Problem 1

Consider the following line element in its general form: $ds^2 = a^2(d\theta^2 + f^2(\theta)d\phi^2)$ and find form of $f(\theta)$ such that

- Line element describes geometry of a peanut.
- Line element describes geometry of an egg.

For $f(\theta) = \sin\theta(1 + \epsilon\sin(2\theta))$, find a and ϵ which will give polar and equatorial radius of earth.

Solution

1. *Peanut:*
   There are many ways to write the equations for a peanut shaped object. One of the simple ways is to consider the following shape,

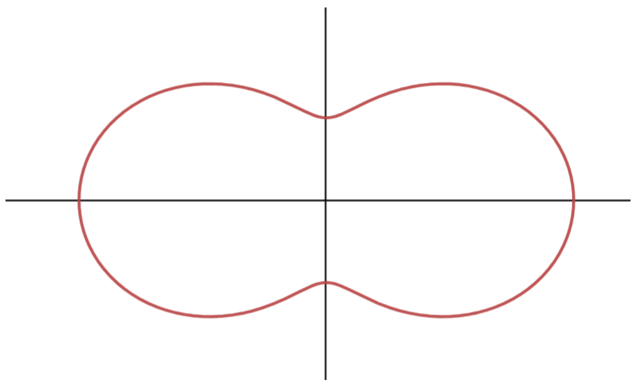

Figure 1: Peanut

in 2D, generated by the polar equation:

$$r(\theta) = 2 + \cos(2\theta) \tag{1}$$

Note that the x-axis here corresponds to the z-axis in 3D, since θ is zero in both. If we rotate this shape about the z-axis, we get the peanut shape. Define the obtained 2D object in $\mathbb{R}^3$ to be $\mathbb{P}$. The map $\psi : \mathbb{P} \to \mathbb{R}^3$ is given by:

$$\psi(\theta, \phi) = (2 + \cos(2\theta), \theta, \phi) \tag{2}$$

and we describe our three dimensional Euclidean space with the usual spherical metric. (the notation of $\mathbb{R}^3$ may seem wrong in this context but that is just for the sake of simplicity).
The Jacobian of this map is given by:

$$J = \begin{pmatrix} \frac{\partial r}{\partial \theta} & \frac{\partial r}{\partial \phi} \\ \frac{\partial \theta}{\partial \theta} & \frac{\partial \theta}{\partial \phi} \\ \frac{\partial \phi}{\partial \theta} & \frac{\partial \phi}{\partial \phi} \end{pmatrix} = \begin{pmatrix} -2\sin(2\theta) & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3}$$

Using this Jacobian, we can find the induced metric on the surface from our three dimensional metric.
Let $y^\alpha = (r, \theta, \phi)$ and $x^\mu = (\theta, \phi)$. The induced metric is given by:

$$(\psi^* g)_{\mu\nu} = \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} g_{\alpha\beta} \tag{4}$$

This simplifies to:

$$(\psi^* g)_{\mu\nu} = \sum_{\alpha=1}^{3} \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\alpha}{\partial x^\nu} g_{\alpha\alpha} \tag{5}$$

Using this, we can find the induced metric:

$$(\psi^* g)_{\theta\theta} = 5 + 4\cos(2\theta) + 3\sin^2(2\theta) \tag{6}$$
$$(\psi^* g)_{\phi\phi} = \sin^2(\theta)(4 + 4\cos(2\theta) + \cos^2(2\theta)) \tag{7}$$

Both components are positive (semi-)definite, so the metric is Riemannian. Below is the Mathematica plot for this surface.

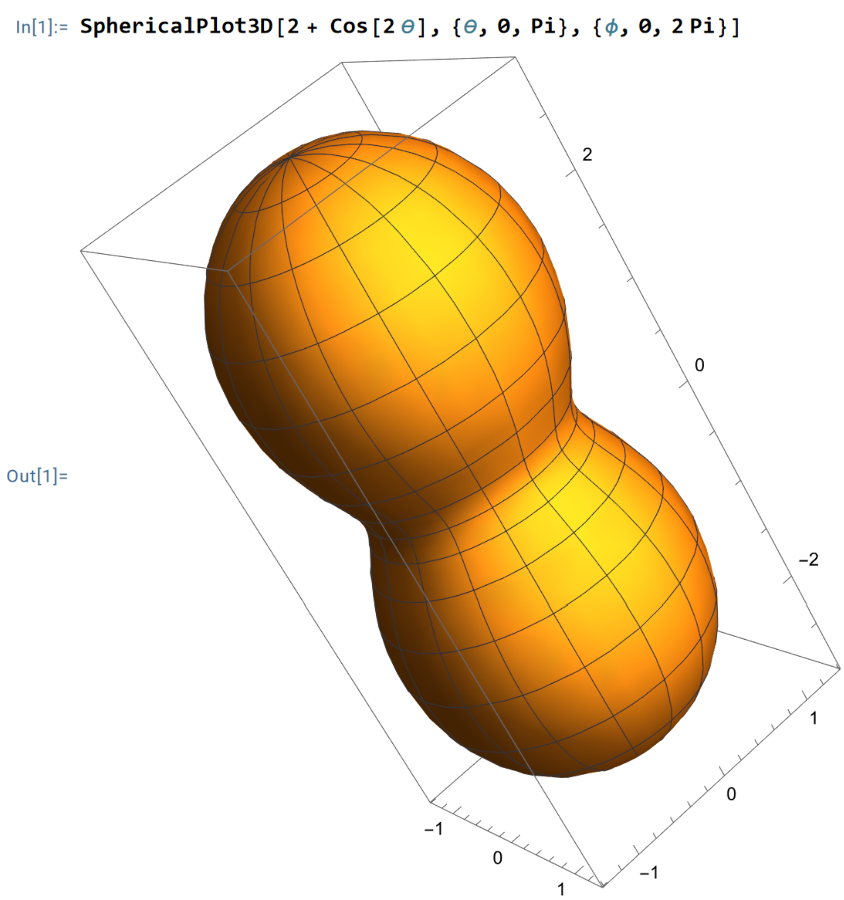

Figure 2: Peanut in 3D

It has nowhere been mentioned to assume that a is a constant, so we can assume that a is a function of θ. We can write the line element as:

$$ds^2 = a^2(d\theta^2 + f^2(\theta)d\phi^2) \tag{8}$$

with

$$f(\theta) = \frac{\sin(\theta)(2+\cos(2\theta))}{\sqrt{5+4\cos(2\theta)+3\sin^2(2\theta)}} \tag{9}$$

$$a(\theta) = \sqrt{5+4\cos(2\theta)+3\sin^2(2\theta)} \tag{10}$$

Here I have ignored scaling factors, and these can be easily incorporated into the analysis.

2. *Egg:*
We proceed using a similar analysis as above. The egg shape can be described by the following polar equation:

$$r(\theta) = \cos\theta(2+\cos(2\theta)) \tag{11}$$

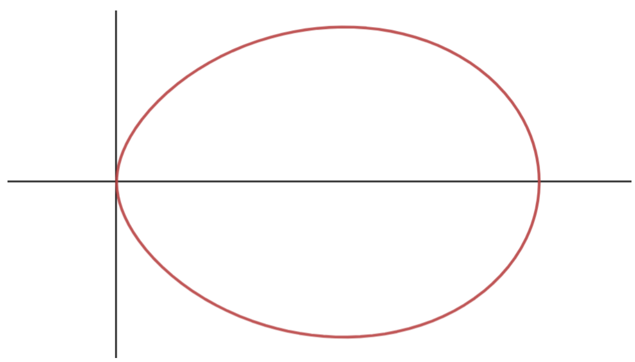

Figure 3: Egg

The map $\psi : \mathbb{E} \to \mathbb{R}^3$ is given by:

$$\psi(\theta,\phi) = (\cos\theta(2+\cos(2\theta)), \theta, \phi) \tag{12}$$

and the Jacobian of this map is given by:

$$\frac{\partial y^\alpha}{\partial x^\mu} = \begin{pmatrix} -\sin\theta(2+\cos(2\theta)) - 2\cos\theta\sin(2\theta) & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{13}$$

which leads to the induced metric:

$$(\psi^* g)_{\theta\theta} = \cos\theta(2+\cos(2\theta))^2 + \frac{1}{4}(5\sin\theta + 3\sin(3\theta))^2 \tag{14}$$

$$(\psi^* g)_{\phi\phi} = \sin^2\theta\cos^2\theta(2+\cos(2\theta))^2 \tag{15}$$

```
In[1]:= SphericalPlot3D[((Cos[θ]) (2 + Cos[2 θ])), {θ, 0, Pi}, {ϕ, 0, 2 Pi}]
```

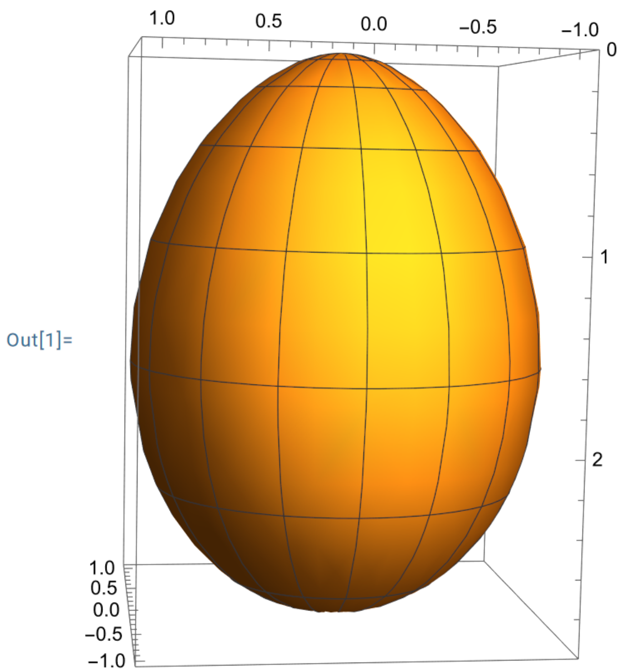

Figure 4: Egg in 3D

Both components are positive (semi-)definite, so the metric is Riemannian. Above is the Mathematica plot for this surface. We find:

$$f(\theta) = \frac{\sin\theta\cos\theta(2+\cos(2\theta))}{\sqrt{\cos\theta(2+\cos(2\theta))^2 + \frac{1}{4}(5\sin\theta + 3\sin(3\theta))^2}} \tag{16}$$

$$a(\theta) = \sqrt{\cos\theta(2+\cos(2\theta))^2 + \frac{1}{4}(5\sin\theta + 3\sin(3\theta))^2} \tag{17}$$

3. *Given metric and Earth:*
   We are given that $f(\theta) = \sin\theta(1+\epsilon\sin(2\theta))$. We need to find a and ϵ such that the polar and equatorial radius of the Earth are obtained. The polar radius of the Earth is 6356.8 km and the equatorial radius is 6378.1 km.
   For polar radius, $\phi = 0$ and for equatorial radius, $\theta = \frac{\pi}{2}$. We can use the given metric to find the polar and equatorial radius.

Problem 2

Calculate the covariant derivative of a tensor $T_{\mu\nu}$, Levi-Civita Contravariant tensor $\epsilon^{\alpha\beta\gamma\delta}$ and a covariant vector V_μ.

Solution

First a few statements are in order:

- Covariant derivative is defined in terms of a connection $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$.
- The object $\nabla(X, Y)$ is written as $\nabla_X Y$. We call the object $\nabla_X Y$ the covariant derivative of Y in the direction of X.
- The connection is linear with respect to all functions in its first argument and is Leibniz and linear in the second argument.
- We define the $\nabla(X, f) = X(f)$ for a vector field X and a function f.

First, let's define:

$$\nabla_{e_\rho} e_\nu = \Gamma^\mu_{\rho\nu} e_\mu \tag{18}$$

in standard notation.
For the covariant vector,

$$\nabla_\nu (V_\mu Y^\mu) = \nabla_\nu V_\mu Y^\mu + V_\mu \nabla_\nu Y^\mu \tag{19}$$

Recall that for a scalar inside the covariant derivative, it is the same as the partial derivative. So,

$$\partial_\nu (V_\mu Y^\mu) = \nabla_\nu V_\mu Y^\mu + V_\mu \nabla_\nu Y^\mu \tag{20}$$

$$Y^\mu \nabla_\nu V_\mu = Y^\mu (\partial_\nu V_\mu - \Gamma^\rho_{\nu\mu} V_\rho) \tag{21}$$

$$\nabla_\nu V_\mu = \partial_\nu V_\mu - \Gamma^\rho_{\nu\mu} V_\rho \tag{22}$$

Next, for $T_{\mu\nu}$,

$$\nabla_\rho (T_{\mu\nu} Y^\mu Z^\nu) = \nabla_\rho T_{\mu\nu} Y^\mu Z^\nu + T_{\mu\nu} \nabla_\rho Y^\mu Z^\nu + T_{\mu\nu} Y^\mu \nabla_\rho Z^\nu \tag{23}$$

$$= \nabla_\rho T_{\mu\nu} Y^\mu Z^\nu + T_{\mu\nu} Z^\nu (\partial_\rho Y^\mu + \Gamma^\mu_{\rho\sigma} Y^\sigma) + T_{\mu\nu} Y^\mu (\partial_\rho Z^\nu + \Gamma^\nu_{\rho\sigma} Z^\sigma) \tag{24}$$

This reduces to

$$\nabla_\rho T_{\mu\nu} = \partial_\rho T_{\mu\nu} - \Gamma^\sigma_{\rho\mu} T_{\sigma\nu} - \Gamma^\sigma_{\rho\nu} T_{\mu\sigma} \tag{25}$$

Following a similar logic,

$$\nabla_\mu \epsilon^{\alpha\beta\gamma\delta} = \partial_\mu \epsilon^{\alpha\beta\gamma\delta} + \Gamma^\alpha_{\mu\sigma} \epsilon^{\sigma\beta\gamma\delta} + \Gamma^\beta_{\mu\sigma} \epsilon^{\alpha\sigma\gamma\delta} + \Gamma^\gamma_{\mu\sigma} \epsilon^{\alpha\beta\sigma\delta} + \Gamma^\delta_{\mu\sigma} \epsilon^{\alpha\beta\gamma\sigma} \tag{26}$$

Problem 3

Show that Ricci Tensor can be written in the form

$$R_{\mu\nu} = \frac{1}{\sqrt{g}} \frac{\partial(\sqrt{g}\Gamma^\alpha_{\mu\nu})}{\partial x^\alpha} - \frac{\partial^2 \ln \sqrt{g}}{\partial x^\mu \partial x^\nu} - \Gamma^\alpha_{\mu\beta} \Gamma^\beta_{\nu\alpha}$$

Solution

The Ricci tensor $R_{\mu\nu}$ is given by:

$$R_{\mu\nu} = R^{\rho}_{\mu\rho\nu} \tag{27}$$
$$= \partial_\rho \Gamma^{\rho}_{\nu\mu} - \partial_\nu \Gamma^{\rho}_{\rho\mu} + \Gamma^{\lambda}_{\nu\mu}\Gamma^{\rho}_{\rho\lambda} - \Gamma^{\lambda}_{\rho\mu}\Gamma^{\rho}_{\nu\lambda} \tag{28}$$

We just have to show that equation (28) is equivalent to the given expression.
The first observation we must make is:

$$\mathrm{Tr}(\ln A) = \ln(\det A) \tag{29}$$

where Tr is the trace of the matrix. From this, we notice that the derivative of both sides must be the same, and this gives us:

$$\Gamma^{\mu}_{\mu\nu} = \frac{1}{\sqrt{g}}\partial_\nu \sqrt{g} \tag{30}$$

Using this identity,

$$\Gamma^{\sigma}_{\sigma\alpha}\Gamma^{\alpha}_{\mu\nu} + \partial_\alpha \Gamma^{\alpha}_{\mu\nu} - \partial_\mu \Gamma^{\sigma}_{\sigma\nu} - \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} \tag{31}$$

By making numerous little substitutions in each term and observing that the Christoffel Symbols are symmetric in the lower indices, we can show that the above expression is equivalent to the given expression.
Let the following transformations be made in the above expression.

First Term	$\sigma \to \rho$	$\alpha \to \lambda$	$\mu \leftrightarrow \nu$
Second Term	$\alpha \to \rho$		$\mu \leftrightarrow \nu$
Third Term	$\sigma \to \rho$		$\mu \leftrightarrow \nu$
Fourth Term	$\alpha \to \lambda$	$\beta \to \rho$	

This gives us (28) exactly.

Problem 4

Prove Bianchi Identities.

Solution

The covariant derivatives satisfy the Jacobi identity

$$[\nabla_\mu, [\nabla_\nu, \nabla_\kappa]] + [\nabla_\nu, [\nabla_\kappa, \nabla_\mu]] + [\nabla_\kappa, [\nabla_\mu, \nabla_\nu]] = 0 \tag{32}$$

This can be verified directly, but it is also known that pretty much any associative algebra will satisfy the Jacobi identity, and the elements $\nabla_1, \ldots \nabla_n$ basically generate a formal associative

algebra.
Then letting the Jacobi identity act on any vector field X_ρ we get for one of the terms

$$[\nabla_\mu, [\nabla_\nu, \nabla_\kappa]]X^\rho = \nabla_\kappa[\nabla_\mu, \nabla_\nu]X^\rho - [\nabla_\mu, \nabla_\nu]\nabla_\kappa X^\rho \tag{33}$$
$$= \nabla_\kappa(R^\rho_{\sigma\mu\nu}X^\sigma) - R^\rho_{\sigma\mu\nu}\nabla_\kappa X^\sigma + R^\sigma_{\kappa\mu\nu}\nabla_\sigma X^\rho \tag{34}$$
$$= \nabla_\kappa R^\rho_{\sigma\mu\nu}X^\sigma + R^\rho_{\sigma\mu\nu}\nabla_\kappa X^\sigma - R^\rho_{\sigma\mu\nu}\nabla_\kappa X^\sigma + R^\sigma_{\kappa\mu\nu}\nabla_\sigma X^\rho \tag{35}$$
$$= \nabla_\kappa R^\rho_{\sigma\mu\nu}X^\sigma + R^\sigma_{\kappa\mu\nu}\nabla_\sigma X^\rho \tag{36}$$

Now writing this into the Jacobi identity gives

$$0 = [\nabla_\kappa R^\rho_{\sigma\mu\nu} + \text{cyclic permutations}]X^\sigma + [R^\sigma_{\kappa\mu\nu}\nabla_\sigma + \text{cyclic permutations}]X^\rho \tag{37}$$

where the cyclic permutations are on κ, μ, ν.

This interesting result can give both the Bianchi Identities in two ways, if we use the following clever manipulations of X^ρ. Consider at any general point x the following choices of X^σ and the covariant derivative $\nabla_\sigma X^\rho$:

First Choice	$X^\sigma(x) = \delta^\sigma_\alpha$	$\nabla_\sigma X^\rho(x) = 0$
Second Choice	$X^\sigma(x) = 0$	$\nabla_\sigma X^\rho(x) = \delta^\rho_\sigma$

The first choice gives the differential Bianchi identity, and the second choice gives the algebraic Bianchi identity.
In this proof I have assumed torsionlessness, but the generalization to the torsionful case is similar, though more laborous.
For clarity, I state these identities below:

1. Differential Bianchi Identity:

$$\nabla_\kappa R^\rho_{\sigma\mu\nu} + \nabla_\mu R^\rho_{\sigma\nu\kappa} + \nabla_\nu R^\rho_{\sigma\kappa\mu} = 0 \tag{38}$$

2. Algebraic Bianchi Identity:

$$R^\rho_{\sigma\mu\nu} + R^\rho_{\nu\kappa\mu} + R^\rho_{\mu\sigma\nu} = 0 \tag{39}$$

Let's move a step further and prove the contracted Bianchi Identities.

$$g^{\rho\mu}g_{\rho\rho}[\nabla_\kappa R^\rho_{\sigma\mu\nu} + \nabla_\mu R^\rho_{\sigma\nu\kappa} + \nabla_\nu R^\rho_{\sigma\kappa\mu}] = 0 \tag{40}$$

By using the fact that the metric is covariantly constant, we can write the above expression as:

$$\nabla_\kappa R_{\sigma\nu} + \nabla_\mu R^\mu_{\sigma\nu\kappa} - \nabla_\nu R_{\sigma\kappa} = 0 \tag{41}$$

Contracting it with $g^{\sigma\nu}$, we get:

$$\nabla_\kappa R - \nabla_\mu R^\mu_\kappa - \nabla_\nu R^\nu_\kappa = 0 \tag{42}$$
$$\nabla_\kappa R - 2\nabla_\mu R^\mu_\kappa = 0 \tag{43}$$

By defining the Einstein Tensor as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \tag{44}$$

We have effectively proven that

$$\nabla_\mu G^{\mu\nu} = 0 \tag{45}$$

Problem 5

Calculate Killing vectors for S^2 sphere. What is the significance of your result?

Solution

The metric for the 2-sphere is given by:

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2 \tag{46}$$

The Killing vector satisfies:

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0 \tag{47}$$

We have to figure out what are the symmetries of the metric. The metric is independent of ϕ, and thus we can expect a Killing vector to denote this symmetry. We can write the Killing vector as:

$$K = a\partial_\phi \tag{48}$$

We can keep going on in this method and try to guess the other Killing vectors, but it will be easier solve the differential equation (47) given our metric (46).
Let's calculate the Christoffel Symbols:

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) \tag{49}$$

for the metric:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \tag{50}$$

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^{-2}\theta \end{pmatrix} \tag{51}$$

We get $\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$ and $\Gamma^\phi_{\phi\theta} = \cot\theta = \Gamma^\phi_{\theta\phi}$ with rest all zero.
Covariant derivative of a covariant vector is given by:

$$\nabla_\mu K_\nu = \partial_\mu K_\nu - \Gamma^\sigma_{\mu\nu}K_\sigma \tag{52}$$

The differential equation is now

$$\partial_\mu K_\nu + \partial_\nu K_\mu = 2\Gamma^\sigma_{\mu\nu}K_\sigma \tag{53}$$

We know that for $(\sigma, \mu, \nu) = (\theta, \phi, \phi)$ or (ϕ, ϕ, θ) the connection is non-zero. First, let's look at all other cases.

$$\partial_\mu K_\nu + \partial_\nu K_\mu = 0 \tag{54}$$

This reduces to

$$\partial_\theta K_\theta = 0 \tag{55}$$

For $(\sigma, \mu, \nu) = (\theta, \phi, \phi)$

$$\partial_\phi K_\phi = -\sin\theta\cos\theta K_\theta \tag{56}$$

For $(\sigma, \mu, \nu) = (\phi, \phi, \theta)$ and symmetric case

$$\partial_\phi K_\theta + \partial_\theta K_\phi = 2\cot\theta K_\phi \tag{57}$$

(55) implies that $K_\theta = f(\phi)$ i.e it is independent of θ.
Putting this in (56) gives:

$$\partial_\phi K_\phi = -\sin\theta\cos\theta f(\phi) \tag{58}$$

Integrating with respect to ϕ gives:

$$K_\phi = -\sin\theta\cos\theta \int_{\phi_0}^{\phi} f(\phi')\phi' + g(\theta) \tag{59}$$

$$= -\sin\theta\cos\theta F(\phi) + g(\theta) \tag{60}$$

which on putting in (57) gives:

$$f'(\phi) + g'(\theta) - \cos(2\theta)F(\phi) + 2\cos^2\theta F(\phi) = 2\cot\theta g(\theta) \tag{61}$$

$$F(\phi) + f'(\phi) = -g'(\theta) + 2\cot\theta g(\theta) \tag{62}$$

which is separable.

$$2\cot\theta g(\theta) - g'(\theta) = \lambda \tag{63}$$

$$F(\phi) + f'(\phi) = \lambda \tag{64}$$

(63) gives:

$$g(\theta) = \lambda\cos\theta\sin\theta + C\sin^2\theta \tag{65}$$

and (64) along with boundary condition gives:

$$F(\phi) = -A\cos\phi + B\sin\phi - \lambda \tag{66}$$

for

$$f(\phi) = A\sin\phi + B\cos\phi \tag{67}$$

The Killing vector is then given by:

$$K_\theta = A\sin\phi + B\cos\phi \tag{68}$$
$$K_\phi = \cos\theta\sin\theta(A\cos\phi - B\sin\phi) + C\sin^2\theta \tag{69}$$

We can make this look more reasonable by converting to vector rather than one-form notation by multiplying by the metric, and then writing in a cleaner manner.

$$K = \sum_{A,B,C} \begin{cases} A\left(\sin\phi\cdot\partial_\theta + \cot\theta\cos\phi\cdot\partial_\phi\right) \\ B\left(\cos\phi\cdot\partial_\theta - \cot\theta\sin\phi\cdot\partial_\phi\right) \\ C\cdot\partial_\phi \end{cases} \tag{70}$$

where A, B, C are arbitrary constants.

Three independent parameters in the general solution: A, B and C. There are three independent basis vectors in the space of solutions of $\nabla_{(\mu}K_{\nu)} = 0$. In other words, there are three independent basis vectors in the space of isometries. These Killing vectors correspond to rotations around the x, y and z axis, respectively, i.e. all possible rotations in the embedding space $\mathbb{R}^3$.

In general, the maximal number of Killing vectors (isometries) in N-dimensional space is $N(N+1)/2$. Here, $N = 2 \implies 2(2+1)/2 = 3$. S^2 is maximally symmetric!

Problem 6

Compute the Riemann tensor and Ricci tensor of the unit sphere S^2.

Solution

The Riemann tensor is given by:

$$R^\rho_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \tag{71}$$

We have already calculated the Christoffel Symbols in the previous problem. We can use these to calculate the Riemann tensor. The only non-zero Christoffel Symbols are:

$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$$
$$\Gamma^\phi_{\phi\theta} = \cot\theta = \Gamma^\phi_{\theta\phi}$$

From the above, and from the algebraic Bianchi identity, we can see that the Riemann tensor must have μ,ν indices must be different, not both θ or both ϕ. Also, a similar argument shows that the ρ,σ indices must be different.

The only non-zero components of the Riemann tensor are:

$$R^{\theta}_{\phi\theta\phi} = \sin^2\theta \tag{72}$$
$$R^{\phi}_{\theta\phi\theta} = 1 \tag{73}$$
$$R^{\theta}_{\phi\phi\theta} = -\sin^2\theta \tag{74}$$
$$R^{\phi}_{\theta\theta\phi} = -1 \tag{75}$$

The Ricci tensor is given by:

$$R_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix} \tag{76}$$

Problem 7

Consider the 1+2 dimensional line element

$$ds^2 = dt^2 - A(t)^2\left(\frac{dx^2}{1-\kappa x^2} + x^2 dy^2\right)$$

where $A(t)$ is a function of time and κ is a constant. Determine Christoffel Symbols and Geodesic equations for this metric.

Solution

This is the Robertson-Walker metric for a 1+2 dimensional space. It would make more sense to change x to r and y to θ for clarity. The metric is then:

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{A(t)^2}{1-\kappa r^2} & 0 \\ 0 & 0 & -A(t)^2 r^2 \end{bmatrix} \tag{77}$$
$$g^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\kappa r^2-1}{A(t)^2} & 0 \\ 0 & 0 & -\frac{1}{A(t)^2 r^2} \end{bmatrix} \tag{78}$$

This calculation was too intense to do by hand and I used Mathematica to calculate the Christoffel Symbols. The input is:

```
                                          ( 1          0                  0          )
In[1]:= ResourceFunction["ChristoffelSymbol"][( 0  -(A[t]^2)/(1 - k*r^2)      0          ), {t, r, θ}] // MatrixForm
                                          ( 0          0              -A[t]^2*r^2     )
```

Figure 5: FRW Christoffel Symbols Input

The corresponding output is on the next page.

To read it, Γ^{i}_{jk} has Mathematica notation {i, 0, 2}, {j, 0, 2}, {k, 0, 2}.

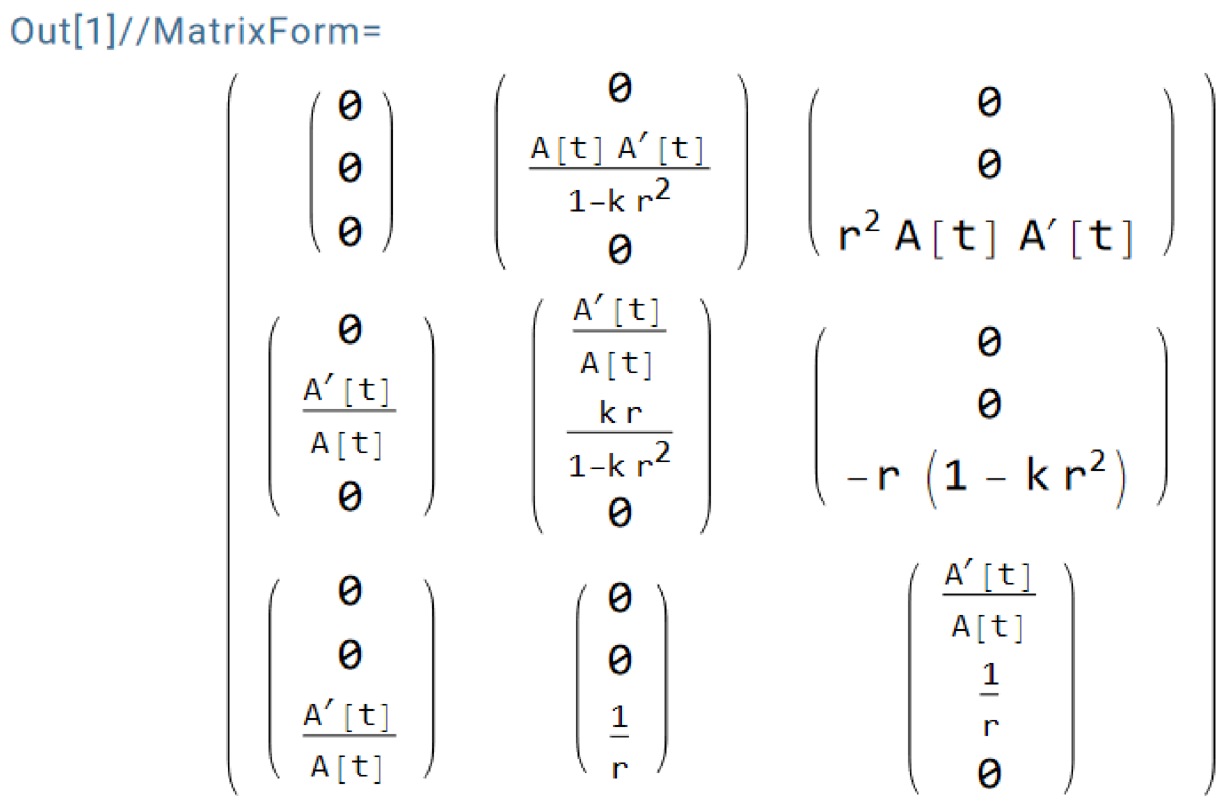

Figure 6: FRW Christoffel Symbols Output

A nice way to obtain both the geodesic equations and the Christoffel symbols is to use the Lagrangian:

$$\mathcal{L} = -\frac{1}{2}\left(-\dot{t}^2 + A(t)^2\frac{\dot{r}^2}{1-\kappa r^2} + A(t)^2 r^2\dot{\theta}^2\right) \tag{79}$$

Here, we can use the Euler-Lagrange equations to obtain the geodesic equations.

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{t}}\right) - \frac{\partial\mathcal{L}}{\partial t} = 0 \tag{80}$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{r}}\right) - \frac{\partial\mathcal{L}}{\partial r} = 0 \tag{81}$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta} = 0 \tag{82}$$

And these lead to

$$\ddot{t} + \frac{A(t)\dot{A}(t)}{1-\kappa r^2}\dot{r}^2 + A(t)\dot{A}(t)r^2\dot{\theta}^2 = 0 \tag{83}$$

for the time part

$$\ddot{r} + \frac{\kappa r}{1-\kappa r^2}\dot{r}^2 + 2\frac{\dot{A}(t)}{A(t)}\dot{r}\dot{t} - r(1-\kappa r^2)\dot{\theta}^2 = 0 \tag{84}$$

for the radial part

$$\ddot{\theta} + 2\frac{\dot{A}(t)}{A(t)}\dot{\theta}\dot{t} + \frac{2}{r}\dot{r}\dot{\theta} = 0 \tag{85}$$

for the angular part. It is easy to check to what Christoffel symbols the terms correspond to.

Problem 8

Find out the symmetry properties of Riemann and Weyl tensors. Determine the number of independent components of Riemann, Ricci and Weyl tensors in 3 and 4 dimensions.

Solution

1. *Riemann Tensor:* these can be derived by considering the abstract definitions of R

Symmetry	Abstract Geometric Notation	Component Notation
Skew Symmetry	$R(u,v) = -R(v,u)$	$R_{ab(cd)} = 0$
Skew Symmetry	$\langle R(u,v)w,z\rangle = -\langle R(u,v)z,w\rangle$	$R_{(ab)cd} = 0$
Interchange	$\langle R(u,v)w,z\rangle = \langle R(w,z)u,v\rangle$	$R_{abcd} = R_{cdab}$
Algebraic Bianchi Identity	$R(u,v)w + R(v,w)u + R(w,u)v = 0$	$R_{a[bcd]} = 0$
Differential Bianchi Identity	$\nabla_u R(v,w) + \nabla_v R(w,u) + \nabla_w R(u,v) = 0$	$R_{ab[cd;e]} = 0$

The number of independent components of the Riemann tensor in n dimensions is given by:

$$\frac{n^2(n^2-1)}{12} \tag{86}$$

which gives 6 in 3 dimensions and 20 in 4 dimensions.

2. *Weyl Tensor:* has exactly the same symmetries as above and is traceless

Symmetry	Abstract Geometric Notation	Component Notation
Skew Symmetry	$C(u,v) = -C(v,u)$	$C_{ab(cd)} = 0$
Skew Symmetry	$\langle C(u,v)w,z\rangle = -\langle C(u,v)z,w\rangle$	$C_{(ab)cd} = 0$
Algebraic Bianchi Identity	$C(u,v)w + C(v,w)u + C(w,u)v = 0$	$C_{a[bcd]} = 0$
Trace Free	$\mathrm{Tr}\ C(u,\cdot)v = 0$	$C^a_{bac} = 0$

The number of independent components of the Weyl tensor in n dimensions is given by:

$$\frac{1}{12}n(n+1)(n+2)(n-3) \tag{87}$$

which gives 0 in 3 dimensions and 10 in 4 dimensions.

The number of independent components of the Ricci tensor is the difference between the number of independent components of the Riemann tensor and the Weyl tensor. This gives 6 in 3 dimensions and 10 in 4 dimensions.

Problem 9

Conformal transformation is defined as $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega^2(x,t)g_{\mu\nu}$. Show that null geodesic will remain null geodesic after conformal transformation.

Solution

I have corrected the question to have only a positive scale factor multiplied instead of some arbitrary $f(x)$.

Another correction is that null geodesics map to null geodesics. Curves retain their causal structure anyway, but all kinds of geodesics may not be necessarily be mapped homeomorphically to corresponding geodesics. For example, a timelike geodesic will map to a timelike curve, not necessarily a geodesic.

We can look at the geodesic equation for light (null curves) and show that we can modify the parameters to obtain a similar equation after a conformal transformation.
The Lagrangian for a null curve is

$$\mathcal{L} = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0 \tag{88}$$

where the coordinates are parametrised by affine proper time.
Let's consider what the Christoffel Symbols are in the conformally transformed metric.

$$\begin{aligned}
\Gamma^\sigma_{\mu\nu}[\tilde{g}] &= \frac{1}{2}\tilde{g}^{\sigma\rho}\left(\partial_\mu \tilde{g}_{\rho\nu} + \partial_\nu \tilde{g}_{\mu\rho} - \partial_\rho \tilde{g}_{\mu\nu}\right) && (89)\\
&= \frac{1}{2}\Omega^{-2}g^{\sigma\rho}\left(\partial_\mu(\Omega^2 g_{\rho\nu}) + \partial_\nu(\Omega^2 g_{\mu\rho}) - \partial_\rho(\Omega^2 g_{\mu\nu})\right) && (90)\\
&= \Gamma^\sigma_{\mu\nu}[g] + \Omega^{-1}\left(\delta^\sigma_\mu \partial_\nu \Omega + \delta^\sigma_\nu \partial_\mu \Omega - g_{\mu\nu}g^{\sigma\rho}\partial_\rho \Omega\right) && (91)\\
&= \Gamma^\sigma_{\mu\nu}[g] + \Omega^{-1}\left(\delta^\sigma_\mu \nabla_\nu \Omega + \delta^\sigma_\nu \nabla_\mu \Omega - g_{\mu\nu}\nabla^\sigma \Omega\right) && (92)
\end{aligned}$$

Given an affinely parametrised geodesic in the metric g we obtain that:

$$\frac{d^2x^\sigma}{d\tau^2} + \Gamma^\sigma_{\mu\nu}[g]\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{93}$$

which reduces to

$$\frac{d^2x^\sigma}{d\tau^2} + \Gamma^\sigma_{\mu\nu}[\tilde{g}]\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = \Omega^{-1}\left(\delta^\sigma_\mu \nabla_\nu \Omega + \delta^\sigma_\nu \nabla_\mu \Omega - g_{\mu\nu}\nabla^\sigma \Omega\right)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \tag{94}$$

We can now use (88) to get that

$$\frac{d^2x^\sigma}{d\tau^2} + \Gamma^\sigma_{\mu\nu}[\tilde{g}]\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 2\frac{1}{\Omega}\frac{dx^\sigma}{d\tau}\frac{d\Omega}{d\tau} \tag{95}$$

We now choose the affine parameter to be such that $\frac{d\Omega}{d\tau} = 0$. This is equivalent of saying that the conformal factor is constant along the geodesic.
Since the RHS of (95) is zero, null geodesics are mapped to null geodesics under a conformal transformation.

Problem 10

Calculate the stress-energy tensor of a free electromagnetic field A_μ.

Solution

In this solution, I am going with the method of pure abstract geometry. We try to find top-forms to integrate over our spacetime to get the action. From this action, we can find the stress energy tensor by varying with respect to the metric.

We need a few notions before going to the final solution:

1. *Wedge Product:* Consider a p-form ω and a q-form η. The wedge product of these two forms is a $(p+q)$-form and is denoted by $\omega \wedge \eta$. The wedge product is anti-symmetric and associative.
$$(\omega \wedge \eta)_{\mu_1 \dots \mu_p \nu_1 \dots \nu_q} = \frac{(p+q)!}{p!q!} \omega_{[\mu_1 \dots \mu_p} \eta_{\nu_1 \dots \nu_q]} \tag{96}$$
As a particular example, the wedge product of two 1-forms is a 2-form and is given by:
$$(\omega \wedge \eta)_{\mu\nu} = \omega_\mu \eta_\nu - \omega_\nu \eta_\mu \tag{97}$$

2. *Exterior Derivative:* The exterior derivative of a p-form is a (p+1)-form and is denoted by $d\omega$. It is defined as:
$$(d\omega)_{\mu_1 \dots \mu_{p+1}} = (p+1) \partial_{[\mu_1} \omega_{\mu_2 \dots \mu_{p+1}]} \tag{98}$$
or equivalently
$$d\omega = \frac{1}{p!} \frac{\partial \omega_{\mu_1 \dots \mu_p}}{\partial x^\nu} dx^\nu \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \tag{99}$$
This operation is antisymmetric and hence $d^2 = 0$ for any p-form.
In particular, the exterior derivative of a 1-form is a 2-form and is given by:
$$d\omega = \frac{1}{2} (\partial_\mu \omega_\nu - \partial_\nu \omega_\mu) dx^\mu \wedge dx^\nu \tag{100}$$

3. *Hodge Star Operator:* The Hodge star operator is a map from p-forms to (n-p)-forms, where n is the dimension of the manifold. It is denoted by $\star$. It is defined as:
$$(\star\omega)_{\mu_1 \dots \mu_{n-p}} = \frac{1}{p!} \sqrt{|g|} \epsilon_{\mu_1 \dots \mu_{n-p} \nu_1 \dots \nu_p} \omega^{\nu_1 \dots \nu_p} \tag{101}$$
Interestingly, it is coordinate independent.
The Hodge Dual helps in defining a metric independent inner product on the space of p-forms.
If $\omega, \eta \in \Lambda^p(M)$, then
$$\langle \omega, \eta \rangle = \int_M \omega \wedge \star\eta \tag{102}$$
we can integrate the latter term over the manifold as it is a top form. For integration of scalar functions over the manifold, we must include the volume form.

Observe that the electromagnetic potential is a one-form. We define the field strength tensor as the exterior derivative of the potential.

$$F = dA \tag{103}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{104}$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \tag{105}$$

which is an antisymmetric 2-form. Seeing that $2+2=4$ and $2+(4-2)=4$, we can define two actions for this spacetime. Note that, there is no mention of metric in the definition of the Wedge Product and the Exterior Derivative. This is a purely topological construction independent of the metric. On the other hand, the Hodge Dual involves the deterimant of metric, and hence is dependent on the geometry.

1. *Topological Action:*

$$S_{\text{top}} = \int_M F \wedge F \tag{106}$$

   which reduces to

$$S_{\text{top}} = -2 \int_M d^4x \, \mathbf{E} \cdot \mathbf{B} \tag{107}$$

   However, since we can integrate (106) by parts to get a total derivative under the integral, we can see that the contribution of the topological action to the classical equations of motion is zero.

2. *Maxwell Action:*

$$\begin{aligned} S_{\text{Maxwell}} &= -\frac{1}{2} \int_M \langle F, F \rangle & (108) \\ &= -\frac{1}{2} \int_M F \wedge \star F & (109) \\ &= -\frac{1}{4} \int_M d^4x \sqrt{-g} g^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} & (110) \\ &= -\frac{1}{4} \int_M d^4x \sqrt{-g} F_{\mu\nu} F^{\mu\nu} & (111) \end{aligned}$$

   This on variation with the parameters gives the equation of motion.
   We can write these equations by (not writing the complete derivation since the question is not about that)

$$d \star F = 0 \tag{112}$$

which is the same as

$$\nabla_\mu F^{\mu\nu} = 0 \tag{113}$$

written in terms of the covariant derivative.

We can do much more with this action, if we include the inner product of electromagnetic potential with the current density (both are one-forms). The new equation would be:

$$\begin{aligned} S_{\text{Maxwell}} &= \int_M -\frac{1}{2}\langle F, F\rangle + \langle A, J\rangle & (114) \\ &= \int_M -\frac{1}{2}F \wedge \star F + A \wedge \star J & (115) \end{aligned}$$

This action is Lorentz invariant by construction, but we also need it be gauge invariant. By acting a transformation on the potential $A \to A + d\alpha$, we see:

$$S \to S + \int_M d\alpha \wedge \star J \tag{116}$$

After integration by parts, we obtain that

$$d \star J = 0 \tag{117}$$

is the condition for gauge invariance. We can observe that this is just conservation of current written in a differential form language. The new equations of motion come out to be:

$$d \star F = \star J \tag{118}$$

which automatically ensures (117).

We can now calculate the stress-energy tensor by varying the action with respect to the metric. For simplicity, let's consider the Maxwell action without the source current density.

$$\begin{aligned} \delta S_{\text{Maxwell}} &= -\frac{1}{\cdot}\int_M \delta(\sqrt{-g}F_{\mu\nu}F^{\mu\nu})d^4x & (119) \\ &= -\frac{1}{4}\int_M d^4x\left[\sqrt{-g}(-\frac{1}{2}g_{\mu\nu}\delta g^{\mu\nu})F_{\alpha\beta}F^{\alpha\beta} + \sqrt{-g}\delta(F_{\mu\nu}F^{\mu\nu})\right] & (120) \end{aligned}$$

We can reduce $\delta(F_{\mu\nu}F^{\mu\nu})$ to $\delta(F_{\alpha\beta}F^{\alpha\beta})$ by using the fact that this is a coordinate independent contraction. Writing $F^{\alpha\beta} = g^{\alpha\gamma}g^{\beta\gamma}F_{\gamma\delta}$ we can solve:

$$\frac{\delta(g^{\alpha\gamma}g^{\beta\gamma}F_{\gamma\delta}F_{\alpha\beta})}{\delta g^{\mu\nu}} = g^{\alpha\gamma}F_{\alpha\mu}F_{\gamma\nu} + g^{\beta\delta}F_{\mu\beta}F_{\nu\delta} = 2g^{\sigma\rho}F_{\mu\rho}F_{\nu\sigma} \tag{121}$$

Using the definition of the stress-energy tensor, we can write:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\text{Maxwell}}}{\delta g^{\mu\nu}} = g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \tag{122}$$

Problem 11

Check whether the 2D space with metric $ds^2 = xdy^2 + ydx^2$ is curved or not by calculating the curvature scalar.

Solution

The metric looks like:

$$g_{\mu\nu} = \begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix} \tag{123}$$

Using Mathematica,

```
In[1]:= g = {{y, 0}, {0, x}};

In[2]:= g // MatrixForm
Out[2]//MatrixForm=
```

$$\begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}$$

```
In[3]:= Rμν = ResourceFunction["RicciCurvature"][g, {x, y}];

In[4]:= Rμν // MatrixForm
Out[4]//MatrixForm=
```

$$\begin{pmatrix} \frac{x+y}{4x^2y} & 0 \\ 0 & \frac{x+y}{4xy^2} \end{pmatrix}$$

```
In[5]:= ginv = Inverse[g];

In[6]:= ginv // MatrixForm
Out[6]//MatrixForm=
```

$$\begin{pmatrix} \frac{1}{y} & 0 \\ 0 & \frac{1}{x} \end{pmatrix}$$

```
In[7]:= R = Tr[ginv.Rμν]
Out[7]=
```

$$\frac{x+y}{2x^2y^2}$$

Figure 7: Curvature Scalar Calculation

which yields that this space is curved.

Problem 12

[**Use Mathematica or Python to calculate**] Write down the metric, Christoffel symbols, Riemann tensor, Ricci tensor, Ricci scalar and Weyl tensor.

Solution

I have attached the Mathematica implementation of these calculations in the following pages. The variable names are easy to follow.

In[1]:= `ThreeEuclidean = ResourceFunction["MetricTensor"]["Euclidean"];`

In[2]:= `ChristoffelThreeEuclidean = ResourceFunction["ChristoffelSymbols"][ThreeEuclidean];`

In[3]:= `RiemannThreeEuclidean = ResourceFunction["RiemannTensor"][ThreeEuclidean];`

In[4]:= `RicciCurvThreeEuclidean = ResourceFunction["RicciTensor"][ThreeEuclidean];`

In[5]:= `ThreeEuclidean["MatrixRepresentation"] // MatrixForm`

Out[5]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In[6]:= `ChristoffelThreeEuclidean["TensorRepresentation"] // MatrixForm`

Out[6]//MatrixForm=

$$\begin{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix}$$

In[7]:= `RiemannThreeEuclidean["ReducedTensorRepresentation"] // MatrixForm`

Out[7]//MatrixForm=

$$\begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{pmatrix}$$

In[8]:= `RicciCurvThreeEuclidean["MatrixRepresentation"] // MatrixForm`

Out[8]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In[9]:= `RicciCurvThreeEuclidean["RicciScalar"]`

Out[9]= 0

In[1]:=
```
TwoSphere =
  ResourceFunction["MetricTensor"][DiagonalMatrix[{R^2, (R * Sin[ψ]) ^2}], {ψ, θ}];
```

In[2]:=
```
ChristoffelTwoSphere = ResourceFunction["ChristoffelSymbols"][TwoSphere];
```

In[3]:=
```
RiemannTwoSphere = ResourceFunction["RiemannTensor"][TwoSphere];
```

In[4]:=
```
RicciCurvTwoSphere = ResourceFunction["RicciTensor"][TwoSphere];
```

In[5]:=
```
TwoSphere["MatrixRepresentation"] // MatrixForm
```

Out[5]//MatrixForm=

$$\begin{pmatrix} R^2 & 0 \\ 0 & R^2 \operatorname{Sin}[\psi]^2 \end{pmatrix}$$

In[6]:=
```
ChristoffelTwoSphere["TensorRepresentation"] // MatrixForm
```

Out[6]//MatrixForm=

$$\begin{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ -\operatorname{Cos}[\psi] \operatorname{Sin}[\psi] \end{pmatrix} \\ \begin{pmatrix} 0 \\ \operatorname{Cot}[\psi] \end{pmatrix} & \begin{pmatrix} \operatorname{Cot}[\psi] \\ 0 \end{pmatrix} \end{pmatrix}$$

In[7]:=
```
RiemannTwoSphere["ReducedTensorRepresentation"] // MatrixForm
```

Out[7]//MatrixForm=

$$\begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & \operatorname{Sin}[\psi]^2 \\ -\operatorname{Sin}[\psi]^2 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$$

In[8]:=
```
RicciCurvTwoSphere["MatrixRepresentation"] // Simplify // MatrixForm
```

Out[8]//MatrixForm=

$$\begin{pmatrix} 1 & 0 \\ 0 & \operatorname{Sin}[\psi]^2 \end{pmatrix}$$

In[9]:=
```
RicciCurvTwoSphere["RicciScalar"] // Simplify
```

Out[9]=

$$\frac{2}{R^2}$$

In[1]:=
```
ThreeSphere = ResourceFunction["MetricTensor"][
   DiagonalMatrix[{R^2, (R * Sin[ψ])^2, (R * Sin[ψ] * Sin[θ])^2}], {ψ, θ, φ }];
```

In[2]:=
```
ChristoffelThreeSphere = ResourceFunction["ChristoffelSymbols"][ThreeSphere];
```

In[3]:=
```
RiemannThreeSphere = ResourceFunction["RiemannTensor"][ThreeSphere];
```

In[4]:=
```
RicciCurvThreeSphere = ResourceFunction["RicciTensor"][ThreeSphere];
```

In[5]:=
```
ThreeSphere["MatrixRepresentation"] // MatrixForm
```

Out[5]//MatrixForm=

$$\begin{pmatrix} R^2 & 0 & 0 \\ 0 & R^2 \mathrm{Sin}[\psi]^2 & 0 \\ 0 & 0 & R^2 \mathrm{Sin}[\theta]^2 \mathrm{Sin}[\psi]^2 \end{pmatrix}$$

In[6]:=
```
ChristoffelThreeSphere["TensorRepresentation"] // MatrixForm
```

Out[6]//MatrixForm=

$$\begin{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ -\mathrm{Cos}[\psi]\,\mathrm{Sin}[\psi] \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ -\mathrm{Cos}[\psi]\,\mathrm{Sin}[\theta]^2\,\mathrm{Sin}[\psi] \end{pmatrix} \\ \begin{pmatrix} 0 \\ \mathrm{Cot}[\psi] \\ 0 \end{pmatrix} & \begin{pmatrix} \mathrm{Cot}[\psi] \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ -\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta] \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ \mathrm{Cot}[\psi] \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ \mathrm{Cot}[\theta] \end{pmatrix} & \begin{pmatrix} \mathrm{Cot}[\psi] \\ \mathrm{Cot}[\theta] \\ 0 \end{pmatrix} \end{pmatrix}$$

In[7]:=
```
RiemannThreeSphere["ReducedTensorRepresentation"] // MatrixForm
```

Out[7]//MatrixForm=

$$\begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & \mathrm{Sin}[\psi]^2 & 0 \\ -\mathrm{Sin}[\psi]^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & \mathrm{Sin}[\theta]^2\,\mathrm{Sin}[\psi]^2 \\ 0 & 0 & 0 \\ -\mathrm{Sin}[\theta]^2\,\mathrm{Sin}[\psi]^2 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathrm{Sin}[\theta]^2\,\mathrm{Sin}[\psi]^2 \\ 0 & -\mathrm{Sin}[\theta]^2\,\mathrm{Sin}[\psi]^2 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{Sin}[\psi]^2 \\ 0 & \mathrm{Sin}[\psi]^2 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{pmatrix}$$

In[8]:=
```
RicciCurvThreeSphere["MatrixRepresentation"] // Simplify // MatrixForm
```

Out[8]//MatrixForm=

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2\,\mathrm{Sin}[\psi]^2 & 0 \\ 0 & 0 & 2\,\mathrm{Sin}[\theta]^2\,\mathrm{Sin}[\psi]^2 \end{pmatrix}$$

In[9]:=
```
RicciCurvThreeSphere["RicciScalar"] // Simplify
```

Out[9]= $\frac{6}{R^2}$

In[1]:=
```
FourSphere = ResourceFunction["MetricTensor"][DiagonalMatrix[{R^2, (R * Sin[ψ]) ^2,
      (R * Sin[ψ] * Sin[θ]) ^2, (R * Sin[ψ] * Sin[θ] * Sin[φ]) ^2}], {ψ, θ, φ , ϕ}];
```

In[2]:=
```
ChristoffelFourSphere = ResourceFunction["ChristoffelSymbols"][FourSphere];
```

In[3]:=
```
RiemannFourSphere = ResourceFunction["RiemannTensor"][FourSphere];
```

In[4]:=
```
RicciCurvFourSphere = ResourceFunction["RicciTensor"][FourSphere];
```

In[5]:=
```
FourSphere["MatrixRepresentation"] // MatrixForm
```

Out[5]//MatrixForm=

$$\begin{pmatrix} R^2 & 0 & 0 & 0 \\ 0 & R^2\,\mathrm{Sin}[\psi]^2 & 0 & 0 \\ 0 & 0 & R^2\,\mathrm{Sin}[\theta]^2\,\mathrm{Sin}[\psi]^2 & 0 \\ 0 & 0 & 0 & R^2\,\mathrm{Sin}[\theta]^2\,\mathrm{Sin}[\varphi]^2\,\mathrm{Sin}[\psi]^2 \end{pmatrix}$$

In[6]:=
```
ChristoffelFourSphere["TensorRepresentation"] // MatrixForm
```

Out[6]//MatrixForm=

$$\begin{pmatrix}
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ -\mathrm{Cos}[\psi]\,\mathrm{Sin}[\psi] \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ -\mathrm{Cos}[\psi]\,\mathrm{Sin}[\theta]^2\,\mathrm{Sin}[\psi] \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\mathrm{Cos}[\psi]\,\mathrm{Sin}[\theta]^2\,\mathrm{Sin}[\varphi]^2\,\mathrm{Sin}[\psi] \end{pmatrix} \\
\begin{pmatrix} 0 \\ \mathrm{Cot}[\psi] \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} \mathrm{Cot}[\psi] \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ -\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta] \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]\,\mathrm{Sin}[\varphi]^2 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \\ \mathrm{Cot}[\psi] \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ \mathrm{Cot}[\theta] \\ 0 \end{pmatrix} & \begin{pmatrix} \mathrm{Cot}[\psi] \\ \mathrm{Cot}[\theta] \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\mathrm{Cos}[\varphi]\,\mathrm{Sin}[\varphi] \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \\ 0 \\ \mathrm{Cot}[\psi] \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mathrm{Cot}[\theta] \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mathrm{Cot}[\varphi] \end{pmatrix} & \begin{pmatrix} \mathrm{Cot}[\psi] \\ \mathrm{Cot}[\theta] \\ \mathrm{Cot}[\varphi] \\ 0 \end{pmatrix}
\end{pmatrix}$$

In[7]:=
```
RiemannFourSphere["ReducedTensorRepresentation"] // MatrixForm
```

Out[7]//MatrixForm=

$$\begin{pmatrix}
\begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0 & \mathrm{Sin}[\psi]^2 & 0 & 0 \\ -\mathrm{Sin}[\psi]^2 & 0 & 0 & 0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & \mathrm{Sin}[\theta]^2\,\mathrm{Sin}[\psi]^2 & 0 \\ 0&0&0&0 \\ -\mathrm{Sin}[\theta]^2\,\mathrm{Sin}[\psi]^2 & 0 & 0 & 0 \\ 0&0&0&0 \end{pmatrix} & \begin{pmatrix} \\ \\ \\ -\mathrm{Sin}[\end{pmatrix} \\
\begin{pmatrix} 0&-1&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0 \\ 0 & 0 & \mathrm{Sin}[\theta]^2\,\mathrm{Sin}[\psi]^2 & 0 \\ 0 & -\mathrm{Sin}[\theta]^2\,\mathrm{Sin}[\psi]^2 & 0 & 0 \\ 0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 & -\mathrm{Si} \end{pmatrix} \\
\begin{pmatrix} 0&0&-1&0 \\ 0&0&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0 \\ 0 & 0 & -\mathrm{Sin}[\psi]^2 & 0 \\ 0 & \mathrm{Sin}[\psi]^2 & 0 & 0 \\ 0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0 \\ 0&0 \\ 0&0 \\ 0&0&- \end{pmatrix} \\
\begin{pmatrix} 0&0&0&-1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0 \\ 0&0&0&-\mathrm{Sin}[\psi]^2 \\ 0&0&0&0 \\ 0&\mathrm{Sin}[\psi]^2&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&-\mathrm{Sin}[\theta]^2\,\mathrm{Sin}[\psi]^2 \\ 0&0&\mathrm{Sin}[\theta]^2\,\mathrm{Sin}[\psi]^2&0 \end{pmatrix} &
\end{pmatrix}$$

In[8]:= `RicciCurvFourSphere["MatrixRepresentation"] // Simplify // MatrixForm`

Out[8]//MatrixForm=

$$\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3\,\mathrm{Sin}[\psi]^2 & 0 & 0 \\ 0 & 0 & 3\,\mathrm{Sin}[\theta]^2\,\mathrm{Sin}[\psi]^2 & 0 \\ 0 & 0 & 0 & 3\,\mathrm{Sin}[\theta]^2\,\mathrm{Sin}[\varphi]^2\,\mathrm{Sin}[\psi]^2 \end{pmatrix}$$

In[9]:= `RicciCurvFourSphere["RicciScalar"] // Simplify`

Out[9]= $\frac{12}{R^2}$

```
In[1]:= schwarzschild = ResourceFunction["MetricTensor"]["Schwarzschild"];

In[2]:= ChristoffelSchwarzschild = ResourceFunction["ChristoffelSymbols"][schwarzschild];

In[3]:= RiemannSchwarzschild = ResourceFunction["RiemannTensor"][schwarzschild];

In[4]:= RicciCurvSchwarzschild = ResourceFunction["RicciTensor"][schwarzschild];

In[5]:= schwarzschild["MatrixRepresentation"] // MatrixForm
```

Out[5]//MatrixForm=

$$\begin{pmatrix} -1+\frac{2M}{r} & 0 & 0 & 0 \\ 0 & \frac{1}{1-\frac{2M}{r}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \text{Sin}[\theta]^2 \end{pmatrix}$$

```
In[6]:= ChristoffelSchwarzschild["TensorRepresentation"] // MatrixForm
```

Out[6]//MatrixForm=

$$
\begin{pmatrix}
\begin{pmatrix} 0 \\ -\frac{r^2 M \mathrm{Sin}[\theta]^2}{\left(1-\frac{2M}{r}\right)\left(-\frac{r^4 \mathrm{Sin}[\theta]^2}{1-\frac{2M}{r}}+\frac{2 r^3 M \mathrm{Sin}[\theta]^2}{1-\frac{2M}{r}}\right)} \\ 0 \\ 0 \end{pmatrix} &
\begin{pmatrix} -\frac{r^2 M \mathrm{Sin}[\theta]^2}{\left(1-\frac{2M}{r}\right)\left(-\frac{r^4 \mathrm{Sin}[\theta]^2}{1-\frac{2M}{r}}+\frac{2 r^3 M \mathrm{Sin}[\theta]^2}{1-\frac{2M}{r}}\right)} \\ 0 \\ 0 \\ 0 \end{pmatrix} &
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} &
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} \frac{M\left(-r^4 \mathrm{Sin}[\theta]^2+2 r^3 M \mathrm{Sin}[\theta]^2\right)}{r^2\left(-\frac{r^4 \mathrm{Sin}[\theta]^2}{1-\frac{2M}{r}}+\frac{2 r^3 M \mathrm{Sin}[\theta]^2}{1-\frac{2M}{r}}\right)} \\ 0 \\ 0 \\ 0 \end{pmatrix} &
\begin{pmatrix} 0 \\ -\frac{M\left(-r^4 \mathrm{Sin}[\theta]^2+2 r^3 M \mathrm{Sin}[\theta]^2\right)}{r^2\left(1-\frac{2M}{r}\right)^2\left(-\frac{r^4 \mathrm{Sin}[\theta]^2}{1-\frac{2M}{r}}+\frac{2 r^3 M \mathrm{Sin}[\theta]^2}{1-\frac{2M}{r}}\right)} \\ 0 \\ 0 \end{pmatrix} &
\begin{pmatrix} 0 \\ 0 \\ -\frac{r\left(-r^4 \mathrm{Sin}[\theta]^2+2 r^3 M \mathrm{Sin}[\theta]^2\right)}{-\frac{r^4 \mathrm{Sin}[\theta]^2}{1-\frac{2M}{r}}+\frac{2 r^3 M \mathrm{Sin}[\theta]^2}{1-\frac{2M}{r}}} \\ 0 \end{pmatrix} &
\begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{r \mathrm{Sin}[\theta]^2\left(-r^4 \mathrm{Sin}[\theta]^2+2 r^3 M \mathrm{Sin}[\theta]^2\right)}{-\frac{r^4 \mathrm{Sin}[\theta]^2}{1-\frac{2M}{r}}+\frac{2 r^3 M \mathrm{Sin}[\theta]^2}{1-\frac{2M}{r}}} \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} &
\begin{pmatrix} 0 \\ 0 \\ \frac{r\left(-\frac{r^2 \mathrm{Sin}[\theta]^2}{1-\frac{2M}{r}}+\frac{2 r M \mathrm{Sin}[\theta]^2}{1-\frac{2M}{r}}\right)}{-\frac{r^4 \mathrm{Sin}[\theta]^2}{1-\frac{2M}{r}}+\frac{2 r^3 M \mathrm{Sin}[\theta]^2}{1-\frac{2M}{r}}} \\ 0 \end{pmatrix} &
\begin{pmatrix} 0 \\ \frac{r\left(-\frac{r^2 \mathrm{Sin}[\theta]^2}{1-\frac{2M}{r}}+\frac{2 r M \mathrm{Sin}[\theta]^2}{1-\frac{2M}{r}}\right)}{-\frac{r^4 \mathrm{Sin}[\theta]^2}{1-\frac{2M}{r}}+\frac{2 r^3 M \mathrm{Sin}[\theta]^2}{1-\frac{2M}{r}}} \\ 0 \\ 0 \end{pmatrix} &
\begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{r^2 \mathrm{Cos}[\theta]\, \mathrm{Sin}[\theta]\left(-\frac{r^2 \mathrm{Sin}[\theta]^2}{1-\frac{2M}{r}}+\frac{2 r M \mathrm{Sin}[\theta]^2}{1-\frac{2M}{r}}\right)}{-\frac{r^4 \mathrm{Sin}[\theta]^2}{1-\frac{2M}{r}}+\frac{2 r^3 M \mathrm{Sin}[\theta]^2}{1-\frac{2M}{r}}} \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} &
\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{r\left(-\frac{r^2}{1-\frac{2M}{r}}+\frac{2 r M}{1-\frac{2M}{r}}\right) \mathrm{Sin}[\theta]^2}{-\frac{r^4 \mathrm{Sin}[\theta]^2}{1-\frac{2M}{r}}+\frac{2 r^3 M \mathrm{Sin}[\theta]^2}{1-\frac{2M}{r}}} \end{pmatrix} &
\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{r^2\left(-\frac{r^2}{1-\frac{2M}{r}}+\frac{2 r M}{1-\frac{2M}{r}}\right) \mathrm{Cos}[\theta]\, \mathrm{Sin}[\theta]}{-\frac{r^4 \mathrm{Sin}[\theta]^2}{1-\frac{2M}{r}}+\frac{2 r^3 M \mathrm{Sin}[\theta]^2}{1-\frac{2M}{r}}} \end{pmatrix} &
\begin{pmatrix} 0 \\ \frac{r\left(-\frac{r^2}{1-\frac{2M}{r}}+\frac{2 r M}{1-\frac{2M}{r}}\right) \mathrm{Sin}[\theta]^2}{-\frac{r^4 \mathrm{Sin}[\theta]^2}{1-\frac{2M}{r}}+\frac{2 r^3 M \mathrm{Sin}[\theta]^2}{1-\frac{2M}{r}}} \\ \frac{r^2\left(-\frac{r^2}{1-\frac{2M}{r}}+\frac{2 r M}{1-\frac{2M}{r}}\right) \mathrm{Cos}[\theta]\, \mathrm{Sin}[\theta]}{-\frac{r^4 \mathrm{Sin}[\theta]^2}{1-\frac{2M}{r}}+\frac{2 r^3 M \mathrm{Sin}[\theta]^2}{1-\frac{2M}{r}}} \\ 0 \end{pmatrix}
\end{pmatrix}
$$

```
In[7]:= RiemannSchwarzschild["ReducedTensorRepresentation"] // MatrixForm
```

Out[7]//MatrixForm=

$$\begin{pmatrix}
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & \frac{2 M}{r^2 (r-2 M)} & 0 & 0 \\ -\frac{2 M}{r^2 (r-2 M)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & -\frac{M}{r} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{M}{r} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & -\frac{M\,\mathrm{Sin}[\theta]^2}{r} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{M\,\mathrm{Sin}[\theta]^2}{r} & 0 & 0 & 0 \end{pmatrix} \\
\begin{pmatrix} 0 & \frac{2 (r-2 M) M}{r^4} & 0 & 0 \\ -\frac{2 (r-2 M) M}{r^4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{M}{r} & 0 \\ 0 & \frac{M}{r} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{M\,\mathrm{Sin}[\theta]^2}{r} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{M\,\mathrm{Sin}[\theta]^2}{r} & 0 & 0 \end{pmatrix} \\
\begin{pmatrix} 0 & 0 & -\frac{(r-2 M) M}{r^4} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{(r-2 M) M}{r^4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{M}{r^2 (r-2 M)} & 0 \\ 0 & -\frac{M}{r^2 (r-2 M)} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2 M\,\mathrm{Sin}[\theta]^2}{r} \\ 0 & 0 & -\frac{2 M\,\mathrm{Sin}[\theta]^2}{r} & 0 \end{pmatrix} \\
\begin{pmatrix} 0 & 0 & 0 & -\frac{(r-2 M) M}{r^4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{(r-2 M) M}{r^4} & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{M}{r^2 (r-2 M)} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{M}{r^2 (r-2 M)} & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{2 M}{r} \\ 0 & 0 & \frac{2 M}{r} & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{pmatrix}$$

```
In[8]:= RicciCurvSchwarzschild["MatrixRepresentation"] // Simplify // MatrixForm
```

Out[8]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

```
In[9]:= RicciCurvSchwarzschild["RicciScalar"] // Simplify
```

Out[9]= 0