# Covariance functions of a recursive system

Stochastic Processes

# Why can we compute?

- The problem of calculating the output covariance and correlation of the general linear system whose operator is L: $Y[n] = L\{X[n]\}$
- If Y [n] depends on the most recent k input values $X[n], \ldots, X[n-k+1]$, then the kth-order pdf of X has to be known in order to compute even the first-order pdf of Y.
- The moments of the output random sequence can be calculated from equal- or lower-order moments of the input, when the system is linear.

# Observations

- The mean function of the output is the response of the linear system to the mean function of the input. (1) $E\{Y[n]\} = L\{E\{X[n]\}\}$
- **Cross-correlation** function between the input and output:

$$R_{XY}[m,n] \triangleq E\{X[m]Y^*[n]\} = E\{X[m]\left(L\{X[n]\}\right)^*\}$$

$$R_{XY}[m,n] \;=\; E\{X[m]L_n^*[X^*[n]]\} \;=\; L_n^*E\{X[m]X^*[n]\}.$$

- we denote with $L_n^*$, the linear operator with time index m, that treats n as a constant.

# Observations cont.

- Let $X[n]$ and $Y[n]$ be two random sequences that are the input and output, respectively, of the linear operator $L_n$ .
- Let the input-correlation function be $R_{XX}[m,n]$
- Then the cross-correlation functions:

$$R_{XY}[m,n] = L_n^* \{R_{XX}[m,n]\}$$

- And output-correlation functions:

$$R_{YY}[m,n] = L_m \{R_{XY}[m,n]\}$$

# Observations cont.

- Covariance Functions:

$$K_{XY}[m,n] = L_n^*\{K_{XX}[m,n]\} \quad (2)$$

$$K_{YY}[m,n] = L_m\{L_n^*\{K_{XX}[m,n]\}\}$$

$$K_{YY}[m,n] = L_m\{K_{XY}[m,n]\} \quad (3)$$

- Superposition summation of $K_{yy}$ :

$$K_{YY}[m,n] = \sum_{k=-\infty}^{+\infty} h[m,k] \left( \sum_{l=-\infty}^{+\infty} h^*[n,l] K_{XX}[k,l] \right)$$

# Example

- Covariance functions of a recursive system: $|\alpha| < 1, \; Y[-1] = 0$

$$Y[n] = \alpha Y[n-1] + (1-\alpha)W[n]$$

- As system is LSI, we can represent L by convolution: $h[n] = (1-\alpha)\alpha^n u[n].$

- Here h[n] is the impulse response of the corresponding deterministic first-order difference equation, that is, h[n] is the solution to obtained from z-transform:

$$h[n] = \alpha h[n-1] + (1-\alpha)\delta[n],$$

- where δ[n] is the discrete-time impulse sequence

# Example cont.

- Using (1) we can get:

$$\mu_Y[n] = \sum_{k=0}^{\infty}(1-\alpha)\alpha^k \mu_W[n-k], \text{ where } \mu_W[n] = 0 \text{ for } n < 0.$$

- Then by applying (2) and (3):

$$K_{WY}[m,n] = \sum_{k=0}^{\infty}(1-\alpha)\alpha^k K_{WW}[m,n-k]$$

$$K_{YY}[m,n] = \sum_{l=0}^{\infty}(1-\alpha)\alpha^l K_{WY}[m-l,n],$$

$$K_{YY}[m,n] = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(1-\alpha)^2\alpha^k\alpha^l K_{WW}[m-l,n-k]$$

# Example cont.

- Now if the input sequence W[n] has covariance function

$$K_{WW}[m,n] = \sigma_W^2 \delta[m-n] \text{ for } m,n \geq 0$$

- Then the output covariance is calculated as:

$$\begin{aligned}
K_{YY}[m,n] &= \sum_{k=0}^{n} (1-\alpha)^2 \alpha^k \alpha^{(m-n)+k} \sigma_W^2 \quad \text{for} \quad m \geq n \geq 0, \\
&= \alpha^{(m-n)} (1-\alpha)^2 \sum_{k=0}^{n} \alpha^{2k} \sigma_W^2 \\
&= \alpha^{(m-n)} \left[ (1-\alpha)^2 / (1-\alpha^2) \right] \sigma_W^2 (1-\alpha^{2n+2}) \qquad \text{for} \quad m \geq n \geq 0 \\
&= [(1-\alpha)/(1+\alpha)] \alpha^{|m-n|} \sigma_W^2 (1-\alpha^{2\min(m,n)+2}) \quad \text{for all } m,n \geq 0,
\end{aligned}$$

# Example cont.

- Where the last step follows from the required symmetry in (m,n). Note that the term $\alpha^{2\min(m,n)+2}$ is a transient that dies away as $m,n \to \infty$, since $|\alpha| < 1$, so that asymptotically we have the steady-state answer

$$K_{YY}[m,n] = \left(\frac{1-\alpha}{1+\alpha}\right)\sigma_W^2\ \alpha^{|m-n|}, \quad m,n \to \infty$$