varphi$ is a transformation $p_2 \mapsto p_1$ which can be represented as $\lambda p_2.p_1$ (using λ-calculus denotation). Hence, $p = \lambda p_1.(\lambda p_2.p_1) : \varphi \to (\psi \to \varphi)$.

(2) $\varphi \to \neg\neg\varphi$

Assume that $p_1 : \varphi$ and $p_2 : \neg\varphi$. According to definition 2.1.3, we have $p_2(p_1) : \bot$.

Since $\neg\neg\varphi$ stands for $\neg\varphi \to \bot$, its proof is $\lambda p_2.p_2(p_1)$. Therefore, the proof of $\varphi \to \neg\neg\varphi$ is $p = \lambda p_1.(\lambda p_2.p_2(p_1)) : \varphi \to \neg\neg\varphi$.

(3) $(\varphi \wedge \psi) \to (\psi \wedge \varphi)$

Assume $q : \varphi \wedge \psi$. According to definition 2.1.3, we have $\pi_1(q) : \varphi$ and $\pi_2(q) : \psi$. Then the pair $\langle \pi_2(q), \pi_1(q) \rangle$ is a proof of $(\psi \wedge \varphi)$. So the proof of $(\varphi \wedge \psi) \to (\psi \wedge \varphi)$ is $p = \lambda q.\langle \pi_2(q), \pi_1(q) \rangle : (\varphi \wedge \psi) \to (\psi \wedge \varphi)$.

(4) $(\neg\varphi \vee \psi) \to (\varphi \to \psi)$

Assume $q : \neg\varphi \vee \psi$. According to definition 2.1.3, we have $\pi_2(q) : \neg\varphi$ if $\pi_1(q) = 0$ or $\pi_2(q) : \psi$ if $\pi_1(q) = 1$. ???

Both the law of excluded middle and double negation elimination are axioms in the system of classical logic. However, in intuitionistic one neither of them can be proved, i.e. there is no constructive proof for them.

(1) $\varphi \vee \neg\varphi$

If $\varphi \vee \neg\varphi$ has a proof p, then p should be a pair $\langle i, a \rangle$ such that $a : \varphi$ if $i = 0$ or $a : \neg\varphi$ if $i = 1$. However, for an arbitrary proposition φ we do not know whether φ or $\neg\varphi$ has a proof. As a result, the value of i cannot be known. Therefore, there is no proof of $\varphi \vee \neg\varphi$ for arbitrary φ.

(2) $\neg\neg\varphi \to \varphi$

Assume that we have a proof $p : \neg\neg\varphi$. According to definition 2.1.3, p tells us that there is no proof of $\neg\varphi$. Then we end here and unable to obtain a proof of φ. Hence, there is no proof of $\neg\neg\varphi \to \varphi$.

From the above examples, one may make a conclusion that for every theorem in intuitionistic logic there is always a closed proof, i.e. a term without variables, while for those which are axioms in classical logic but not in intuitionistic one, it is possible to find a corresponding closed proof.

1. Wikipedia mentions the non-interdefiability of connectives in IL. Is implicational logic as powerful as propositional logic?
2. Add a section with some counter-examples to show that something do not correspond to each other in intuitionistic logic and lambda calculus
3. Add a section to demonstrate the connection between provability and type-inhabitation and the one between proof checking and type checking.