Law of Large Numbers

STRONG AND WEAK LLN

Definitions

- Weak Law of Large Numbers:
  - $\overline{X_n}$ converges ***in probability*** to μ as $n \to \infty$.

$$\lim_{n\to\infty} P(|\overline{X_n} - \mu| \leq \varepsilon) = 1$$

- Strong Law of Large Numbers:

$$P(\lim_{n\to\infty} \overline{X_n} = \mu) = 1$$

  - $\overline{X_n}$ converges ***almost surely*** to μ as $n \to \infty$.

Almost Surely

- In probability theory, an event is said to happen ***almost surely*** if it happens with probability 1.
- This definition exists for infinite sample spaces. In finite sample states it is surely.
- In finite sample spaces, all subsets with probability of 0 are null sets (empty sets) but in infinite sample spaces, a non-empty subset with probability of 0 may exist.
- In math terms: event E is almost surely if $P(E) = 1$ or $P(E^C) = 0$ or E^C contained a null set.

Almost Surely cont.

- Example:
  - Consider tossing a coin where observing a H has probability of p.
  - If we toss the coin for infinite times where each toss is i.i.d. RVs, the probability of event “at least one T” will be almost surely because $1 - p^{\infty} = 1$
  - But if we stop tossing for a large N such as 1000000, then the probability of aforementioned event won’t be 1 anymore
- This is same definition of intuitive concept of probability of a limit in infinity.

Strong vs. Weak

- Strong a generalized and *stronger* form of weak LLN.
- Let X_i be i.i.d. Bernouli RVs where:
  - $P(X_i) = p,\ P(X_i = 0) = 1 - p = q$
  - And, $k = X_1 + X_2 + \cdots + X_n$ represents the number of **successes** in n trails.
- Weak LLN:
  - $P\left\{\left|\frac{k}{n} - p\right| > \varepsilon\right\} \le \frac{pq}{n\varepsilon^2} \qquad (1)$
  - The ratio **total number of successes to total number of trials** tends to p **in probability** as n increases.
- Strong LLN:
  - This ratio $\frac{k}{n}$ tends to p no only **in probability**, but **with probability 1**.

Strong vs. Weak cont.

- Strong LLN:
  - $\sum_{n=1}^{\infty} P\left\{\left|\frac{k}{n} - p\right| > \varepsilon\right\} < \infty$
  - Event $\left\{\left|\frac{k}{n} - p\right| < \varepsilon\right\}$ occurs infinitely w.r.t. $\left\{\left|\frac{k}{n} - p\right| > \varepsilon\right\}$
  - So, event $\frac{k}{n}$ converges to p ***almost surely***.

Strong vs. Weak cont.

- The weak law states that for every n that is large enough, the ratio $\frac{\sum_{i=1}^{n} X_i}{n} = \frac{k}{n}$ is likely to be near p with certain probability that tends to 1 as n increases.
- it does not say that $\frac{k}{n}$ is bound to stay near p if the number of trials is increased.
- Suppose (1) is satisfied for a given in a certain number of trials n_0. If additional trials are conducted beyond n_0 the weak law does not guarantee that the new $\frac{k}{n}$ is bound to stay near p for such trials.
- Weak LLN unbale to say anything about sum of small probabilities of events that violet (1).

Strong vs. Weak

- However, the strong law states that not only all such sums converge, but the total number of all such events, is in fact finite.
- This implies that the probability $\left\{\left|\frac{k}{n}-p\right|>\varepsilon\right\}$ of the events as n increases becomes and remains small, since with probability 1 only finitely many violations to the above inequality takes place as $n \to \infty$
- Example:
  - $\varepsilon = 0.1$, for tossing for 1000 times we have: $P\left\{\left|\frac{k}{n}-\frac{1}{2}\right|>0.01\right\} \leq \frac{1}{40}$

Reference

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