Poisson Random Process

PART 2 – ALTERNATIVE DERIVATION OF POISSON PROCESS

Intro

- Rederive the Poisson counting process from the elementary properties of random points in time
- Asymptotic behavior of binomial law: the Poisson law

Asymptotic Behavior of Binomial Law

- Suppose $b(k; n, p)$, $n >> 1$, $p << 1$, $np = \mu$, $q = p - 1$
- Hence, $$\binom{n}{k} p^k (1-p)^{n-k} \simeq \frac{1}{k!}\mu^k \left(1 - \frac{\mu}{n}\right)^{n-k}$$
- If k fixed and n grows large enough: $n(n-1)\ldots(n-k+1) \simeq n^k$
- If $n -> \inf$, $p -> 0$ and $k << n$, we obtain:

$$b(k; n, p) \simeq \frac{1}{k!}\mu^k \left(1 - \frac{\mu}{n}\right)^{n-k} \xrightarrow{n\to\infty} \frac{\mu^k}{k!}e^{-\mu}$$

- Approximation on binomial law:

$$b(k; n, p) \simeq \frac{\mu^k}{k!}e^{-\mu}$$

Example 1

- The Poisson probability law, with parameter $\mu(> 0)$, is given as:

$$p(k) = \frac{\mu^k}{k!}e^{-\mu}, \quad 0 \le k < \infty$$

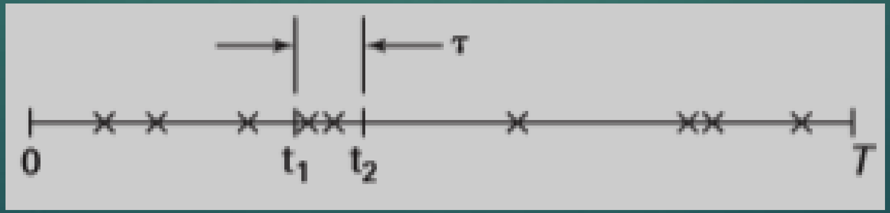
- Example 1.10-1: time to failure
- A computer contains 10,000 components. Each component fails independently from the others and the yearly failure probability per component is 10−4. What is the probability that the computer will be working one year after turn-on? Assume that the computer fails if one or more components fail.

$$p = 10^{-4}, \qquad n = 10{,}000, \qquad k = 0, \qquad np = 1.$$

$$b(0; 10{,}000, 10^{-4}) = \frac{1^0}{0!}e^{-1} = \frac{1}{e} = 0.368.$$

Example 2

- Example 1.10-2: Random points in time
- Suppose that n independent points are placed at random in an interval $(0, T)$
- Let $0 \ < t\,1 \ < t\,2 \ < T$ and $t_2 \ - t_1 = \ \tau$
- Let $\frac{\tau}{T} << 1$ and $n \ >> 1$
- Each point is placed with equal likelihood anywhere along the line
- What is the probability of observing exactly k points in τ seconds?



Example 2 - cont.

- Consider a single point placed at random in $(0, T)$. The probability of the point appearing in τ is $\frac{\tau}{T}$, hence, $p = \frac{\tau}{T}$
- Every other point has the same probability of being in τ seconds. Hence, finding k points in τ seconds:

$$P[k \text{ points in } \tau \text{ sec}] = \binom{n}{k} p^k q^{n-k}$$

- With $n \gg 1$, we use approximation in slide 3:

$$b(k; n, p) \simeq \frac{\mu^k}{k!} e^{-\mu}$$

$$b(k; n, p) \simeq \left(\frac{n\tau}{T}\right)^k \frac{e^{-(n\tau/T)}}{k!}$$

- Where $\frac{\tau}{T}$ can be interpreted as the "average" number of points per unit interval.

Asymptotic Behavior of Binomial Law – cont.

- Replacing the average rate in this example with parameter $\mu\ (\mu > 0)$,

$$P[k \text{ points}] = e^{-\mu}\frac{\mu^k}{k!}$$

  - Where $k = 0, 1, 2, \ldots$. With $\mu = \lambda\tau$, (rate*interval width)
  - λ is the average number of points per unit time
  - τ is the length of the interval $(t, t + \tau]$
- Hence,

$$P(k; t, t+\tau) = e^{-\lambda\tau}\frac{(\lambda\tau)^k}{k!}$$

- For the Poisson law, we assume that numbers of points arriving in disjoint time intervals constitute independent events
  - from an underlying set of Bernoulli trials, which are always independent

Alternative Derivation of Poisson Process – cont.

$$P(k; t, t+\tau) = e^{-\lambda\tau}\frac{(\lambda\tau)^k}{k!}$$

- For Δt small:
  - $P_N(1;\ t, t+\Delta t) = \lambda(t)\Delta t + o(\Delta t)$
  - $P_N(k;\ t, t+\Delta t) = o(\Delta t)$
  - Events in nonoverlapping time intervals are statistically independent
- $o(\Delta t)$, denotes any quantity that goes to zero at a faster than linear rate in such a way that:

$$\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0$$

- We want to compute the probability $P_N(k;\ t, t+\tau)$ of k events in $(t, t+\tau)$

Alternative Derivation of Poisson Process – cont.

- Consider $P_N(k;\ t, t+\tau+\Delta t)$, if Δt is very small (and little-o), then for k events in this interval:

$$E_1 = \{k \text{ in } (t, t+\tau) \text{ and } 0 \text{ in } (t+\tau, t+\tau+\Delta t)\} \quad \text{or}$$
$$E_2 = \{k-1 \text{ in } (t, t+\tau) \text{ and } 1 \text{ in } (t+\tau, t+\tau+\Delta t)\}.$$

- E_1 and E_2 are disjoint:

$$\begin{aligned} P_N(k;t,t+\tau+\Delta t) &= P_N(k;t,t+\tau)P_N(0;t+\tau,t+\tau+\Delta t) \\ &\quad + P_N(k-1;t,t+\tau)P_N(1;t+\tau,t+\tau+\Delta t) \\ &= P_N(k;t,t+\tau)[1-\lambda(t+\tau)\Delta t] \\ &\quad + P_N(k-1;t,t+\tau)\lambda(t+\tau)\Delta t. \end{aligned}$$

Alternative Derivation of Poisson Process – cont.

- Rearrange terms, divide by Δt, and take limits, we obtain the linear differential equations (LDEs):

$$\frac{dP_N(k;t,t+\tau)}{d\tau} = \lambda(t+\tau)[P_N(k-1;t,t+\tau) - P_N(k;t,t+\tau)]$$

- Set $P_N(-1;t,t+\tau) = 0$, since this is the probability of the impossible event
- When $k = 0$, $$\frac{dP_N(0)}{d\tau} = -\lambda(t+\tau)P_N(0).$$
  - Which is first-order, homogeneous differential equation
  - Solution:

Alternative Derivation of Poisson Process – cont.

- Solution: $$P_N(0) = C\exp\left[-\int_t^{t+\tau}\lambda(\xi)d\xi\right]$$
- Since $P_N(0;t,t) = 1$, $C = 1$ and $$P_N(0) = \exp\left[-\int_t^{t+\tau}\lambda(\xi)d\xi\right]$$
- Define μ $$\mu \triangleq \int_t^{t+\tau}\lambda(\xi)d\xi$$
- Then $$P_N(0) = e^{-\mu}$$
- When $k = 1$ $$\frac{dP_N(1)}{d\tau} + \lambda(t+\tau)P_N(1) = \lambda(t+\tau)P_N(0)$$
- Then $$P_N(1) = \mu e^{-\mu}$$

Alternative Derivation of Poisson Process – cont.

- General Case
- LDE

$$\frac{dP_N(k)}{d\tau} + \lambda(t+\tau)P_N(k) = \lambda(t+\tau)P_N(k-1)$$

- Proceeding by induction,

$$P_N(k) = \frac{\mu^k}{k!}e^{-\mu} \qquad k = 0, 1, \ldots$$

- Recalling the definition of μ,

$$P_N(k; t, t+\tau) = \frac{1}{k!}\left[\int_t^{t+\tau} \lambda(\xi)d\xi\right]^k \exp\left[-\int_t^{t+\tau} \lambda(\xi)d\xi\right]$$

- We thus obtain the nonuniform Poisson counting process

QA

Thank you

References

- Henry Stark, John William Woods, Probability, Statistics, and Random Processes for Engineer, Pearson, 2012
- KhanAcademy