The Mackey - Glass timeseries prediction

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The Mackey-Glass equation, widely used to model biological rhythms, is a time-delay differential system described as follows:

$$\frac{dx}{dt} = \frac{\alpha x(t-\tau)}{1 + x^c(t-\tau)} - \beta x(t). \tag{1}$$

To obtain the time series data set, one could solve (1) using a 4th order Runge–Kutta method. Either way, we were given the noisy time series produced by (1), for unknown parameters α, β, c and τ. Our goal is to create a Convolutional Neural Network (CNN) in order to do the following:

- Perform single step predictions. Here, we always use as input the previous L real states of the time series and we try to predict the next $L+1$ state.
- Perform iterative autoregressive prediction. In this case, for each trajectory in the test data we start from the first L initial timesteps and then iteratively a) feed the network with these input timesteps, b) perform a prediction, c) append the prediction to the input data, and d) feed the network with the updated input data and predict again.

In both cases, we must plot samples of our predictions, calculate the mean squared error (MSE) and finally plot the error with respect to the number of timesteps. Before we begin, we will do a brief error analysis of the Mackey - Glass.

Error Analysis: First assume that f is the analytical solution to a dynamical system, x_0 is the initial condition and $f^t(x_0)$ is the evolution of the dynamical system after time t. Now assume that $x_0 + \delta_0$ is a small perturbation in the initial condition. Then, after time t, we have

$$|\delta_t| = |f^t(x_0 + \delta_0) - f^t(x_0)| \approx |\delta_0| e^{\lambda t}, \tag{2}$$

where λ is the Lyapunov exponent. If $\lambda > 0$ then this is signature of *chaos*, as it signify the exponential separation of orbits with close initial conditions. In a few words, relation (2) says that no matter how small the perturbation δ_0 is, it is just a matter of time for an orbit to diverge exponentially from the original state through time in a chaotic system. Thus, it is impossible to predict chaotic timeseries accurately through a long time.

It is well known that Mackey-Glass is a chaotic system and we expect that our itterative predictions for our time series will diverge exponentially from the true solution.

The architecture of our CNN: Our model has the following simple architecture:

```
CNN(
  (conv1): Conv1d(1, 16, kernel_size=(3,), stride=(1,), padding=(1,))
  (conv2): Conv1d(16, 32, kernel_size=(3,), stride=(1,), padding=(1,))
  (fc1): Linear(in_features=768, out_features=50, bias=True)
  (fc2): Linear(in_features=50, out_features=1, bias=True)
)
```

For activation function we have chosen RELU for the convolution layers and also for the first Linear layer (fc1). As for the optimiser, we chose Adam with learning rate 0.001. Our Loss on

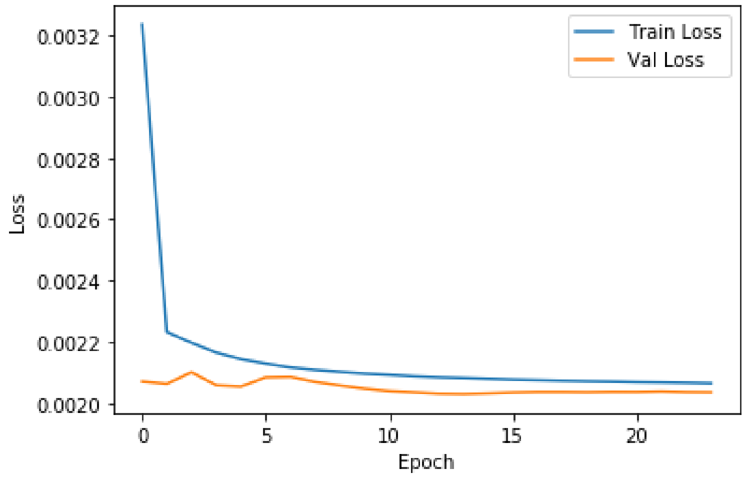

Figure 1: Train and Validation losses.

train and validation set is captured at the following figure:

We made a lot of simulations, but here we present only one for $L = 24$ for both one step predictions and iterative (autoregressive) predictions. We are now ready to present our results.

L=24

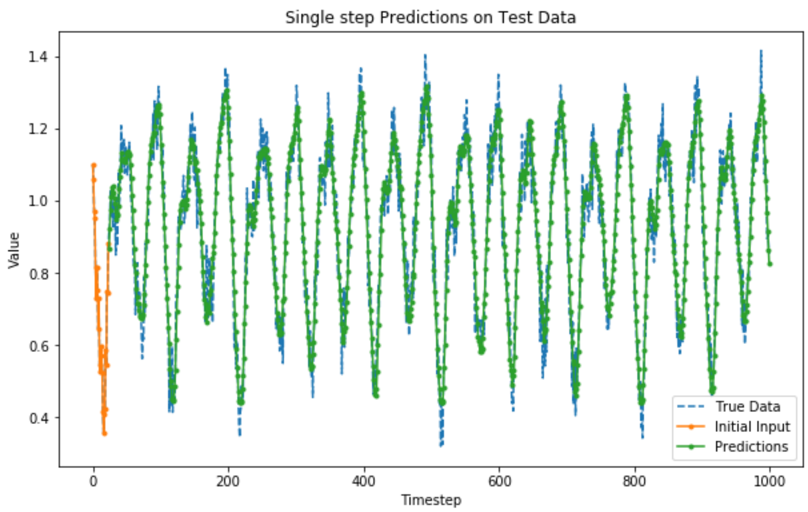

Figure 2: The single step predictions.

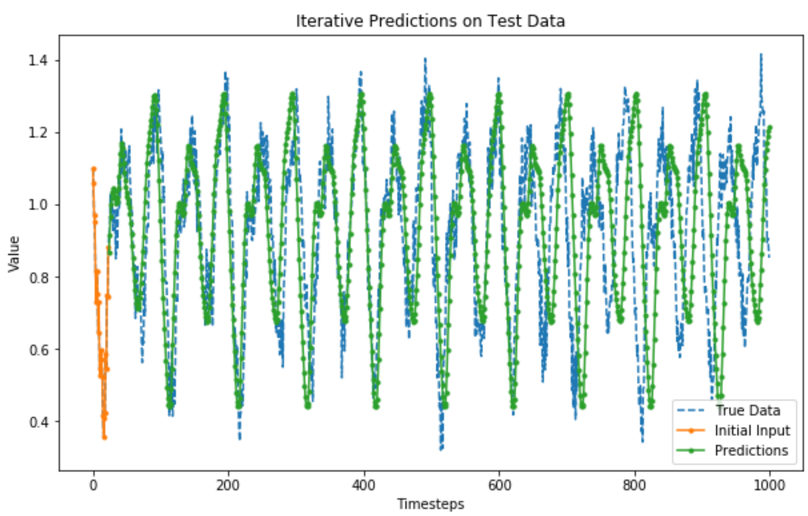

Figure 3: The iterative (autoregressive) predictions.

We see that after some timesteps, our predictions start to differ greatly from the true trajectory. This is expected, as the system is chaotic. The further examine the exponential error growth of our predictions and the Lyapunov exponent, we present the average error at each time step for all trajectories. For example, at timestep 37, we sum the error of all trajectories at timestep 37 and then we divide by 100 to find the average. This procedure give us the average error of the model at each time step for all trajectories. The plots of this average error are the following:

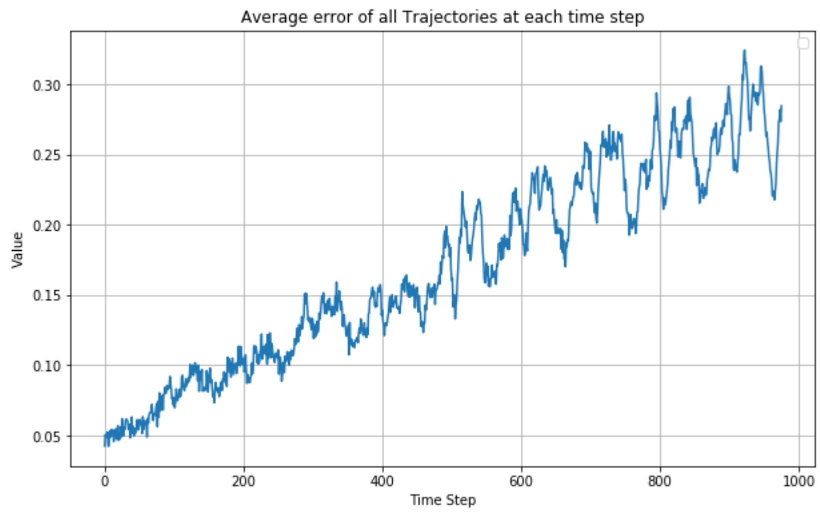

Figure 4: Average Error of all Trajectories at each time step.

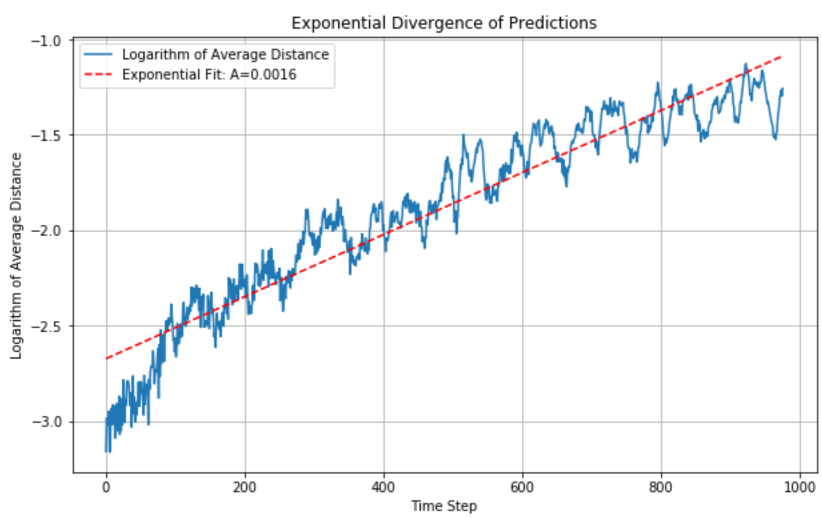

Figure 5: The Lyapunov exponent shows the exponential error growth.