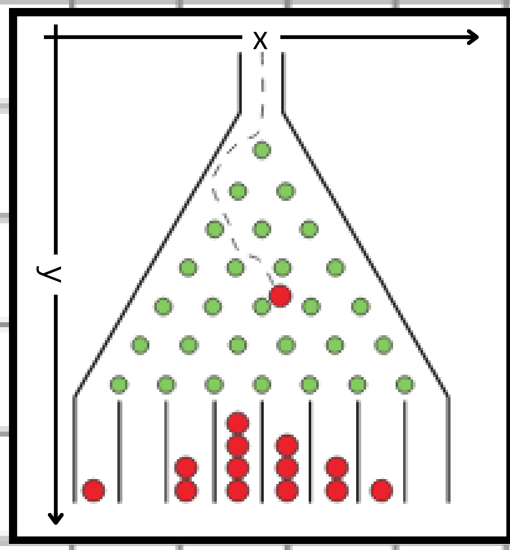
NON-GAUSSIAN STATISTICS IN STATIC AND DYNAMIC GALTON BOARDS

Non-trivial time dependent and independent non-gaussian distributions are generated in Galton boards by perturbing the lattice of scatterers, or by applying external time dependent forcing.



INTRODUCTION

A galton board is a toy composed of balls falling vertically through a lattice of scatterers. Balls are released from the top, get scattered, and are allowed to fall into a set of adjacent bins.

Steady-state solutions for the static board : Anisotropic Diffusion

Varying scatterer density in the continuum limit corresponds to anisotropic diffusion (D(x) as the diffusion constant) described by the general Fokker-Planck equation. Few stationary normalizable solutions can be obtained for specific cases. For instance, $D(x) = \frac{D_0}{l^2 + x^2}$ yields the soln. $\rho_0 \exp\left(-\frac{\rho_0}{l} \arctan \frac{x}{l}\right) \cdot \left(2 \sinh\left(\frac{\rho_0 \pi}{2l}\right)(x^2 + l^2)\right)^{-1}$ This is a classic heavy tailed Landau distribution with diverging mean squared displacement and mean.

The Dynamic Board : external forcing

In the continuum limit the time evolution for a single ball is given by a set of coupled SDE's. We assume 2D sinusoidal forcing.

$$\mathrm{d}x(t) = \gamma f_1 \sin(w_1 t + \phi)\,\mathrm{d}t + \sqrt{\frac{2D\tau}{a}}\mathrm{d}W(y) \qquad \mathrm{d}y(t) = \gamma f_2 \cos(w_2 t + \phi)\,\mathrm{d}t + \gamma g \mathrm{d}t$$

Exact results for 1D solution for general time dependent forcing

In this case the vertical forcing amplitude is zero. The time dependent distribution can be exactly written in terms of the fourier transform of of the horizontal forcing f(t).

$$\widetilde{P}(k,t) \propto e^{-Dk^2 t} \int D\Phi(\omega) \times \exp\left(ik\gamma \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{i\omega t} - 1}{\omega} \tilde{f}(\omega) e^{i\Phi(\omega)}\right)$$

Notably, there is an additive contribution to the variance, and a nonzero fourth cumulant which is the characteristic for departure from Gaussian behavior.

$$\langle x^4(t) \rangle_c = \frac{3}{8}\gamma^4 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left(\frac{\tilde{f}(\omega)}{\omega}\right)^4 \sin^4\left(\frac{\omega t}{2}\right)$$

The results above are for a square lattice. However, for a sharp spectrum (near-monochromatic) the contribution of periodic forcing only results in an additive contribution to the static cumulant generating function, irrespective of peg distribution. This correction depends on the peak height F as :

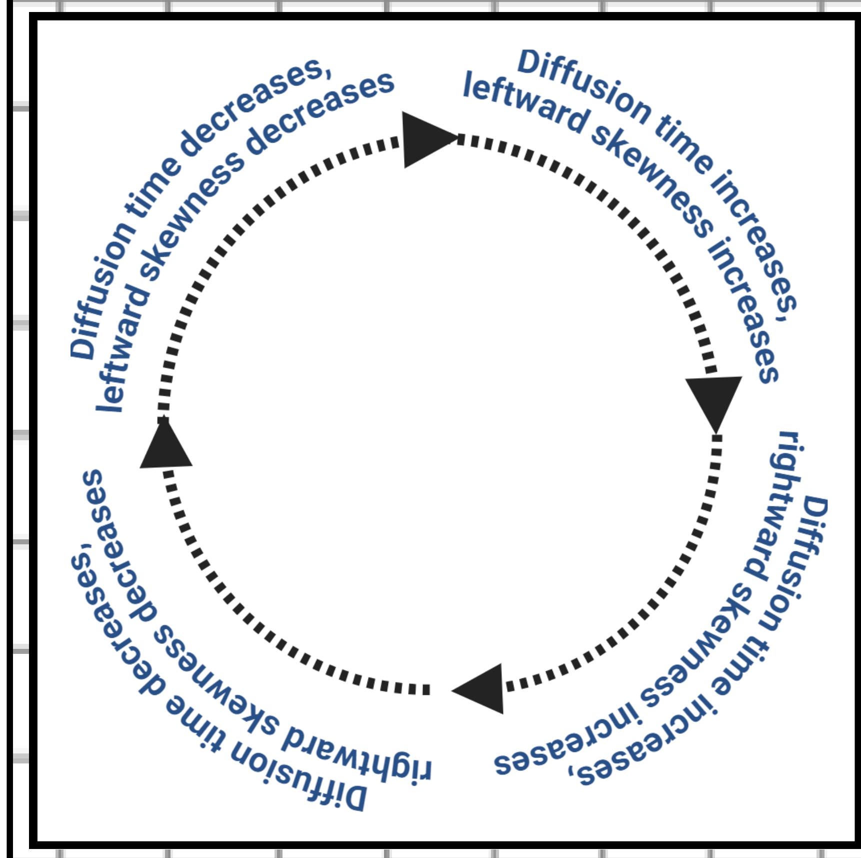
$$C_{forced}(k,t) = C_{static}(k,t) + \ln J_0(Fk)$$

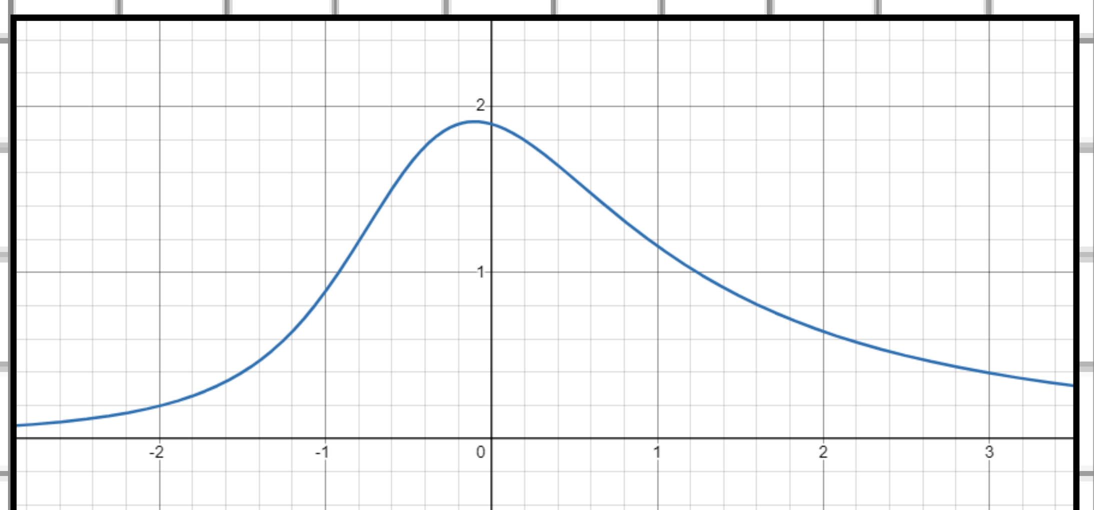
Results for 2D sinusoidal vibration

The case of 2D forcing cannot be solved exactly. To first order in the vertical forcing we get that the distribution is skewed, and this 'skewness' oscillates with the vertical height of the board. This skewness is proportional to the forcing amplitude and falls with frequency.:

$$\langle x^3(y) \rangle_c = \frac{\gamma^3 f_1^3 f_2}{8\omega_2 g}\left(\cos\left(\frac{\omega_1 y}{\gamma g}\right) - \cos\left(\frac{(\omega_1 - \omega_2)y}{\gamma g}\right)\right) + O(f_2^3)$$

A simple intuitive case : rotating the board in an ellipse. The centrifugal force in 2D leads to the same EOM's as above. There are four quarter-cycles : in the upward motion, effectively there is greater 'time' for diffusion, so the distribution skews towards either the left (right) side depending on whether the board accelerates to the right (left).





Skewed distribution produced at some finite height of the board

DISCUSSION

The primary source for departure from gaussian behavior in the forced galton boards in either case is the collective behavior of the large number of balls in the large N limit. Varying release times of the balls yields to different initial phases. The superposition of the individual out-of-phase Gaussians leads to an overall Non-Gaussian distribution, as shown by the nonzero third and fourth cumulants.

WattsUp

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More details : https://arxiv.org/abs/2404.19478 or the QR code here