1 Introduction

In a homogeneously distributed Galton board with pegs of finite size, we get that the distribution of beads at the bottom is a normal distribution. We want to investigate whether we can find a peg distribution which will give rise to a non-normal distribution. For these, we assume that the bead is point sized, and peg is finite sized. In the following sections, a detailed discussion of various methods to obtain the final targeted distribution is given. In this paper, we write PD to mean peg distribution, rest all abreviations are standard. A discussion the i.i.d nature of the pegs is presented, along with other methods to obtain a target pdf. We also discuss the uniqueness of such PD. We also want to investigate the impact of various kinds of vibrations on the final distribution. For these we consider both the beads and pegs to be point sized but mass of peg is significantly greater than that of bead. We will first look at the case when the beads have no memory, and then look at the case when the beads have memory. Deterministic and indeterministic vibrations are considered.

2 Theoretical Model

There can exist many possible distributions for the pegs, and many of them give rise to a non-gaussian distribution. We present a method to find possible peg distributions which generate a general class of probability distributions. We also describe the conditions under which we get a Gaussian distribution. For this section, a key assumption is that the pegs are triangular, and very large compared to the beads. Our idea of a PD follows that of a countably infinite complete binary tree, where 'countably' corresponds to row index. Since each LST and RST of a node is also a CBT, each row clearly represent independent Ber(0.5) variables (if we ignore memory). We can now think of two parameters that can change the distribution of the beads. The first is the height (height of each row, keeping all pegs equilateral) of the pegs, and the second is the shape (keeping the pegs of a same height) of the pegs. We will look at both of them in detail.

We treat the case when the bead has no memory (as described in 3.2). We can treat the top vertex of each peg to be a Ber(0.5) random variable, amounting to $P(\text{left}) = 0.5 = P(\text{right})$. The final distribution is some linear combination of these random variables (since the levels are independent, we do not expect any sort of coupling between them and hence no higher order terms), with the coefficients determined by PD. To make a PD which is i.i.d. Ber(0.5) at each row, only height change is allowed because a shape change will not leave the pegs identically distributed.

Let us now construct a random variable X as follows. Let X_i be the random variable which is 1 if the bead falls right at the ith row and 0 otherwise. Then $X = \lim_{n\to\infty} \sum_{i=1}^{n} c_i X_i$ ($c_i \geq 0$). The physical significance of c_i becomes apparent in 3.3.2. We are given with a target marginal distribution $f(x)$ of X, and we want to find the PD which will give rise to this pdf. We impose the following constraints on the pdf of X: it should be confined in a finite interval, or at least be sufficiently small after a characterstic lengthscale. This is to ensure that we can either physically make or simulate it easily to verify our arguments. Clearly, X and X_i are not independent because X is a linear combination of X_i. We also do not know the joint probability distribution of X and X_i, and hence cannot compute $\text{Cov}(X, X_i)$ to find c_i. A method to bypass this would be to take large samples from both X and X_i and then compute their correlation, but since this is computationally intensive for practical purposes, we ignore that.

If the c_i are all identically 1, then X is just the sum of n i.i.d. Bernoulli random variables, and hence X is a binomial random variable, which has a normal approximation. But if the c_i are not identically 1, then X is not a binomial random variable, and hence it is not guaranteed that X will have a normal approximation.

2.1 Uniform Distribution

We take an alternate approach to find a representation of a uniform random variable X as sum of Ber(0.5) variables. Take any number between 0 and 1, and write it in binary. Then the ith digit of the binary representation is 1 with probability 0.5. So we can write X as a sum of Ber(0.5) variables. But this is not a good representation because the number of variables required is infinite (especially for irrational numbers). We can use the fact that the binary representation of a number is unique, and hence we can use the following representation: $X = \sum_{i=1}^{\infty} 2^{-i} X_i$, where X_i is the ith digit of the binary representation of X.

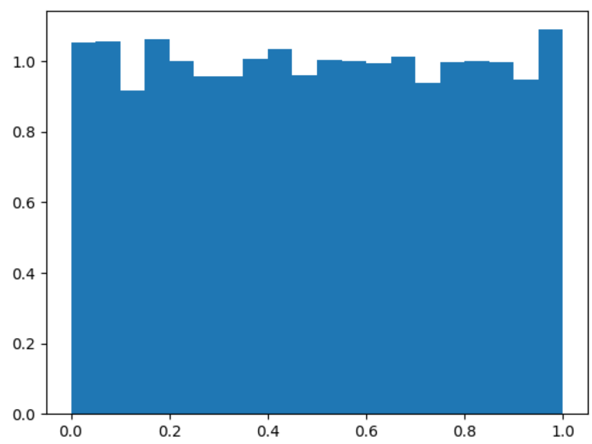

Figure 1: Obtained distribution

We can now interpret the above representation in terms of the pegs. We can think of the pegs as a binary tree, where each node is a peg. The height difference between parent and child nodes, and hence their base, since they are equilateral triangles, is half of the height difference between grandparent and parent nodes. Because of this exponentially decreasing height, it would be difficult to construct such a modified Galton board physically. We however can simulate this and obtain that the distribution is indeed uniform for many experiments.(Figure 1)

2.2 Log-Normal Distribution

Keeping the height of each row the same, we prescribe, following [1] a PD analogous to the normal distribution. For a normal galton board, if we have a peg with base $= 2a$ then for all pegs, if the center of the said peg is at x then the daughters will have top vertex at $x \pm a$. For a lognormal distribution, we have that X is lognormal if e^X is normal.

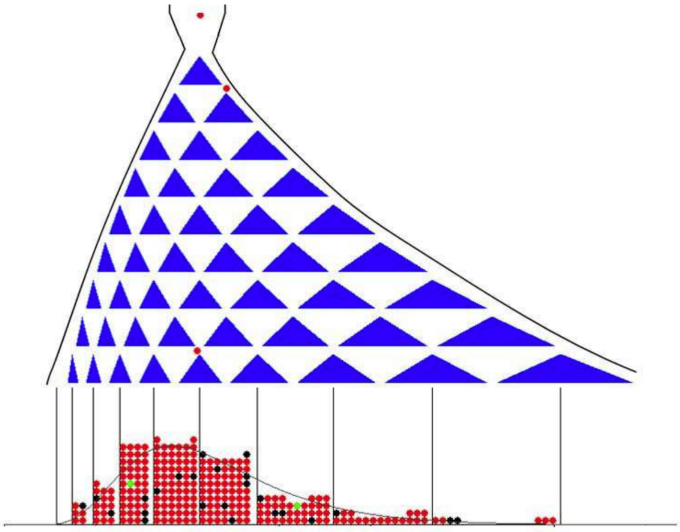

Figure 2: Fa Wang *et al.* 2018

We can say that addition and subtration of a constant in the normal distribution corresponds to multiplication and division by a constant in the lognormal distribution. This gives the prescription that for a peg at x, the daughter pegs will be at $x \cdot c$ and x/c where c is a constant.

Ber(0.5) pegs are still independent, but not identically distributed, so we cannot directly use the summation method to describe this PD. Is this PD unique? To answer this, we must first answer the question: can we represent it, somehow, in the form of a sum of Ber(0.5) variables? The answer to this is provided in the following subsections.

2.3 General Case

For a general case with $f(x)$ following the constraints mentioned in the beginning of this section, we can construct two methods to find the PD. Both of these are numerical methods, but give different solutions for the same pdf of X.

2.3.1 Keeping Row Heights Constant

We place the top peg at the median, the next row pegs at the 1st and 3rd quartiles, the next row at 1st, 3rd, 5th and 7th octiles, and so on. We can define the base of a parent peg by connecting the top vertices of the child pegs. This distribution can be efficiently represented as a binary tree, with distance between the parent and child node being defined as the value of a node, and the root node being identified with the median of X: this uniquely defines the distribution since the height of each level is constant.

We can now use the same method as in the uniform distribution and find a representation of $X \sim$ Unif[0,1] as something other than a sum of Ber(0.5) variables. These peg distributions are clearly very different.

In fact, this method gives a different PD for lognormal distribution as well. Simulation gives that the final distribution is the same for both solutions of peg distributions.

We can draw a conclusion from these two examples that the PD is not unique.

Perhaps the best way to show the non-uniqueness is to consider the normal distribution and apply this method. The number of pegs has changed, but still we get the same distribution. We need to place the pegs at x = ... , -4.5, 0, 4.5 ... approximately for $\mathcal{N}(0,1)$. Here the base of the two pegs on LST root and RST root do not touch but still we get desired standard normal.

2.3.2 Keeping Shape Constant

Construct $X = \sum_{i=1}^{n} c_i X_i$ for large n, where X_i are independent Ber(0.5) variables. Right now we do not know the c_i. Assume that $\max(c_i) = 1$, and $c_i < c_j$ for $i < j$. This physically corresponds to fixing the maximimum height at 1, and the height of the pegs decreasing as we go down the rows. In this case, even if $f(x)$ does not have any apparent symmetry, we still have that the pegs, and hence X_i are identically distributed. [confirm, and check by simulation]

Construct an objective function $F(c_1, c_2, \ldots, c_n) = \int_{-\infty}^{\infty} |f(x) - \sum_{i=1}^{n} c_i f_i(x)| dx$, where $f_i(x)$ is the pmf of X_i. We choose a tuple of $(c_1, \ldots c_n)$ from their n-dimensional vector space which minimises F. The PD is now defined by the c_i, which correspond to the height of rows, or alternatively the base length of pegs.

3 Vibrational Analysis

3.1 Vibration

We use simulations to check what happens when we vibrate the galton board. To model vibration mathematically, we say at a particular peg the probability of falling to the right is $0.5 - \delta(t)$ and to the left is $0.5 + \delta(t)$, where $\delta(t)$ is some sinusoid. But we don't yet have any parameter to represent time. For time, we use the level one particular bead has travelled. We take 500 to be the mean.

We look at the distribution at some given frequencies(Fig 3 and 4):

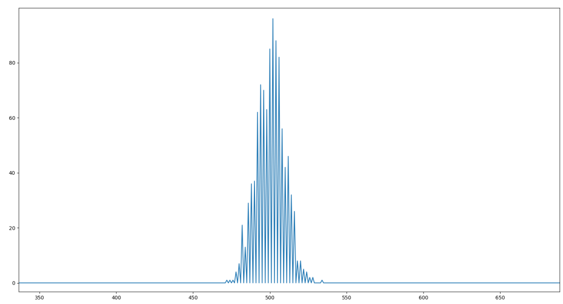

Figure 3: Frequency 0.01

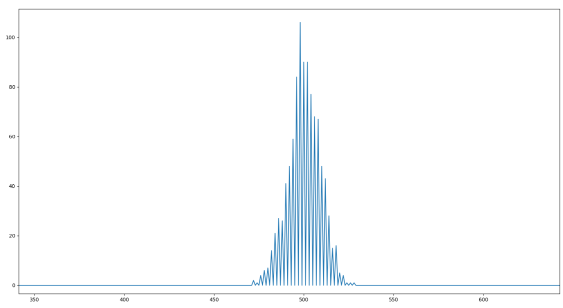

Figure 4: Frequency 0.1

Much changes are not observed so we plot the mean as a function of frequency(Fig 4).

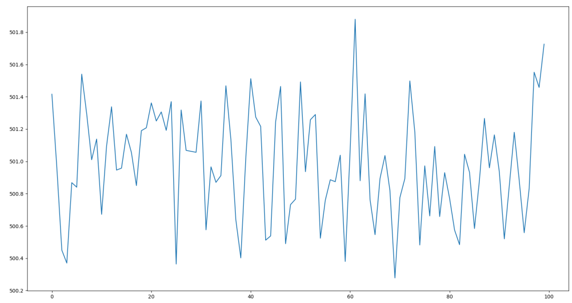

Figure 5: Frequency v/s Mean

Here we see the mean is oscillating about its original mean with a low variance, so in this model we can say vibration does not affect the normal distribution. (Fig 5)

If we try to model vibration by picking δ from a distribution, similar results are obtained if the given distribution is even. Otherwise, a shift in mean is observed depending on the bias of δ.

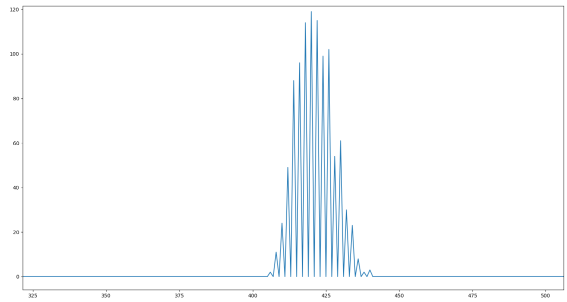

Figure 6: δ taken from gamma distribution

3.2 Giving memory

But what happens when we give the memory to the bead if it came from right or left. Let's try to simulate the case where in the first collision probability of going right and left is 0.5 but from the next time $P(\text{right}|\text{right}) = x$ and $P(\text{right}|\text{left}) = y$. If we simulate that with some arbitrary x, y values(Fig 7) it does not seems particularly illuminating except for the shift in mean. But there is a trivial case where normal distribution is not obtained with $x = 1, y = 0$, that is once it fell right it will continue right and if it fell left it will continue left. (Fig 8)

Also, another trivial case that does not give normal distribution is $x = 0, y = 1$, i.e it will stay at the mean.

3.3 Anaytical solution to vibrations in Galton board

Vibration can be of various types. It can be deterministic or indeterministic (depending on whether we know what will be the changes in probability/velocity of the bead upon collision with a peg). The effect of vibration can be time-dependent or independent. Time-independent vibrations

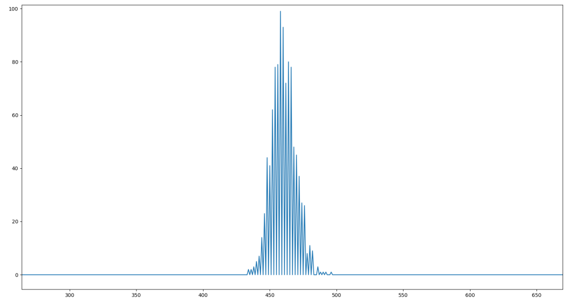

Figure 7: $x = 0.3$, $y = 0.7$

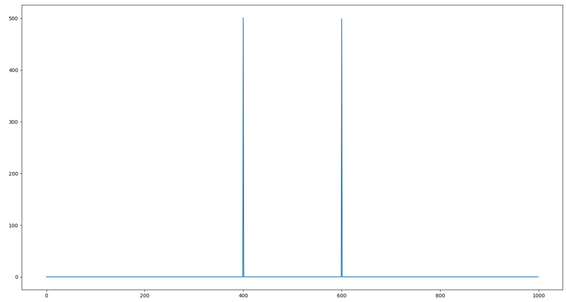

Figure 8: $x = 1$, $y = 0$

are those where the change in probability of a bead going left/right upon hitting a peg does not depend on when the bead hits the peg. Impact of various kinds of vibration on the is given below:

3.3.1 Indeterminitic collisions

Let us consider that there is a galton board with n levels (or rows) and the final distribution is some f(x) (pmf is discrete and pdf in continuous limit) where x is the distance from the the vertical line passing through the point from where the beads started to fall and perpendicular to the horizontal line made by joining pegs in any row. Let us consider each peg to be independent and having an identical probability distribution of going towards right (equals a). Let the vibration impact the probability of each peg in a way that the probability of going toward right is given by $a + \delta$ such that δ belong to the interval $[-\epsilon_1, \epsilon_2]$ with a pdf given by $g(\epsilon, t)$. The distribution of right and left steps (let's call it a string on word made of two words, namely, right and left) that a particle needs to reach a particular x at the bottom still remains unchanged. But the probability corresponding to each string changes.

For any string let there be r rights and l lefts. Then the original probability will be $a^r \cdot (1-a)^l$. Let the changed probability be p. Then, $ln(p) = \sum_{i=1}^{r} ln(a + \delta_i) + \sum_{j=1}^{l} ln(1 - a - \delta_j)$.

As n tends to infinity r and l also tends to infinity. Let the mean (calculated by calculating the time average of the mean of the curve $g(\epsilon, t)$ at various time) of the distribution of ln(a+δ) be μ_1 and the mean of the distribution of ln(a+δ)) be μ_2. Then the two summations written above approaches the $r \cdot \mu_1$ and $l \cdot \mu_2$ respectively. The probability $p = (e^{\mu_1})^r (e^{\mu_2})^l$. Note $e^{\mu_1} + e^{\mu_2}$ may not be equal to 1 because when two beads hit the same peg (at different time) if the first bead had probability of going toward right equal to (let's say) 0.4 then the next bead need not necessarily have probability of going toward left equal to 0.6 when it hit the peg.

There for the impact of vibration on the distribution is such that a change to $\frac{e^{\mu_1}}{e^{\mu_1}+e^{\mu_2}}$ and 1-a change to $\frac{e^{\mu_2}}{e^{\mu_1}+e^{\mu_2}}$. Let us check this theory with the following example. Let a=1/2 and right step size = left step size = 1 in the usual galton board. Then for n=10000 the distribution look like the one given in Fig 6 (a normal curve).

Let $\epsilon_1 = 0.3$ and $\epsilon_2 = 0.3$ and the distribution $g(\epsilon, t)$ be a normalized but truncated Cauchy distribution (which is independent of time in thise case). Then according to our theory there should be no change in the "mean" and "variance" of the final distribution (i.e. it should look exactly like the normal distribution as before) which is what we see in the Fig 6.

3.3.2 Time-dependent deterministic vibrations

We consider the impact of time-dependent sinusoidal vibrations of the Galton board and inelastic collisions of a bead with pegs (inelastic because it takes the velocity of peg and it forgets its initial velocity, this corresponds to $m_{peg} >> m_{bead}$, but both are still point particles). We consider a usual Galton board which has pegs spread homogeneously, unlike the CBT model in the previous section. This is to ensure that beads always hit a peg at each level, no matter their initial velocity.

We assume that the external vibrations are with frequency ω and that each bead takes time τ to fall between each row and each collision is independent. (τ is a constant) We also assume that the bead under consideration is dropped at time t = T and that the vibrations start at time t = 0 with amplitude $\frac{v_0}{\omega}$.

The distance travelled by the bead after colliding with a peg can then be represented as d = $v_0 \sin(\omega t)\tau$, where t is the time at which it falls on the peg.

And hence, the overall distance travelled by the bead after colliding with n pegs (same as travelling down n rows) can be represented as:

$$d(n, T) = \sum_{j=0}^{n-1} v_0 \tau \sin(\omega(T + j\tau))$$

or,

$$d(n, T) = v_0 \tau \frac{\sin(\frac{(n-1)\omega\tau}{2}) \sin(\omega T + \frac{(n-2)\omega\tau}{2})}{\sin(\frac{\omega\tau}{2})}$$

Now, we obtain T as a function of d as:

$$T = \frac{1}{\omega} \left[\arcsin\left(\frac{d}{A}\right) - \frac{(n-2)\omega\tau}{2} \right]$$

where,

$$A = v_0 \tau \frac{\sin(\frac{(n-1)\omega\tau}{2})}{\sin(\frac{\omega\tau}{2})}$$

Here we must observe that if we get $T < 0$ for any case, then we should add the appropriate $2m\pi/\omega$ (where $m \in \mathbb{Z}$) to obtain a positive T.

Since the number of particles that land at x $\geq$ d is proportional to the area of the d vs T graph above d, the number of particles will be $\frac{\pi}{\omega} - \frac{2}{\omega} \arcsin\left(\frac{d}{A}\right)$.

This gives us the probability of a particle landing at x = d when it is on the n$^{\text{th}}$ row as $f_n(d) = \frac{1}{\pi\sqrt{A_n^2 - d^2}}$.

If we now consider a board with finite n then we get the final pdf not as a Gaussian but as above, with A_n a function of $\omega\tau$ and n. Note that only $\omega\tau$ is important for the pdf and not ω and τ individually.

Suppose ω goes to zero, i.e. no oscillation, then we should again get the Gaussian as a limit. This highlights an implicit assumption that this model works only when the velocity imparted by the external vibrations is much larger than the random horizontal velocities attained when beads hit pegs and go left or right as Ber(0.5).

We can correct this by adding a small v to our model, which contributes a distance $v\tau$. If we assume that this system has no bias for very slow oscillations, then $v \in [-\epsilon, \epsilon]$ is uniformly distributed in some maximum possible horizontal speed. If we now follow the same steps, we can simply drop the oscillatory terms and apply CLT to get a Gaussian. Anyway, the contribution for large n terms is high only at ends, and by our approximation for slow vibrations, we can safely ignore that. (Fig 9,10)

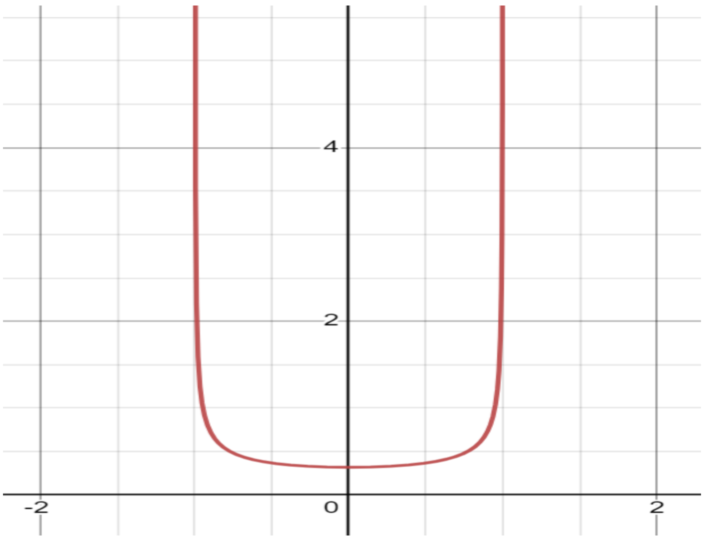

Figure 9: A $\sim$ 1

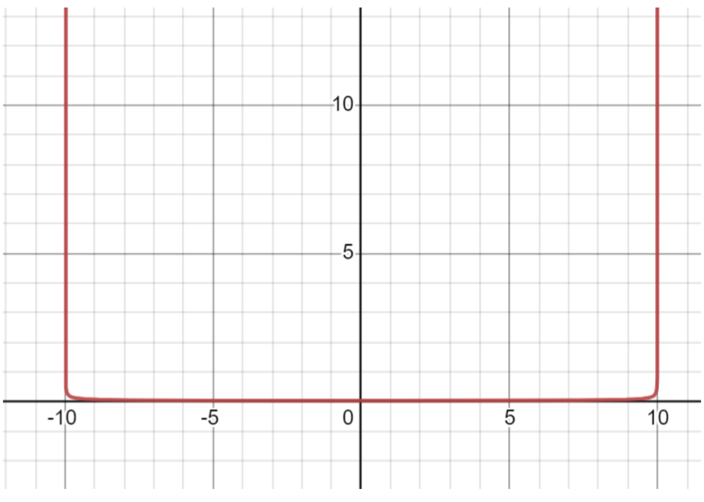

Figure 10: A $\sim$ 10

More generally, for any oscillations which can be decomposed into a Fourier series, we can modify the above equation to $d_n = \sum_k A_k \sin(\omega_k T + \phi_k)$ and solve for T as a function of d.

4 Further Ideas

4.1 A Theorem

Let the probability of going toward left or right be independent for each peg. When we see the usual Galton board the 1st,3rd,5th... row and column are same and 2nd,4th,6th... row and columns are the same. We shall say the periodicity of the row and column equals 2 in this case.

Claim: Any peg distribution that has finite row and column periodicity and independent pegs will lead to a normal distribution.

Proof: Let the periodicity of row be R and column be C. Then any bead that start at the top, after some hitting m pegs, the bead has to come back to a peg same as the first peg it hit because there are only $R \cdot C$ distinct pegs (at max) and assuming probability of going to right is neither zero nor 1 for each peg the bead will hit a peg similar to the first peg. Contribution thereafter will just the same as if it has started again from the top. As n tends to infinity, this will imply that for each bead the ratio of the pegs (of each type) that it collides with will converse to a particular number. This guarantees that the distance random variable (X) which can be written as the sum of distance random variable (corresponding to each row) X_i as $\sum_{i=1}^{n} X_i$ can be written as sum of distance random variable (corresponding to each peg type) X_p as $n \cdot \sum_{p=1}^{R \cdot C} r_p \cdot X_p$, where r_p is equal to the ratio of that type of peg as n tends to infinity. We know that the second summation will converse to a normal as n tends to infinity cause its just like adding n random variable (Y) given by $Y = \sum_{p=1}^{R \cdot C} r_p \cdot X_p$. Below is a curve for the case when the periodicity of the row is 1 and for column it is 5. We can see that it does come out to be a normal as n tends to infinity.

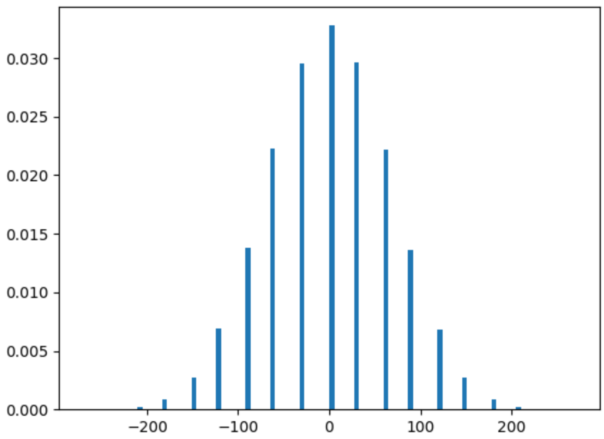

Figure 11: R=1, C = 5

4.2 Heavy Tailed Distribution

In this section, we try to construct pdfs which do not follow the constraint described in section 2. One obvious way to do this to construct some pdf which does not have a finite variance. This would be ideally impossible to construct in a realistic Galton board, so we present a method to construct such a pdf using a modified system.

4.2.1 Attempt at Infinite Variance

In a realistic Galton Board, we can't achieve infinite variance distribution as the distance parameters are finite and attainable velocities are also finite. However, consider a Galton board that has a sufficiently long base and the first peg that vibrates randomly. This is similar to the experiment with light flashing on a wall and a plane mirror randomly rotating in front of it. Since the first peg can deflect the bead to any angle, the bead can travel to any distance from the center. Now the remaining pegs are stationary, so the remaining choices are random once a bead is fired by the first peg. As the pegs are randomly shot with a high velocities, a high proportion of the beads would be shot toward the extremes, resulting in a heavy tailed and infinite variance distribution. (Fig 12) Our method to construct target pdfs fails here, warranting further investigation.

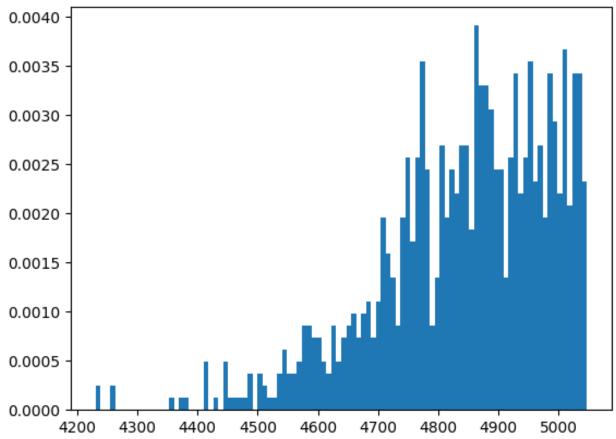

Figure 12: Heavy Tailed distribution, pdf vs x

4.2.2 Biased-pegged heavy tail

Consider a Galton Board constructed as follows: at each nth row, the probabilty of the peg sending the peg left is 1/n and right is 1-1/n. One way to implement this biased peg could be with the use of a small right triangle, its altitude on the hypotenuse splitting it in the ration 1:n-1. This kind of biased-pegs distribution would lead to a heavy tailed distribution for every n as the probability of going to the right extreme increases with each iteration.

5 References

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