Term Paper for HE392: String Theory

Minimal Models in 2D Conformal Field Theory

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Abstract: We present a concise introduction to minimal models in two-dimensional conformal field theory. The first section deals with reviewing concepts in quantum field theory and statistical mechanics, followed by an overview of CFT in dimensions greater than two. We restrict to 2D CFTs in the subsequent sections. In the second section, we discuss the notion of local fields, prove the conformal Ward identities, and compute the operator product expansions for some simple examples. The third section introduces radial quantization and the state-operator correspondence. This leads to the Virasoro algebra, the construction of the Hilbert space via conformal families, and the concept of fixing the operator algebra using conformal blocks and crossing symmetry. The fourth section is devoted to the representation theory of the Virasoro algebra, where we define Verma modules, null states, and Kac determinants. Unitarity constraints on the central charge and conformal dimensions are derived, leading to the classification of minimal models. The fifth section applies these concepts to minimal models, showing how null states generate differential equations for correlators. The Ising model is analyzed in detail as the primary example, demonstrating the calculation of its correlators, fusion rules, and conformal blocks.

Appendices are included, with a detailed treatment of the quantization of free boson and fermion theories on a cylinder. Other appendices outline topics such as the physical meaning of the central charge, the trace anomaly, and fusion rules.

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1 Review of basic concepts

During phase transitions, systems exhibit scale invariance at the critical point. This scale invariance often extends to conformal invariance, particularly in two dimensions. CFTs provide a powerful framework to study such systems, allowing us to compute critical exponents and correlation functions exactly.

Since quantization of fields on the worldsheet leads to operators following the Virasoro algebra with some central charge, and because we can gauge fix the worldsheet metric to work in the conformal gauge, CFT techniques become essential for simple and elegant computations in string theory. Vertex operators on Riemann surfaces of various genera play a crucial role in string scattering amplitudes.

Before studying conformal invariance in two dimensions, we need to fix some notation for conservation laws and conserved currents for general continuous symmetries. Many physically observable quantities exhibit divergent behaviour near the critical point of phase transitions. We fix notation for critical exponents of such divergences. A brief note on renormalization follows.

We end this section with an overview of conformal field theories in dimensions greater than two. They are of less interest in this article because they have finite number of generators in their conformal algebra, unlike the infinite-dimensional case in two dimensions.

1.1 Symmetries and Conservation Laws

For a set of fields denoted by $\Phi : \mathbb{R}^d \to \mathcal{M}$, where $\mathcal{M}$ is some target space[1], the action is given by:

$$S = \int d^d x \, \mathcal{L}(\Phi, \partial_\mu \Phi) \tag{1.1}$$

where $\mathcal{L}$ is the Lagrangian density. Let's suppose that action is invariant under a continuous transformation of the fields:

$$\begin{aligned} x &\to x' \\ \Phi(x) &\to \Phi'(x') \end{aligned} \tag{1.2}$$

It is customary to assume that the new fields at the new coordinates can be represented as a function of the old fields at the old coordinates: $\Phi'(x') = \mathcal{F}(\Phi(x))$. We can compute the new action as:

$$\begin{aligned} S' &= \int d^d x' \, \mathcal{L}(\Phi', \partial'_\mu \Phi') \\ &= \int d^d x \, \mathcal{L}(\mathcal{F}(\Phi), \partial'_\mu \mathcal{F}(\Phi)) \\ &= \int d^d x \, \mathrm{Jac}(x', x) \, \mathcal{L}(\mathcal{F}(\Phi), (\partial x^\nu / \partial x'^\mu) \partial_\nu \mathcal{F}(\Phi)) \end{aligned} \tag{1.3}$$

Here $\mathrm{Jac}(x', x)$ is the Jacobian determinant of the transformation from x to x'. Since many continuous symmetries in physical systems can be smoothly connected to the identity trans-

[1] As can be seen from the domain of the fields, we are working Euclidean spacetime right now for sake of simplicity and can Wick-rotate to Minkowski at relevant points.

formation, it is useful to consider the special case where the transformation can be represented as an element of a Lie group.The identity element corresponds to no transformation and the Lie algebra elements are related to generators of infinitesimal transformations. In general, we can write an infinitesimal transformation as:

$$\begin{aligned} x'^{\mu} &= x^{\mu} + \omega_a \frac{\delta x^{\mu}}{\delta \omega_a} \\ \Phi'(x') &= \Phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x) \end{aligned} \tag{1.4}$$

to first order in the infinitesimal parameters $\{\omega_a\}$. We reserve the word *generators* $\{G^a\}$ for those parameters that govern the transformation of the fields at a fixed point.

$$\delta_\omega \Phi(x) \equiv \Phi'(x) - \Phi(x) \equiv -i\omega_a G^a \Phi(x) \tag{1.5}$$

By Taylor expanding $\Phi(x)$ around x' in (1.4), and noting that since the transformation is infinitesimal we can replace x by x' in the second term on the RHS of the field transformation in (1.4) without any first order error, we obtain the explicit form of the generators:

$$iG^a\Phi = \frac{\delta x^{\mu}}{\delta \omega_a}\partial_\mu \Phi - \frac{\delta \mathcal{F}}{\delta \omega_a} \tag{1.6}$$

It is instructive to note some important examples of generators.

Transformation	$\delta x^{\mu}/\delta\omega_a$	$\mathcal{F}(\Phi)$	$\delta\mathcal{F}/\delta\omega_a$	G^a
Translations	$\delta^{\mu\nu}$	Φ	0	$P^{\nu} = -i\partial^{\nu}$
Lorentz	$\frac{1}{2}(\eta^{\rho\mu}x^{\nu} - \eta^{\nu\mu}x^{\rho})$	$L_\Lambda \Phi \approx (1 - i\frac{1}{2}\omega_{\rho\nu}S^{\rho\nu})\Phi$	$-i\frac{1}{2}S^{\rho\nu}$	$M^{\rho\nu} = i(x^{\rho}\partial^{\nu} - x^{\nu}\partial^{\rho}) + S^{\rho\nu}$
Scale	x^{μ}	$\lambda^{-\Delta}\Phi$	$-i\Delta$	$D = -i(x^{\mu}\partial_\mu + \Delta)$

Table 1. Some examples of generators of continuous symmetries. Note that there is a conventional factor of half compensated in the Lorentz generator. $S^{\rho\nu}$ are Hermitian matrices that follow the Lorentz algebra. Δ is called the scaling dimension of the field.

Having established the generators of infinitesimal transformations, we can now derive the conserved currents associated with continuous symmetries by noting that the action is invariant under such transformations. The Jacobian matrix is:

$$\frac{\partial x'^{\nu}}{\partial x^{\mu}} = \delta^{\nu}_{\mu} + \partial_\mu \left(\omega_a \frac{\delta x^{\nu}}{\delta \omega_a} \right) \tag{1.7}$$

Assuming that the change in coordinates is small, we can use the approximation $\det(1 + M) \approx 1 + \mathrm{Tr}(M)$ for small matrices M. Thus, the Jacobian determinant is:

$$\mathrm{Jac}(x', x) = 1 + \partial_\mu \left(\omega_a \frac{\delta x^{\mu}}{\delta \omega_a} \right) \tag{1.8}$$

Inverting the Jacobian matrix to first order, we get:

$$\frac{\partial x^{\nu}}{\partial x'^{\mu}} = \delta^{\nu}_{\mu} - \partial_\mu \left(\omega_a \frac{\delta x^{\nu}}{\delta \omega_a} \right) \tag{1.9}$$

We now introduce the notion of *rigid* transformations. These are transformations where the infinitesimal parameters $\{\omega_a\}$ are constants, i.e., they do not depend on spacetime coordinates. For such transformations, the derivatives of ω_a vanish and the action is manifestly invariant. We can then consider *local* transformations, where the parameters $\{\omega_a(x)\}$ depend on spacetime coordinates. In this case, the action is no longer trivially invariant. Substituting these expressions into (1.3) and expanding to first order in $\partial_\mu \omega_a$, we find:

$$\delta S \equiv S' - S = -\int d^d x\, j^{\mu,a} \partial_\mu \omega_a \tag{1.10}$$

where

$$j^{\mu,a} = \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \partial_\nu \Phi - \delta^\mu_\nu \mathcal{L} \right) \frac{\delta x^\nu}{\delta \omega_a} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \frac{\delta \mathcal{F}}{\delta \omega_a} \tag{1.11}$$

is the Noether current associated with the symmetry generated by G^a. Integrating by parts and neglecting boundary terms, we have:

$$\delta S = \int d^d x\, (\partial_\mu j^{\mu,a})\, \omega_a \tag{1.12}$$

Since the action is invariant under arbitrary infinitesimal transformations generated by G^a, we conclude that the Noether current is conserved (if the system obeys the equations of motion):

$$\partial_\mu j^{\mu,a} = 0 \tag{1.13}$$

The conserved charge associated with the current is given by:

$$Q^a = \int d^{d-1} x\, j^{0,a} \tag{1.14}$$

It is important note that this charge is computed in the Euclidean formalism. Wick-rotating to Minkowski spacetime, we can interpret $Q^a \to -iQ^a$ as the conserved charge of the symmetry transformations.

All that we have done so far is in the classical field theory context. In quantum field theory, all this does not carry forward trivially. Particularly, if the measure of the path integral is not invariant under the symmetry transformation, but the action is, we have what is called an *anomaly*. Still, classical symmetries of the action lead to Ward identities in the quantum theory, which are relations between correlation functions.

Here, we assume that the reader is familiar with the path integral formalism of quantum field theory, and we use usual notation to quickly derive the Ward identities in QFT. Suppose we have a product of fields at different points, denoted by $X = \Phi(x_1)\Phi(x_2)\cdots\Phi(x_n)$. Assuming that the measure is invariant under the symmetry transformation[2], we have:

$$\langle X \rangle = \frac{1}{Z} \int \mathcal{D}\Phi' \,(X + \delta_\omega X) \exp - \left\{ S[\Phi] + \int d^d x\, (\partial_\mu j^{\mu,a}) \omega_a \right\} \tag{1.15}$$

where $Z = \int \mathcal{D}\Phi\, e^{-S[\Phi]}$ is the partition function, $\delta_\omega X$ is the variation of X under the infinitesimal transformation, and we have used (1.12) to express the change in action.

[2]Equivalently, we can say that there is no anomaly associated with the symmetry.

Expanding to first order in ω_a, we get:

$$\langle \delta_\omega X \rangle = \int d^d x \, \partial_\mu \langle j^{\mu,a}(x) X \rangle \, \omega_a(x) \tag{1.16}$$

It is easy to find $\delta_\omega X$ exactly because we know how each field transforms. Using (1.5), we have:

$$\delta_\omega X = -i \sum_{i=1}^{n} \omega_a(x_i) \left(\Phi(x_1) \cdots G^a \Phi(x_i) \cdots \Phi(x_n)\right) \tag{1.17}$$

Since $\omega_a(x)$ is arbitrary, we can equate the integrands in (1.16) to obtain the Ward identities:

$$\partial_\mu \langle j^{\mu,a}(x) X \rangle = -i \sum_{i=1}^{n} \delta^{(d)}(x - x_i) \langle \Phi(x_1) \cdots G^a \Phi(x_i) \cdots \Phi(x_n) \rangle \tag{1.18}$$

An important application is when we choose one direction to be special, say time, and integrate over a spatial slice. Let $Y = \Phi(x_2) \cdots \Phi(x_n)$ be the product of fields at all points except x_1. Integrating (1.18) over a spatial slice containing x_1, with some special care given to the 'time' direction, we get, over a very small time interval ϵ:

$$\langle Q^a(x_1^0 + \epsilon/2) \Phi(x_1) Y \rangle - \langle \Phi(x_1) Q^a(x_1^0 - \epsilon/2) Y \rangle = -i \langle (G^a \Phi)(x_1) Y \rangle \tag{1.19}$$

In the limit $\epsilon \to 0$, we can interpret this as the commutation relation of operators:

$$[Q^a, \Phi] = -i G^a \Phi \tag{1.20}$$

Note that this is in the Euclidean formalism. Wick-rotating to Minkowski spacetime, and hence replacing $Q^a \to -iQ^a$ we have:

$$[Q^a, \Phi] = G^a \Phi \tag{1.21}$$

We can define the energy-momentum tensor $T^{\mu\nu}$ as the Noether current associated with spacetime translations. For a theory that is invariant under rotations and translations, the energy-momentum tensor can be symmetrized by employing a Belinfante tensor. The Belinfante tensor is not unique, nor is the energy-momentum tensor itself. However, physical observables such as total energy and momentum, which are integrals of the energy-momentum tensor over space, remain invariant under such redefinitions. Details of the proof can be found in chapter 2 of [1].

1.2 Critical Exponents

Widom and Fischer postulated that at the critical point of a phase transition, the free energy density f exhibits homogeneity under scaling transformations of its parameters. Each of the parameters has an associated scaling dimension. In CFT, we can assign scaling dimensions to fields and products of fields (with certain qualifications). It is natural to be familiar with the critical exponents that describe the behaviour of various physical quantities near the critical point to relate them to scaling dimensions in CFT.

For the simplest system, a two-dimensional classical Ising model with an external magnetic field h, we can list out the various critical exponents as follows:

Exponent	Definition	Ising Value
α	$C \sim \lvert T - T_c\rvert^{-\alpha}$	0
β	$M \sim \lvert T - T_c\rvert^{\beta}$	1/8
γ	$\chi \sim \lvert T - T_c\rvert^{-\gamma}$	7/4
δ	$M \sim \lvert h\rvert^{1/\delta}$ at $T = T_c$	15
ν	$\xi \sim \lvert T - T_c\rvert^{-\nu}$	1
η	$G(r) \sim r^{-d+2-\eta}$ at $T = T_c$	1/4

Table 2. Critical exponents for the 2D Ising model. Here, C is the specific heat, M is the magnetization, χ is the magnetic susceptibility, ξ is the correlation length, and $G(r)$ is the two-point correlation function of spins separated by distance r.

These exponents are not all independent; they satisfy scaling relations such as:

$$\begin{aligned} \alpha &= 2 - \nu d \\ \beta &= \frac{\nu}{2}(d - 2 + \eta) \\ \gamma &= \nu(2 - \eta) \\ \delta &= \frac{d + 2 - \eta}{d - 2 + \eta} \end{aligned} \tag{1.22}$$

which were discovered empirically before the advent of renormalization group techniques. We have then effectively expressed four of the six exponents in terms of the other two and the dimension d. It should be noted that both η and ν are related to correlations and the length scale associated with them, at and near the critical point respectively. A detailed discussion of critical exponents and scaling relations can be found in [2].

The only thing about renormalization that we need to know for this article is that renormalization can be thought of as *zooming out* of a system. As we zoom out, we integrate out the high-energy (short-distance) degrees of freedom and obtain an effective theory that describes the low-energy (long-distance) behaviour. The parameters of the theory, such as coupling constants, masses, and fields, get redefined in this process. Let's say we are dealing with a theory defined with fields that take continuous values at each point in space, and that the theory has some symmetries. As we zoom out, we redefine the fields and parameters, and add all possible symmetry allowed terms to the Lagrangian. This leads to a flow in the space of theories, called the renormalization group (RG) flow. Fixed points of this flow are scale-invariant theories. For theories which are Lorentz invariant, scale invariances implies conformal invariance. Thus, CFTs are the fixed points of RG flows when dealing with QFTs[3].

Observing that a phase is defined by which fixed point the RG flow ends up at, we can say that studying conformal field theories is equivalent to studying possible transition points between different phases of matter. Having a general theory of CFTs allows us to classify phases of matter and phase transitions between them. As we will see, two-dimensional

[3] This should be treated with a pinch of salt because the theory needs to be invariant under *special conformal transformations* as well, which is not a trivial requirement.

CFTs are are governed by the central charge c and the conformal dimensions of primary operators h and $\bar{h}$. These quantities can be related to critical exponents of physical systems at the critical point, allowing us to classify phases of matter in two dimensions.

1.3 Conformal Group and CFTs in dimension greater than 2

A conformal transformation is a spacetime coordinate transformation that preserves angles but not necessarily lengths. Mathematically, a conformal transformation is defined as a coordinate transformation $x \to x'$ such that the metric transforms as:

$$g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x) \tag{1.23}$$

where $\Omega(x)$ is a positive function called the conformal factor.

1.3.1 The Conformal Group and Classical Field Theory

Poincare transformations are conformal transformations with $\Omega(x) = 1$ and can be shown to form a subgroup of the conformal group. We need to constrain the infinitesimal $\epsilon^\mu(x) = x'^\mu - x^\mu$ such that the transformation is conformal.

$$\begin{aligned} g'_{\mu\nu} &= \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} \\ &= \left(\delta^\alpha_\mu - \partial_\mu \epsilon^\alpha\right) \left(\delta^\beta_\nu - \partial_\nu \epsilon^\beta\right) g_{\alpha\beta} \\ &= g_{\mu\nu} - \left(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu\right) \end{aligned} \tag{1.24}$$

Demanding that this transformation is conformal,

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = f(x) g_{\mu\nu} \tag{1.25}$$

Taking the trace, we find:

$$f(x) = \frac{2}{d} \partial_\rho \epsilon^\rho \tag{1.26}$$

Applying ∂_μ to (1.25), permuting the μ and ν indices, and taking a linear combination, we get:

$$2\partial_\mu \partial_\nu \epsilon_\rho = g_{\mu\rho} \partial_\nu f + g_{\nu\rho} \partial_\mu f - g_{\mu\nu} \partial_\rho f \tag{1.27}$$

Consider for simplicity the case when $g_{\mu\nu} = \eta_{\mu\nu}$, the flat Minkowski metric. Contracting with $\eta^{\mu\nu}$, we find:

$$2\Box \epsilon_\rho = (2 - d) \partial_\rho f \tag{1.28}$$

Applying ∂_ν to (1.28) and a box operator to (1.25), we find:

$$\begin{aligned} (2 - d) \partial_\mu \partial_\nu f &= \eta_{\mu\nu} \Box f \\ (d - 1) \Box f &= 0 \end{aligned} \tag{1.29}$$

For $d = 1$, there are no constraints on $f(x)$ and this implies that all smooth transformations are conformal. For $d > 2$, we have $\Box f = 0$, and hence $\partial_\mu \partial_\nu f = 0$. Thus, $f(x)$ is at most linear in x. The general solution for ϵ_μ can be found to be:

$$\epsilon_\mu = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho \tag{1.30}$$

with $c_{\mu\nu\rho} = c_{\mu\rho\nu}$. It is obvious that none of the above equations impose any constraints on a_μ. By looking at the linear term, we can see that:

$$b_{\mu\nu} + b_{\nu\mu} = \frac{2}{d}\eta_{\mu\nu}b^\rho_\rho \tag{1.31}$$

which in turn implies that $b_{\mu\nu}$ can be decomposed into an antisymmetric part and a trace part. The antisymmetric part generates Lorentz transformations, while the trace part generates scale transformations.

$$b_{\mu\nu} = \alpha\eta_{\mu\nu} + m_{\mu\nu} \tag{1.32}$$

with $m_{\mu\nu} = -m_{\nu\mu}$. Finally, looking at the quadratic term, we find that:

$$c_{\mu\nu\rho} = \eta_{\mu\nu}b_\rho + \eta_{\mu\rho}b_\nu - \eta_{\nu\rho}b_\mu \tag{1.33}$$

where $b_\mu = \frac{1}{d}c^\nu_{\mu\nu}$. We can see that for a transformation generated by only the quadratic term, the infinitesimal change is given by: $\epsilon_\mu = 2(b\cdot x)x_\mu - b_\mu x^2$. This is called a special conformal transformation (SCT).

From our discussion so far, we can infer that for a d-dimensional spacetime with $d > 2$, the conformal group is generated by:

Transformation	**Generator**
Translations	$P_\mu = -i\partial_\mu$
Lorentz	$L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) + S_{\mu\nu}$
Scale	$D = -i(x^\mu\partial_\mu + \Delta)$
SCT	$K_\mu = -i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu + 2\Delta x_\mu) - x^\nu S_{\mu\nu}$

Table 3. Generators of the conformal group in $d > 2$ dimensions. $S_{\mu\nu}$ are the spin matrices corresponding to the representation of the Lorentz group under which the fields transform. Δ is the scaling dimension of the field. K_μ are the generators of SCT.

It can be shown very easily, by setting:

$$\begin{aligned} J_{\mu\nu} &= L_{\mu\nu} \\ J_{\mu d} &= \frac{1}{2}(P_\mu - K_\mu) \\ J_{\mu(d+1)} &= \frac{1}{2}(P_\mu + K_\mu) \\ J_{d(d+1)} &= D \end{aligned} \tag{1.34}$$

and noting that $\eta_{dd} = 1$, $\eta_{(d+1)(d+1)} = -1$, and $\eta_{\mu\nu}$ is the flat Euclidean metric, that the conformal algebra in d dimensions is isomorphic to the algebra of $SO(d+1,1)$ for a field transforming trivially under all conformal transformations, i.e. a field with $\Delta = 0$ and $S_{\mu\nu} = 0$.

The canonical energy-momentum tensor $T^{\mu\nu}$ can be made traceless by including a term with the derivative of the Belinfante tensor and adding a term with the derivative of the *virial* of the field[4]. We have no general proof to show that $T^{\mu}_{\mu} = 0$ for all conformal field theories in two dimensions however. We believe this to be true. Moreover, it will be shown in the next section that in two dimensions, the expectation value of the square of the trace of the energy-momentum tensor vanishes.

Quasi-primary fields. A quasi-primary (spinless) field is one that transforms like:

$$\Phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\Delta/d} \Phi(x) \tag{1.35}$$

and where the Jacobian determinant is computed to be $\mathrm{Jac}(x', x) = \Omega(x)^{-d/2}$ from (1.23). Here, Δ is the scaling dimension of the field. Note that quasi-primary fields transform covariantly under global conformal transformations. It is impossible to define *local* conformal transformations in dimensions greater than two because the conformal group is finite-dimensional. There exists a notion of *primary fields* based on their properties under local transformations. We will discuss them in the next chapter[5].

1.3.2 Conformal Invariance in Quantum Field Theory

Ehrenfest's theorem suggests that our analysis of the classical theory should carry over to correlation functions and expectation values in the quantum theory, albeit with some modifications.

For simplicity, we present some general results for scalar fields and their correlation functions.

1. **Two-point function:** The two-point correlation function of two scalar quasi-primary fields ϕ_1 and ϕ_2 must transform as:

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \left|\frac{\partial x'}{\partial x}\right|^{\Delta_1/d}_{x=x_1} \left|\frac{\partial x'}{\partial x}\right|^{\Delta_2/d}_{x=x_2} \langle \phi_1(x_1')\phi_2(x_2')\rangle \tag{1.36}$$

Noting that the correlation function can only depend on the distance $|x_1 - x_2|$ and is invariant under dilations, we must have:

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} \tag{1.37}$$

We now invoke the fact that the correlation function must be invariant under special conformal transformations as well. Under an SCT generated by b_μ, we have:

$$\mathrm{Jac}(x_i', x_i) = \frac{1}{(1 - 2(b\cdot x_i) + b^2 x_i^2)^d} \tag{1.38}$$

Demanding invariance under SCTs, it is obtained that unless $\Delta_1 = \Delta_2$, $C_{12} = 0$. Thus, the two-point function is non-zero only when the scaling dimensions of the two fields are equal.

From 2, we can see that $\eta = 2\Delta - d + 2$.

[4] $V^\mu \equiv \frac{\delta\mathcal{L}}{\delta(\partial^\rho\Phi)}(\eta^{\mu\rho}\Delta + iS^{\mu\rho})\Phi$ is defined to be the virial of a field, and our major underlying assumption is that it is the divergence of some tensor $\sigma^{\alpha\mu}$: $V^\mu = \partial_\alpha\sigma^{\alpha\mu}$. A large class of theories obey this.

[5] The distinction between quasi-primary and primary fields is noted lucidly in [3].

2. **Three-point function:** A similar analysis for the three-point function shows that:

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{C_{123}}{|x_1-x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_2-x_3|^{\Delta_2+\Delta_3-\Delta_1}|x_3-x_1|^{\Delta_3+\Delta_1-\Delta_2}} \tag{1.39}$$

3. **Four-point function and higher:** The four-point function is not completely fixed by conformal invariance. This is because we can construct independent cross ratios of any four points that are invariant under conformal transformations. For four points x_1, x_2, x_3, x_4, we can define two independent *anharmonic* cross ratios:

$$u = \frac{|x_1-x_2||x_3-x_4|}{|x_1-x_3||x_2-x_4|} \quad , \quad v = \frac{|x_1-x_4||x_2-x_3|}{|x_1-x_3||x_2-x_4|} \tag{1.40}$$

   Thus, any function constructed from u and v formed from any four points chosen from the set $\{x_i\}$ will be invariant under conformal transformations. We can multiply this arbitrary function to the part of the four-point function that is fixed by conformal invariance.

Ward identities for the global conformal group are straightforward generalizations of (1.18). For a product of n fields $X = \Phi_1(x_1)\Phi_2(x_2)\cdots\Phi_n(x_n)$, we can show that[6]:

1. **Translations:** The current associated with translations is the energy-momentum tensor $T^{\mu\nu}$, which leads us to:

$$\partial_\mu\langle T^\mu_\nu(x)X\rangle = -i\sum_{i=1}^{n}\delta^{(d)}(x-x_i)\partial_\nu\langle X\rangle \tag{1.41}$$

2. **Lorentz transformations:** The associated current is given by $j^{\mu\nu\rho} = T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu$. The Ward identity is:

$$\langle(T^{\rho\nu} - T^{\nu\rho})(x)X\rangle = -i\sum_{i=1}^{n}\delta^{(d)}(x-x_i)S_i^{\nu\rho}\langle X\rangle \tag{1.42}$$

   which states that the energy-momentum tensor is symmetric except at the position of operator insertions.

3. **Scale transformations:** It is easy to derive:

$$\langle T^\mu_\mu(x)X\rangle = -\sum_{i=1}^{n}\delta^{(d)}(x-x_i)\Delta_i\langle X\rangle \tag{1.43}$$

Using (1.43), and choosing X to be $T^\nu_\nu(y)$, we find that in a 2d CFT[7] the expectation value of the square of the trace of the energy-momentum tensor vanishes. Particularly, in the ground state, the trace of the energy-momentum tensor is zero as an operator equation.

[6] Special Conformal Transformations are dealt with in Appendix A.

[7] We can restrict the form of the correlation (due to scale invariance) without any freedom for a trace term in 2d. There are trace terms in this correlation for higher dimensions.

2 Conformal Field Theory in Two Dimensions

We have encountered conformal transformations in two dimensions before. Recall how in [4], we saw that we could map solutions of the Laplace equation in two dimensions to other solutions using analytic functions. A particular example is the mapping of the upper half-plane to the unit disk using a Mobius transformation, which meant converting the problem of solving for the potential of a point charge near a conducting plane to that of a point charge at the center of a conducting circular ring. We used analytic functions because of the Cauchy-Riemann equations, which dictate that angles between functions on the complex plane, and their corresponding vector fields, are preserved under such mappings. Two dimensional Laplace equation is invariant under conformal transformations.

Motivated by such examples, and interested to study one dimensional quantum systems evolving in time (for example quantum spin chains or strings propagating in time), we study two dimensional conformal field theories. These are very different from higher dimensional CFTs because any analytic function defines a conformal transformation in two dimensions[8], in contrast to only certain quadratic functions in the former. Thus, the conformal group in two dimensions is infinite-dimensional.

2.1 Local Fields and Correlation Functions

Since we are dealing with complex analysis, we must fix notation with respect to the original coordinates ($(z^0,\ z^1)$ in our notation).

$$
\begin{aligned}
z &= z^0 + iz^1 & z^0 &= \frac{1}{2}(z+\bar{z}) \\
\bar{z} &= z^0 - iz^1 & z^1 &= \frac{1}{2}(z-\bar{z}) \\
\partial_z &= \frac{1}{2}(\partial_0 - i\partial_1) & \partial_0 &= \partial_z + \partial_{\bar{z}} \\
\partial_{\bar{z}} &= \frac{1}{2}(\partial_0 + i\partial_1) & \partial_1 &= i(\partial_z - \partial_{\bar{z}})
\end{aligned}
\tag{2.1}
$$

Usually we adopt the convention $\partial \equiv \partial_z$ and $\bar{\partial} \equiv \partial_{\bar{z}}$. In terms of the coordinates z and $\bar{z}$, the metric is represented as:

$$
g_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \tag{2.2}
$$

As is customary and often useful to do, we will analytically continue our domain so that both z and $\bar{z}$ are independent complex numbers, and impose the constraint that we are working on the real slice, i.e. $z^\star = \bar{z}$, when *necessary*. Such situations often arise at the end of some tedious algebra, and it will be useful to note that our theory becomes separable until that point. In some cases we will encounter subtlety associated with the choice of branch cuts, which will hopefully be resolved when we restrict to the reality constraint. Smooth functions in z are called holomorphic and those in $\bar{z}$ are called anti-holomorphic.

[8] Proof is omitted for brevity.

2.1.1 Global Transformations

These are transformations that are smooth, invertible, and map the whole plane plus the point at infinity[9] into itself. They are distinct from local transformations that may not be well-defined everywhere and are analytic except at poles.

The requirements stated above all but fix the form that the group elements can take. Consider $f(z)$ to be some global confrmal transformation. Any possibility of branch points and essential singularities are rules out f cannot be uniquely defined in the neighborhood of the former, and sweeps out the entire complex plane in a small neighborhood of the latter (is therefore non-invertible). Poles are the only allowed singularities. This fixes the form of f to be a rational function of polynomials in z. Further, we note that the denominator cannot have multiple distinct zeroes lest it be non-invertible. It also cannot have the same zero with an order different than 1 because that needs multiple Riemann sheets to be defined over the whole plane and hence is, again, non-invertible. Similar arguments for a point near infinity restrict the numerator to be linear in z as well.

We can then write:

$$f(z) = \frac{az+b}{cz+d} \tag{2.3}$$

and fix the determinant[10] to be $ad - bc = 1$ because the scale is arbitrary and can be normalized. This forms the group of scale-preserving, invertible, linear transformations of vectors in $\mathbb{C}^2$, i.e. $SL(2, \mathbb{C})$.

2.1.2 Local Transformations

The local conformal group is the set of all, not necessarily invertible, holomorphic maps. Consider a field $\phi(z, \bar{z})$ with scaling dimension Δ and a *planar* spin s. By a field in CFT, we mean any operator that is *local*, not only those fundamental fields that appear in the action. For example, ψ for the action of a free boson is the field that appears in the action in the form of its derivative $\partial_\mu \psi$. We consider any combination or product of the fields or their derivatives to be a local field.

Under a conformal map $z \to w = z + \epsilon(z)$ (and its anti-holomorphic counterpart[11]), the field transforms as:

$$\phi'(w, \bar{w}) = \left(\frac{dw}{dz}\right)^{-h} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}} \phi(z, \bar{z}) \tag{2.4}$$

where h and $\bar{h}$ are some real numbers called the *conformal* (anti-)*holomorphic dimension*. We call fields that transform like this *primary*. For infinitesimal transformations, we define the variation as:

$$\begin{aligned} \delta_{\epsilon,\bar{\epsilon}}\phi &\equiv \phi'(z, \bar{z}) - \phi(z, \bar{z}) \\ &= -(h\phi\partial\epsilon + \epsilon\partial\phi) - (\bar{h}\phi\bar{\partial}\bar{\epsilon} + \bar{\epsilon}\bar{\partial}\phi) \end{aligned} \tag{2.5}$$

[9]The whole Riemann sphere mapped into itself.

[10]It should be obvious that the matrices formed by these entries represent the group. Matrix multiplication is equivalent to composition of functions in this case. This also makes trivial the fact these matrices need are non-singular to be invertible.

[11]A phrase we will not repeat much. It will be assumed subsequently and mentioned only when necessary.

In going from the first line to the second, we have used (2.4) and the fact that the variation is computed at the same point and have hence Taylor expanded the field around z. Fields that transform like anything else are called *secondary*. Observe that equations (2.5), (2.1), and Table 1 together imply that $h + \bar{h} = \Delta$, and $h - \bar{h} = s$.

For a spinless and dimensionless field Φ, and writing $\epsilon(z) = \sum_{-\infty}^{\infty} c_n z^{n+1}$, and defining generators:

$$\ell_n = -z^{n+1}\partial \qquad \bar{\ell}_n = -\bar{z}^{n+1}\bar{\partial} \tag{2.6}$$

We can then write the variation as:

$$\delta\phi = \sum_n \left\{ c_n \ell_n + \bar{c}_n \bar{\ell}_n \right\} \phi \tag{2.7}$$

The generators follow the commutation relations, known as the Witt algebra:

$$\begin{aligned} [\ell_n, \ell_m] &= (n-m)\ell_{n+m} \\ [\bar{\ell}_n, \bar{\ell}_m] &= (n-m)\bar{\ell}_{n+m} \\ [\ell, \bar{\ell}_m] &= 0 \end{aligned} \tag{2.8}$$

This algebra is the direct sum of two isomorphic algebras. Each of these two infinite-dimensional algebras have finite subalgebras generated by ℓ_{-1}, ℓ_0, and ℓ_1. These constitute the global transformations: ℓ_0 generates dilations, ℓ_{-1} generates translations, ℓ_1 generates SCT. $\ell_n + \bar{\ell}_n$ and $i(\ell_n - \bar{\ell}_n)$ preserve the real surface; the zeroeth term of the former generates dilations on it and that of the latter generates rotations on it.

Correlations of primary fields, specifically two-point and three-point functions, have their form fixed like in (1.37) and (1.39). We can include the contribution of the planar spin using the differences $s_i = h_i - \bar{h}_i$. We simply state the results:

$$\begin{aligned} \langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)\rangle &= \frac{C_{12}}{z_{12}^{2h}\bar{z}_{12}^{2\bar{h}}} \quad \text{for} \quad \begin{cases} h_1 = h_2 = h \\ \bar{h}_1 = \bar{h}_2 = \bar{h} \end{cases} \\ \langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle &= \frac{C_{123}}{\prod_{\text{cyclic}} z_{12}^{h_1+h_2-h_3}\bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3}} \end{aligned} \tag{2.9}$$

Higher-order correlations have functions of anharmonic ratios mutliplied to the form fixed by global conformal transformations[12]. It is important to note that to avoid branch cuts in the two-point correlation, the conformal dimensions should be integers or half integers[13]. This agrees well with the spin-statistics theorem.

2.2 Conformal Ward Identities

Right off the bat, we introduce some notation. We need to talk about the antisymmetric tensor $\varepsilon_{\mu\nu}$ in holomorphic form, with the Jacobian determinant factor incorporated:

$$\varepsilon_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2}i \\ -\frac{1}{2}i & 0 \end{pmatrix} \qquad \varepsilon^{\mu\nu} = \begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix} \tag{2.10}$$

[12] Though in two dimensions, the two ratios constructed from four points are not independent. The multiplied factor ends up being a function of just this unique cross ratio.

[13] This can be bypassed in two dimensions using *parafermions*, something that I shall not discuss here.

In this notation, the holomorphic Cauchy-Riemann equation becomes $\bar{\partial} w(z, \bar{z}) = 0$.

Equation (1.42) is modified to:

$$\varepsilon_{\mu\nu}\langle T^{\mu\nu}(x)X\rangle = -i\sum_{i=1}^{n} s_i \delta(x - x_i)\langle X\rangle \tag{2.11}$$

We restrict to the assumption that the energy-momentum tensor can be made symmetric and traceless in two dimensions. We also use the identity:

$$\delta(x) = \frac{1}{\pi}\bar{\partial}\frac{1}{z} = \frac{1}{\pi}\partial\frac{1}{\bar{z}} \tag{2.12}$$

In holomorphic coordinates, the Ward identities can be explicitly written as:

$$\begin{aligned}
2\pi\partial\langle T_{\bar{z}z}X\rangle + 2\pi\bar{\partial}\langle T_{zz}X\rangle &= -\sum_{i=1}^{n}\bar{\partial}\frac{1}{z - w_i}\partial_{w_i}\langle X\rangle \\
2\pi\partial\langle T_{\bar{z}\bar{z}}X\rangle + 2\pi\bar{\partial}\langle T_{z\bar{z}}X\rangle &= -\sum_{i=1}^{n}\partial\frac{1}{\bar{z} - \bar{w}_i}\partial_{\bar{w}_i}\langle X\rangle \\
2\langle T_{z\bar{z}}X\rangle + 2\langle T_{\bar{z}z}X\rangle &= -\sum_{i=1}^{n}\delta(x - x_i)\Delta_i\langle X\rangle \\
-2\langle T_{z\bar{z}}X\rangle + 2\langle T_{\bar{z}z}X\rangle &= -\sum_{i=1}^{n}\delta(x - x_i)s_i\langle X\rangle
\end{aligned} \tag{2.13}$$

The n points x_i are now described by the $2n$ complex coordinates $(w_i, \bar{w}_i)$ on which the primary fields X depends. Adding and subtracting the last two equations leads us to:

$$\begin{aligned}
2\pi\langle T_{\bar{z}z}X\rangle &= -\sum_{i=1}^{n}\bar{\partial}\frac{1}{z - w_i}h_i\langle X\rangle \\
2\pi\langle T_{z\bar{z}}X\rangle &= -\sum_{i=1}^{n}\partial\frac{1}{\bar{z} - \bar{w}_i}\bar{h}_i\langle X\rangle
\end{aligned} \tag{2.14}$$

Introducing the notation $T = -2\pi T_{zz}$, we get:

$$\bar{\partial}\left(\langle T(z, \bar{z})X\rangle - \sum_{i=1}^{n}\left[\frac{\partial_{w_i}}{z - w_i} + \frac{h_i}{(z - w_i)^2}\right]\langle X\rangle\right) = 0 \tag{2.15}$$

This implies that T is a function of z only. We could equivalently note that:

$$\langle T(z)X\rangle = \sum_{i=1}^{n}\left[\frac{\partial_{w_i}}{z - w_i} + \frac{h_i}{(z - w_i)^2}\right]\langle X\rangle + \text{reg.} \tag{2.16}$$

where "reg." stands for an analytic function of z, regular at all points of insertions of fields in X. It is possible to write all identities in (2.13) in one equation. This is done if we consider the variation of the expectation value of X w.r.t. infinitesimal variations $\epsilon_\nu(x)$.

$$\partial_\mu(\epsilon_\nu T^{\mu\nu}) = \epsilon_\nu\partial_\mu T^{\mu\nu} + \frac{1}{2}(\partial_\sigma\epsilon^\sigma)T^\mu_\mu + \frac{1}{2}\varepsilon^{\alpha\beta}\partial_\alpha\epsilon_\beta\varepsilon_{\mu\nu}T^{\mu\nu} \tag{2.17}$$

We can then integrate the usual Ward identity (1.16) to obtain:

$$\delta_\epsilon \langle X \rangle = \int_M d^2x \partial_\mu \langle \epsilon_\nu T^{\mu\nu} X \rangle \tag{2.18}$$

where the integral is taken over a domain which includes all x_i. We can then use Stokes' law to convert this into a line integral. Only T_{zz} and $T_{\bar{z}\bar{z}}$ contribute to the integral since, by (1.42), the lines do not go through the points of insertion of the fields in X and hence the cross terms vanish. We can then write, after changing coordinates and including contribution from ϵ and $\bar{\epsilon}$:

$$\delta_{\epsilon,\bar{\epsilon}} \langle X \rangle = -\frac{1}{2\pi i} \oint_c dz \cdot \epsilon \langle TX \rangle + \frac{1}{2\pi i} \oint_c d\bar{z} \cdot \bar{\epsilon} \langle \bar{T} X \rangle \tag{2.19}$$

This is called the *Conformal Ward Identity*. Its effects are much more general than the limited scope of (2.13). In particular, we can apply (2.16) and (2.19) together on a primary field ϕ to obtain, by method of residues, (2.5). This strongly suggests that we can take this to be the definition of variation of arbitrary local fields (within a correlation function, of course).

An interesting insight can be gained if we apply the conformal Ward identity to infinitesimal $SL(2,\mathbb{C})$ transformations. Such maps have the form:

$$f(z) = \frac{(1+\alpha)z + \beta}{\gamma z + (1-\alpha)} \tag{2.20}$$

which when expanded to first order in $\{\alpha, \beta, \gamma\}$ gives: $\epsilon(z) = \beta + 2\alpha z - \gamma z^2$. For arbitrary $\{\alpha, \beta, \gamma\}$, we have:

$$\begin{aligned} \sum_i \partial_{w_i} \langle X \rangle &= 0 \\ \sum_i (w_i \partial_{w_i} + h_i) \langle X \rangle &= 0 \\ \sum_i (w_i^2 \partial_{w_i} + 2 w_i h_i) \langle X \rangle &= 0 \end{aligned} \tag{2.21}$$

corresponding to translations, Lorentz rotations, and SCT respectively. These are easy to check for two-point and three-point functions given in (2.9).

2.3 Operator Product Expansions

At critical points, autocorrelation of order parameter fields diverges as the fields get closer. This behaviour is also seen in correlations of different fields if there is more than one order parameter field. Even with just one fuch field, we can compute the divergences of correlations of multiple (for any order) derivatives of this field. We want to quantify this divergence as a function of the closeness of the positions of insertion of operators.

Wilson introduced the idea that it may be possible for two fields $A(z)$ and $B(w)$ at nearby points z and w to admit a product represented by a sum of (possibly different) fields at one point, say w, multiplied by a c-number function of $z - w$, possibly diverging as $z \to w$. We can write:

$$A(z)B(w) = \sum_{n=-\infty}^{N} \frac{\{AB\}_n(w)}{(z-w)^n} \tag{2.22}$$

where the composite fields $\{AB\}_n$ are non-singular at $w = z$. Generally, we are interested in the divergent behaviour, so we would deal with terms having $n \leq 0$ with little care. An important point to note here is that this *operator product expansion* (or OPE) is meaningful only within correlation functions.

In fact, we have already encountered OPEs earlier in this article. For a single primary field ϕ of conformal dimensions h and $\bar{h}$, we have:

$$T(z)\phi(w,\bar{w}) \sim \frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{z-w}\partial_w\phi(w,\bar{w}) \tag{2.23}$$

We expect to impose stringent constraints on the singular terms of the OPE using the conformal Ward identity. To this extent, we ignore the regular differences and use the similarity notation to represent just the singular part.

2.3.1 Free Boson

In two dimensions, a free boson theory is given by

$$S = \frac{1}{2}g\int d^2x\, \partial_\mu\varphi\partial^\mu\varphi \tag{2.24}$$

This theory is conformally invariant with scaling dimension of the field φ given by $\Delta = 0$, and no planar spin. The propagator of this theory is given by:

$$\langle\varphi(x)\varphi(y)\rangle = -\frac{1}{4\pi g}\ln(x-y)^2 + \text{const.} \tag{2.25}$$

which in holomorphic coordinates is

$$\langle\varphi(z,\bar{z})\varphi(w,\bar{w})\rangle = -\frac{1}{4\pi g}\left[\ln(z-w) + \ln(\bar{z}-\bar{w})\right] + \text{const.} \tag{2.26}$$

Observe that the OPE of $\partial\varphi$ with itself is

$$\partial\varphi(z)\partial\varphi(w) \sim -\frac{1}{4\pi g}\frac{1}{(z-w)^2} \tag{2.27}$$

The energy-momentum tensor is

$$T_{\mu\nu} = g(\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}\eta_{\mu\nu}\partial_\rho\varphi\partial^\rho\varphi) \tag{2.28}$$

which in holomorphic coordinates translates to

$$T(z) = -2\pi g : \partial\varphi\partial\varphi : \tag{2.29}$$

where the normal ordering is defined such that the vaccum expectation value vanishes. It is written explicitly as

$$T(z) = -2\pi g \lim_{z\to w}\left(\partial\varphi(z)\partial\varphi(w) - \langle\partial\varphi(z)\partial\varphi(w)\rangle\right) \tag{2.30}$$

We now compute the first non-trivial thing; the OPE of $T\partial\varphi$. Wick's theorem comes to our rescue. In going from the second line to the third, we drop the regular term.

$$
\begin{aligned}
T(z)\partial\varphi(w) &= -2\pi g : \partial\varphi(z)\partial\varphi(z) : \partial\varphi(w) \\
&= -4\pi g : \partial\varphi(z)\partial\varphi(z) : \partial\varphi(w) - 2\pi g : \partial\varphi(z)\partial\varphi(z)\partial\varphi(w) : \\
&\sim -4\pi g : \partial\varphi(z)\partial\varphi(z) : \partial\varphi(w) \\
&\sim \frac{\partial\varphi(z)}{(z-w)^2} \\
&\sim \frac{\partial\varphi(w)}{(z-w)^2} + \frac{\partial_w^2\varphi(w)}{(z-w)}
\end{aligned}
\tag{2.31}
$$

which helps us identify $h = 1$. This is expected since φ is dimensionless and spinless, so its derivative has scaling dimension 1. We can also compute the OPE of the energy-momentum tensor with itself:

$$
\begin{aligned}
T(z)T(w) &= 4\pi^2 g^2 : \partial\varphi(z)\partial\varphi(z) :: \partial\varphi(w)\partial\varphi(w) : \\
&\sim 8\pi^2 g^2 \Big[: \partial\varphi(z)\partial\varphi(z) :: \partial\varphi(w)\partial\varphi(w) : + 2 : \partial\varphi(z)\partial\varphi(z) :: \partial\varphi(w)\partial\varphi(w) : \Big] \\
&\sim \frac{1/2}{(z-w)^4} - \frac{4\pi g : \partial\varphi(z)\partial\varphi(w) :}{(z-w)^2} \\
&\sim \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}
\end{aligned}
\tag{2.32}
$$

2.3.2 Free Fermion

The action is given by

$$
S = \frac{1}{2} g \int d^2x \Psi^\dagger \gamma^0 \gamma^\mu \partial_\mu \Psi \tag{2.33}
$$

where the Dirac matrices follow the Clifford algebra for a Euclidean metric, and take the form

$$
\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^1 = i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{2.34}
$$

This implies that

$$
\gamma^0(\gamma^0\partial_0 + \gamma^1\partial_1) = 2 \begin{pmatrix} \bar\partial & 0 \\ 0 & \partial \end{pmatrix} \tag{2.35}
$$

which lets us break the Majorana two-component spinor Ψ into holomorphic and anti-holomorphic parts $(\psi, \bar\psi)$, with the action in holomorphic coordinates given by

$$
S = g \int d^x (\bar\psi \partial \bar\psi + \psi \bar\partial \psi) \tag{2.36}
$$

The classical equations of motion imply that ψ is holomorphic and $\bar\psi$ is anti-holomorphic, as expected.

We can now find the correlation functions. The propagator can be broken down, using usual methods of Green's functions, such that we obtain the equation

$$2g\begin{pmatrix}\bar\partial & 0\\ 0 & \partial\end{pmatrix}\begin{pmatrix}\langle\psi(z)\psi(w)\rangle & \langle\psi(z)\bar\psi(w)\rangle\\ \langle\bar\psi(z)\psi(w)\rangle & \langle\bar\psi(z)\bar\psi(w)\rangle\end{pmatrix} = \delta(x) = \frac{1}{\pi}\begin{pmatrix}\bar\partial\frac{1}{z-w} & 0\\ 0 & \partial\frac{1}{\bar z-\bar w}\end{pmatrix} \tag{2.37}$$

The solutions (and OPEs constructed of ψ fields and derivatives thereof) are easily read off.

The energy-momentum tensor can be computed to be $T(z) = -\pi g : \psi(z)\partial\psi(z) :$ and now we can find the OPE with the fields to find their conformal dimension.

$$\begin{aligned} T(z)\psi(w) &= -\pi g : \psi(z)\partial\psi(z) : \psi(w) \\ &\sim -\pi g : \psi(z)\partial\psi(z) : \psi(w) - \pi g : \psi(z)\partial\psi(z) : \psi(w) \\ &\sim \frac{1/2\partial(z)}{z-w} + \frac{1/2\psi(z)}{(z-w)^2} \\ &\sim \frac{\frac{1}{2}\psi(w)}{(z-w)^2} + \frac{\partial\psi(w)}{(z-w)} \end{aligned} \tag{2.38}$$

which implies that ψ is a primary field with $h = 1/2$.

The OPE of the energy-momentum tensor with itself is:

$$\begin{aligned} T(z)T(w) &= \pi^2 g^2 : \psi(z)\partial\psi(z) :: \psi(w)\partial\psi(w) : \\ &\sim (\pi^2 g^2)\Big[- : \psi(z)\partial\psi(z) :: \psi(w)\partial\psi(w) : \\ &\quad + : \psi(z)\partial\psi(z) :: \psi(w)\partial\psi(w) : \\ &\quad + : \psi(z)\partial\psi(z) :: \psi(w)\partial\psi(w) : \\ &\quad - : \psi(z)\partial\psi(z) :: \psi(w)\partial\psi(w) : \Big] \\ &\sim \frac{1/4}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \end{aligned} \tag{2.39}$$

Observing that both (2.32) and (2.39) have similar forms, and it can be speculated that the OPE of T with itself always takes the form:

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \tag{2.40}$$

where we postulate a theory-dependent constant c called the central charge. This form of the OPE is well motivated since we know that T is a quasi-primary field with scaling dimension 2 because it is an energy density on a two dimensional system. We will study the behavior of $T(z)$ near origin in the following section.

The central charge contributes to the central extension of the Witt algebra resulting in the Virasoro algebra. This will be covered in detail in the following chapters.

2.3.3 Generalized Ghosts

For sake of completeness, we include an application of string theory. When we fix the gauge on the worldsheet, this introduces the Faddeev-Popov determinant in the path integral.

This can be represented as by an action obtained after integration over fermionic *ghost* fields. The action is given by:

$$S = \frac{1}{2}g \int d^2x b_{\mu\nu}\partial^\mu c^\nu \tag{2.41}$$

where the field $b_{\mu\nu}$ is a traceless symmetric tensor, where both b and c are fermionic. The propagator can be computed by

$$\frac{1}{2}g\delta^\mu_\alpha \partial^\nu K^\beta_{\mu\nu}(x,y) = \delta(x-y)\delta_{\alpha\beta} \tag{2.42}$$

where the index α is also two dimensional. $K^z_{zz}(z,w)$ represents the correlation $\langle b(z)c(w)\rangle$ and it is computed to be

$$\langle b(z)c(w)\rangle = \frac{1}{\pi g}\frac{1}{z-w} \tag{2.43}$$

The canonical energy-momentum tensor is not identically symmetric or traceless for this theory. However, since there is Poincare symmetry, it can be made symmetric by adding the derivative of a Belinfante tensor. After some tedious algebra, it can be shown that[14]

$$T^{\mu\nu}_B = \frac{1}{2}g\left[b^{(\mu|\alpha|}\partial^{\nu)}c_\alpha + \partial_\alpha b^{\mu\nu}c^\alpha - \eta^{\mu\nu}b^{\alpha\beta}\partial_\alpha c_\beta\right] \tag{2.44}$$

is symmetric as well as traceless. The normal-ordered holomorphic component can be written down as:

$$T(z) = \pi g : 2\partial c b + c\partial b : \tag{2.45}$$

The system can be further generalised. If instead we had the OPE defined by

$$c(z)b(w) \sim \frac{1}{z-w} \qquad b(z)c(w) \sim \frac{\epsilon}{z-w} \tag{2.46}$$

where we have absorbed the coefficient into the definition of b and c, and $\epsilon = \pm 1$ depending on whether the ghosts anticommute or commute respectively.

The associated energy-momentum tensor is then given by $T(z) = (1-\lambda) : \partial bc : -\lambda : b\partial c :$ and we can compute the OPEs of T with the fields b and c to obtain that they have conformal weights λ and $1-\lambda$ respectively.

The structure of the generalized ghost system depends on the conformal weight λ of the b-field, which in turn determines the weights of both fields and the total central charge. Various important systems in string theory and conformal field theory arise as special cases of this general construction. The following table summarizes the most relevant examples:

System	$h_b = \lambda$	$h_c = 1-\lambda$	c_{ghost}	**Context**
Reparametrization ghosts	2	-1	-26	Bosonic string theory
Superconformal ghosts (β,γ)	$\frac{3}{2}$	$-\frac{1}{2}$	$+11$	Superstring theory
Scalar fermions (Dirac fields)	$\frac{1}{2}$	$\frac{1}{2}$	$+1$	Free fermion CFT

Table 4. Representative examples of the generalized ghost system for conformal weights λ.

[14] This subsection is intentionally light on the algebra.

These examples illustrate that the conformal weight directly determines the contribution of the ghost sector to the total central charge of the theory. This contribution plays a crucial role in ensuring the consistency of the quantized theory, as we discuss next.

In any string worldsheet theory, conformal invariance at the quantum level requires that the total central charge vanish. Denoting the matter and ghost contributions as c_{matter} and c_{ghost}, the condition is

$$c_{\text{matter}} + c_{\text{ghost}} = 0. \tag{2.47}$$

For the bosonic string, the matter sector contributes $c_{\text{matter}} = D$, where D is the spacetime dimension. Using the reparametrization ghost value $c_{\text{ghost}} = -26$, this gives the well-known critical dimension

$$D - 26 = 0 \quad \Rightarrow \quad D = 26. \tag{2.48}$$

For the superstring, the matter fields include both bosons and worldsheet fermions, contributing $c_{\text{matter}} = \frac{3}{2}D$, while the combined ghost systems (reparametrization and superconformal) give $c_{\text{ghost}} = -26 + 11 = -15$. Hence, conformal anomaly cancellation requires

$$\tfrac{3}{2}D - 15 = 0 \quad \Rightarrow \quad D = 10. \tag{2.49}$$

Thus, the vanishing of the total central charge dictates the critical spacetime dimension in which the theory is consistent and anomaly-free.

The generalized ghost system can be realized with either fermionic (anticommuting) or bosonic (commuting) statistics. The fermionic case corresponds to the familiar (b, c) system, while the bosonic version is described by (β, γ) fields. Although their stress tensors share the same structural form, the central charge changes sign due to their statistics. The following table summarizes the two cases:

Type	**Statistics**	**Weights** (h_b, h_c)	**OPE**	**Central Charge**
(b, c)	Fermionic	$(\lambda,\, 1-\lambda)$	$b(z)c(w) \sim \dfrac{1}{z-w}$	$c = 1 - 3(2\lambda - 1)^2$
(β, γ)	Bosonic	$(\lambda,\, 1-\lambda)$	$\beta(z)\gamma(w) \sim \dfrac{1}{z-w}$	$c = -1 + 3(2\lambda - 1)^2$

Table 5. Comparison between fermionic and bosonic generalized ghost systems.

The sign reversal in the central charge is a direct reflection of the Grassmann parity of the fields. Fermionic ghosts are anticommuting and yield a negative central charge for typical integer λ (as in the reparametrization case), while bosonic ghosts produce positive contributions, such as in the superconformal $\beta\gamma$ system of the RNS superstring.

From a geometric point of view, the (b, c) system can be interpreted in terms of sections of vector bundles over the Riemann surface that represents the worldsheet. If K denotes the canonical bundle, then the b-field and c-field can be viewed as

$$b \in \Gamma(K^{\lambda}), \qquad c \in \Gamma(K^{1-\lambda}), \tag{2.50}$$

where $\Gamma(K^{\lambda})$ denotes the space of sections of the λ-th tensor power of the canonical bundle. For $\lambda = 2$, corresponding to the reparametrization ghosts, c transforms as a vector field on the worldsheet, generating infinitesimal diffeomorphisms, while b transforms as a rank-2 covariant tensor. This geometric characterization provides a natural framework for understanding the role of ghosts in the moduli space of Riemann surfaces and in the computation of string scattering amplitudes.

2.4 Properties of the Energy-Momentum Tensor

The infinitesimal variation of $T(w)$ under an infinitesimal $\epsilon(z)$ can be computed by first applying the conformal Ward identity, writing $T(z)T(w)$ in its OPE, and finally computing the integral by the method of residues. This results in

$$\begin{aligned}\delta_\epsilon T(w) &= -\frac{1}{2\pi i}\oint dz\, \epsilon(z)T(z)T(w) \\ &= -\frac{c}{12}\partial_w^3\epsilon(w) - 2T(w)\partial_w\epsilon(w) - \epsilon(w)\partial_w T(w)\end{aligned} \tag{2.51}$$

The exponentiation of this infinitesimal variation, through non-trivial algebra, yields the exact form:

$$\begin{aligned}T'(w) &= (\partial w)^{-2}\left[T(z) - \frac{c}{12}\{w;z\}\right] \\ &= (\partial w)^{-2}T(z) + \frac{c}{12}\{z;w\}\end{aligned} \tag{2.52}$$

where $\{w;z\}$ is the *Schwarzian* derivative given by

$$\{w;z\} = \frac{\partial^3 w}{\partial w} - \frac{3}{2}\left(\frac{\partial^2 w}{\partial w}\right)^2 \tag{2.53}$$

One can verify that this transformation forms a group. It is also easy to show that the Schwarzian derivative vanishes for any global conformal transformation[15].

Particularly, the behavior of $T(z)$ near the origin can be mapped to its behavior at infinity. For $w = 1/z$

$$T'(w) = z^4 T(z) \tag{2.54}$$

The condition that $T'(1/z)$ is just as regular as as $T(z)$ and that $T'(0)$ is finite together imply that $T(z)$ must decay as z^{-4} for $z \to \infty$.

c has important interpretations. When we impose periodic boundary conditions, let's say in the form of a cylinder brought about from a map from the plane, the central charge contributes linearly to the expectation value of the *Casimir* energy[16]. If we were working on a two-dimensional Riemannian manifold with non-zero Ricci scalar curvature, then the expectation value of the trace of the energy-momentum tensor picks up a linear contribution from c[17]. Entropy of high energy states, as computed by Cardy [6], is proportional to the square root of c. In fact, in the space of all theories, Zamalodchikov's c-theorem [7] proves that there exists (in all dimensions) a c-function in QFT that under the RG flow monotonically converges to the central charge (only in two dimensions)[18].

[15] This is to be expected since global conformal transformations are true symmetries of the theory and the energy-momentum tensor is a quasi-primary field with conformal dimension two.

[16] Refer to Appendix B

[17] Refer to Appendix C and [5].

[18] Extensions of this idea to four dimensions, primarily due to [8–11], lead to the A-theorem.

3 Operator Algebra and Conformal Families

We have so far restricted our attention to just correlation functions and the constraints imposed on them due to the conformal Ward identities. In principle, we could have obtained all information about the correlation functions from the path integrals.

There exists a far more powerful correspondence in CFT: between states and operators, and the goal of this chapter is to introduce this. We do this by first noting that since the choice of time in Euclidean formalism is arbitrary, we can choose the radial coordinate to be time. Using this, one can talk about states in the far past and the far future w.r.t operators at the origin and at very large radius, respectively. Mode expansions of primary fields and of the energy-momentum tensor then motivate the need to construct a new notion of normal ordering. We end the first section by taking the example of vertex operators and applying our formalism to obtain that they are primary fields.

Next, we formally derive the Virasoro algebra given an energy-momentum tensor with central charge c, and construct the Hilbert state. The free boson is solved on the cylinder and vertex operators are used to find vacuum states with fixed center-of-mass momentum.

This chapter ends with a lengthy discussion of descendant fields, conformal families, and the algebra of operators. We illustrate how we can obtain the complete operator algebra of primary fields, given just the central charge, the conformal dimensions of the primary fields, and the coefficient of the three-point correlation function.

3.1 Ordering

3.1.1 Time as Radius

Choosing a time direction in Minkowski spacetime breaks the Lorentz invariance but is not as arbitrary as choosing one in Euclidean spacetime. In context of statistical mechanics, the problem can be thought of as choosing the direction in which transfer matrices act, and on what direction do we define the configurations of the degrees of freedom. The limit of lattice spacing going to zero then admits the possibility of choosing the transfer matrices in a radial direction.

Say the flat spacetime with PBC of space-length L is labelled by $\xi = t + ix$ (the theory lies on a cylinder), then we can explode the cylinder into the Riemann sphere (complex plane plus the point at infinity).

$$z = e^{2\pi\xi/L} \tag{3.1}$$

The far past is now equivalent to the origin and the far future becomes the point at infinity. We can similarly define an antiholomorphic variable and impose the reality constraints at the end of the calculation.

In order to make a sensible map between states and operators, we must assume that states exist in the first place. The key assumption is that there exists a vacuum state $|0\rangle$ upon which the Hilbert space is constructed by the application of creation operators[19]. For a free theory, the vacuum state is defined to be the one that is annihilated by the positive frequency part of the field. Assuming that interactions are not switched on from

[19]Whatever may be the form that they take in this theory.

past infinity to future infinity, and that the theory is free at those points, we can define asymptotic in-states and out-states.

$$|\phi_{\text{in}}\rangle = \lim_{z,\bar{z}\to 0} \phi(z,\bar{z})\,|0\rangle \tag{3.2}$$

We will define the Hermitian inner product by directly specifiying the out-state and defining the action of the Hermitian conjugation on conformal fields. Let ϕ be a quasi-primary field with conformal dimensions $(h,\bar{h})$. We motivate the definition of the inner product by introducing the mode expansion of $\phi(z,\bar{z})$:

$$\begin{aligned} \phi(z,\bar{z}) &= \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} z^{-m-h}\bar{z}^{-n-\bar{h}}\phi_{m,n} \\ \phi_{m,n} &= \frac{1}{2\pi i}\oint dz\, z^{m+h-1}\frac{1}{2\pi i}\oint d\bar{z}\,\bar{z}^{n+\bar{h}-1}\phi(z,\bar{z}) \end{aligned} \tag{3.3}$$

Straightforward Hermitian conjugation on the real surface yields

$$\phi(z,\bar{z})^\dagger = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \bar{z}^{-m-h}z^{-n-\bar{h}}\phi^\dagger_{m,n} \tag{3.4}$$

The demand that $\phi^\dagger_{m,n} = \phi_{-m,-n}$, as is usual for free fields in QFT, requires that:

$$\phi(z,\bar{z})^\dagger = \bar{z}^{-2h}z^{-2\bar{h}}\phi(1/\bar{z},1/z) \tag{3.5}$$

and this helps us identify $\langle\phi_{\text{out}}| = |\phi_{\text{in}}\rangle^\dagger$. This also shows that the inner product on asymptotic states exists as

$$\begin{aligned} \langle\phi_{\text{out}}|\phi_{\text{in}}\rangle &= \lim_{z,\bar{z},w,\bar{w}\to 0} \langle 0|\phi(z,\bar{z})^\dagger\phi(w,\bar{w})|0\rangle \\ &= \lim_{z,\bar{z},w,\bar{w}\to 0} \bar{z}^{-2h}z^{-2\bar{h}}\,\langle 0|\phi(1/\bar{z},1/z)\phi(w,\bar{w})|0\rangle \\ &= \lim_{\xi,\bar{\xi}\to\infty} \bar{\xi}^{2h}\xi^{2\bar{h}}\,\langle 0|\phi(\xi,\bar{\xi})\phi(0,0)|0\rangle \end{aligned} \tag{3.6}$$

which, by (1.37), makes perfect sense.

If the in-states and out-states are well-defined, then the mode expansion of the field should act on the vacuum as:

$$\phi_{m,n}\,|0\rangle = 0 \qquad (m > -h,\, n > -\bar{h}) \tag{3.7}$$

It is obvious that in this scheme, time ordering operators corresponds to ordering them radially[20]. Since all field operators within a correlation must be time ordered, the OPEs that we have discussed before make sense only when the operators are radially ordered. The procedure is called *radial quantization*.

We now relate the OPEs to commutation relations. We consider two types of commutators.

$$\begin{aligned} [A,b(w)] &\quad \text{where} \quad A \equiv \oint dz\, a(z) \\ [A,B] &\quad \text{where} \quad A \equiv \oint dz\, a(z) \text{ and } B \equiv \oint dz\, b(z) \end{aligned} \tag{3.8}$$

[20]Fermionic fields pick up a sign based on the permutation when being radially ordered.

These obviously correspond to anti-commutators if the fields are mutually fermionic. Note that any contour integral around a non-origin point w can be broken into the difference of two integrals: one with a radius just slightly larger than that of w and the other with a radius slightly smaller. In the limit that the both these radii converge, we get the exact operator equality[21]. This amounts to an equal time commutator. Written formally:

$$\begin{aligned} [A, b(w)] &= \oint_w dz\, a(z)b(w) \\ [A, B] &= \oint_0 dw \oint_w dz\, a(z)b(w) \end{aligned} \tag{3.9}$$

3.1.2 Prescription for Normal Ordering

Having constructed the time-ordering, one would expect the normal ordering to be trivial like in (2.30). However, this prescription only works for very simple examples, and in most cases is restricted to just the normal ordering of local fields constructed from primary fields explicitly. Consider for example the same normal ordering procedure applied to $T(z)T(w)$.

A slight detour follows. If we consider (1.43) more generally, we can construct the *Schwinger* function, defined as $S_{\mu\nu\rho\sigma}(x) = \langle T_{\mu\nu}(x)T_{\rho\sigma}(0)\rangle$ which in two dimensions can be represented as[22]:

$$S_{\mu\nu\rho\sigma}(x) = \frac{c}{4\pi^2}\frac{\left[(3g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})(x^2)^2 - 4x^2(g_{\mu\nu}x_\rho x_\sigma + g_{\rho\sigma}x_\mu x_\nu) + 8x_\mu x_\nu x_\rho x_\sigma\right]}{(x^2)^4} \tag{3.10}$$

Converted to holomorphic coordinates, we get

$$\langle T(z)T(0)\rangle = \frac{c/2}{z^4} \tag{3.11}$$

Using this and (2.40), we obtain that the normal ordering prescription fails and terms with second order and first order poles still remain. We must modify our notion of contraction to include all singular terms of the OPE. For two local fields $A(z)$ and $B(w)$, we define the contraction to be:

$$\contraction{}{A(z)}{}{B}A(z)B(w) \equiv \sum_{n=1}^{N} \frac{\{AB\}_n(w)}{(z-w)^n} \tag{3.12}$$

following the notation of (2.22). We want the normal ordering to yield $(AB)(w)$

$$\begin{aligned} (AB)(w) &= \{AB\}_0 \\ &= \lim_{z\to w}\left[A(z)B(w) - \contraction{}{A(z)}{}{B}A(z)B(w)\right] \end{aligned} \tag{3.13}$$

This can be computed using multiple different and equivalent methods. One such method is the method of residues.

$$(AB)(w) = \frac{1}{2\pi i}\oint_w \frac{dz}{(z-w)} A(z)B(w) \tag{3.14}$$

[21] Even though a finite difference in these radii would lead to the correct equation, it would be sensible only if these were the only fields in the correlation function. If there are fields other than these, then one must take the limit of converging radii to get the operator equality.

[22] Details are omitted since this is irrelevant to the main line of inquiry in this article.

We can now use mode expansions to figure out interesting relations between mode generators of local fields and mode generators of the energy-momentum tensor. As we shall see in the next section, the mode generators of the energy-momentum tensor form the Virasoro algebra.

Around an arbitrary point w, we have for a field with conformal dimension h,

$$\phi(z) = \sum_{n\in\mathbb{Z}} (z-w)^{-n-h} \phi_n(w) \tag{3.15}$$

The usual convention for the mode generators of $T(z)$ is to call them L_n. These are very simply computed to be

$$L_n(w) = \frac{1}{2\pi i} \oint_w dz\, (z-w)^{n+1} T(z) \tag{3.16}$$

When we construct the Hilbert space of the CFT, we will need to use $(L_{-n}A)(w)$ for some fields $A(z)$ and non-negative n. This can be computed by the OPE

$$T(z)A(w) \sim \sum_{n\in\mathbb{Z}} \frac{(L_n A)(w)}{(z-w)^{n+2}} \tag{3.17}$$

Comparing the singular terms upto order $n = \{0, -1\}$, and comparing the regular terms for more negative n, we get

$$\begin{aligned} (L_0 A)(w) &= h_A A(w) \\ (L_{-1} A)(w) &= \partial A(w) \\ (L_{-n-2} A)(w) &= \frac{1}{n!} (\partial^n T A)(w) \end{aligned} \tag{3.18}$$

We state without proof the following result about the mode expansion of the normal ordered product of fields:

$$\begin{aligned} (AB)(z) &= \sum_{n\in\mathbb{Z}} z^{-n-h_A-h_B} (AB)n \\ (AB)_n &= \sum_{m\leq -h_A} A_m B_{n-m} + \sum_{m>-h_A} B_{n-m} A_m \end{aligned} \tag{3.19}$$

3.2 Operating

The conformal Ward identity can be equivalently written in terms of the conformal generator:

$$\begin{aligned} \delta_\epsilon \Phi(w) &= -[Q_\epsilon, \Phi(w)] \\ Q_\epsilon &= \frac{1}{2\pi i} \oint dz\, \epsilon(z) T(z) \end{aligned} \tag{3.20}$$

We can then write out the mode generators at the origin and observe that $Q_\epsilon = \sum_{n\in\mathbb{Z}} \epsilon_n L_n$. This means that L_n and $\bar{L}_n$ are the generators of the local conformal transformations on the Hilbert space, exactly like ℓ_n and $\bar{\ell}_n$ were in the case of conformal mappings of functions.

We now compute their algebra using (3.9) and (2.40). Skipping some steps, we obtain:

$$
\begin{aligned}
&[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \\
&[\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \\
&[L_n, \bar{L}_m] = 0
\end{aligned}
\tag{3.21}
$$

This is the Virasoro algebra. The mathematical content of this is identical to the conformal Ward identity, but since this deals with mode expansions it has a major application in constructing the Fock space of states. The second half of this article deals with mostly constructing representations of this algebra.

3.2.1 The Hilbert Space

The demand that $T(z)\left|0\right\rangle$ is well defined as z approaches the origin implies:

$$
\begin{aligned}
L_n \left|0\right\rangle = 0 \\
\bar{L}_n \left|0\right\rangle = 0
\end{aligned}
\tag{3.22}
$$

for $n \geq -1$ and no constraint on $n < -1$ yet. Important subconditions from (3.22) are that the vacuum is invariant under L_{-1}, L_0, and L_1, which are the generators of global conformal transformations. This fixes the ground state energy to be zero. In fact, computing $\langle 0|T(z)|0\rangle$ by breaking it down into modes acting on the bra and the ket will single out $\langle 0|L_0|0\rangle$ and we know that this is 0.

For a primary field $\phi(z, \bar{z})$ with conformal dimensions $(h, \bar{h})$, we can compute

$$
[L_n, \phi(w)] = h(n+1)w^n\phi(w) + w^{n+1}\partial\phi(w)
\tag{3.23}
$$

This suggests that $\phi(0,0)$ acting on the vacuum creates a state with eigenvalue of L_0 given by h and which is annihilated by all $L_{n>0}$.

$$
\begin{aligned}
\left|h, \bar{h}\right\rangle &\equiv \phi(0,0)\left|0\right\rangle \\
L_0 \left|h, \bar{h}\right\rangle &= h \left|h, \bar{h}\right\rangle \\
\bar{L}_0 \left|h, \bar{h}\right\rangle &= \bar{h} \left|h, \bar{h}\right\rangle \\
L_n \left|h, \bar{h}\right\rangle &= 0 \quad \text{if} \quad n > 0 \\
\bar{L}_n \left|h, \bar{h}\right\rangle &= 0 \quad \text{if} \quad n > 0
\end{aligned}
\tag{3.24}
$$

Properties of L_0 make it a strong candidate for something on which the Hamiltonian solely depends. We call $\left|h, \bar{h}\right\rangle$ the *highest-weight state*. Expanding into ϕ into modes leads to:

$$
\begin{aligned}
[L_n, \phi_m] &= [n(h-1) - m]\phi_{n+m} \\
[L_0, \phi_m] &= -m\phi_m
\end{aligned}
\tag{3.25}
$$

This means that ϕ_m act as raising and lowering operators for eigenstates of L_0. Each application of ϕ_{-m} raises the conformal dimension by m. Similarly, the generators L_{-m} also increase the conformal dimension by m: $[L_0, L_{-m}] = mL_{-m}$.

We can thus construct states of conformal dimension $h' = h + N$ by partitioning N into integers and applying L_{-k_i} on $|h\rangle$ successively such that $\sum_i k_i = N$. By convention, we order higher k_i to the right, because other permutations can be brought into a linear combination of the *well-ordered* states by applying the Virasoro commutation relations as necessary. These can written in shorthand as $L_{-\vec{k}}\,|h\rangle$ so that the vector is interpreted to be applied as mentioned.

Such states are called *descendants* of the asymptotic state $|h\rangle$. N is called the *level* of the descendant. The number of distinct, linearly independent states at level N is given by $p(N)$ which is the number of partitions of N. The generating function of partitions is given by $1/(q;q)_\infty$. The subset of the Hilbert space generated by the asymptotic state $|h\rangle$ and its descendants is closed under the action of the Virasoro generators and hence the action of any conformal transformation restricts us to this particular subset of the Hilbert space; it forms a *module* of the Virasoro algebra. It is caled the *Verma Module*. Table 6 lists out states of a Verma module upto level 4.

Level N	$p(N)$	**States**
0	1	$\|h\rangle$
1	1	$L_{-1}\,\|h\rangle$
2	2	$L_{-2}\,\|h\rangle\,,\ L_{-1}^2\,\|h\rangle$
3	3	$L_{-3}\,\|h\rangle\,,\ L_{-2}L_{-1}\,\|h\rangle\,,\ L_{-1}^3\,\|h\rangle$
4	5	$L_{-4}\,\|h\rangle\,,\ L_{-3}L_{-1}\,\|h\rangle\,,\ L_{-2}^2\,\|h\rangle\,,\ L_{-2}L_{-1}^2\,\|h\rangle\,,\ L_{-1}^4\,\|h\rangle$

Table 6. Lowest states of a Verma Module up to level 4 .

Inner product on the Verma module can be defined by the discussion in Section 3.1.1. This immediately leads us to the following results. Firstly, the inner product of two states vanishes unless they belong to the same level. Second, the eigenspace of L_0 having different eigenvalues are orthogonal. Third, Hermiticity forces the conformal dimension to be real.

In general, the Verma module constructed on the family[23] generated by a highest-weight h conformal dimension field for a theory with central charge c is represented by $V(c, h)$. The full Hilbert space is then:

$$\mathcal{H} = \sum_{h,\bar{h}} V(c,h) \otimes \bar{V}(c,\bar{h}) \tag{3.26}$$

3.2.2 Conformal Families

We noted in the previous section that a Verma module is closed under the Virasoro generators. The elements of a Verma module are states in the CFT. We have not yet rigorously defined a totally valid correspondence between states and operators. Our correspondence would be totally valid if for all states in CFT, we could identify some field in the CFT which when acting on the vacuum generates that state.

[23]This word will be cleared up in a minute.

This can be achieved by constructing what are called *descendant fields*. Let's start with something simple, say $L_{-n}|h\rangle$:

$$L_{-n}|h\rangle = L_{-n}\phi(0)|0\rangle = \frac{1}{2\pi i}\oint dz\,\frac{T(z)\phi(0)}{z^{n-1}}|0\rangle \tag{3.27}$$

which when taken into account with (3.14) and (3.17) motivates the definition:

$$\phi^{(-n)}(w) \equiv (L_{-n}\phi)(w) = \frac{1}{2\pi i}\oint dz\,\frac{T(z)\phi(w)}{(z-w)^{n-1}} \tag{3.28}$$

We can recursively define the descendant fields[24]:

$$\phi^{(-k_s,\cdots,-k_1)}(w) = \frac{1}{2\pi i}\oint dz\,\frac{T(z)\phi^{(-k_{s-1},\cdots,-k_1)}(w)}{(z-w)^{k_s-1}} \tag{3.29}$$

which can be written in shorthand as $\phi^{(-\vec{k})}(w)$. A primary field ϕ, along with all its descendant fields, forms what is called the *conformal family*, denoted by $[\phi]$.

We should now consider the OPE of these descendant fields with some X.

$$\begin{aligned}\langle(L_{-n}\phi)(w)X\rangle &= \langle\phi^{(-n)}(w)X\rangle\\ &= \frac{1}{2\pi i}\oint_w dz\frac{\langle T(z)\phi(w)X\rangle}{(z-w)^{n-1}}\\ &\equiv \mathcal{L}_{-n}\langle\phi(w)X\rangle \qquad (n\geq 1)\end{aligned} \tag{3.30}$$

where we have defined the differential operator:

$$\mathcal{L}_{-n} = \sum_i\left[\frac{(n-1)h_i}{(w_i-w)^n} - \frac{\partial_{w_i}}{(w_i-w)^{n-1}}\right] \tag{3.31}$$

It is obvious that $\mathcal{L}_{-1}$ annihilates any correlator because of translation invariance. We finally note:

$$\langle\phi^{(-\vec{k})}(w)X\rangle = \mathcal{L}_{-\vec{k}}\langle\phi(w)X\rangle \tag{3.32}$$

This equation implies that correlators of only primary fields can determine all possible correlators in the conformal family.

We now need to show that the conformal family is closed, and we do that by stating[25] the OPE of any descendant field with the energy-momentum tensor.

$$\begin{aligned}T(z)\phi^{(-n)}(w) &= \frac{c\,n(n^2-1)/12}{(z-w)^{n+2}}\,\phi(w)\\ &+\sum_{k=1}^{n}\frac{n+k}{(z-w)^{k+2}}\,\phi^{(k-n)}(w)\\ &+\sum_{k\geq 0}(z-w)^{k-2}\,\phi^{(-k,-n)}(w)\end{aligned} \tag{3.33}$$

The state-operator map is then equivalent to the correspondence between conformal families and Verma modules.

[24] Descendants of primary fields are called *secondary*.

[25] The proof is irrelevant to the flow of this article.

3.3 Fixing the Operator Algebra

For a given CFT, the complete set of OPEs among all its primary fields (including the regular terms) is known as the operator algebra. Possessing knowledge of this algebra is equivalent to "solving" the CFT, as it allows for the determination of all correlation functions.

The problem of fixing the operator algebra boils down to finding two sets of data:

1. The spectrum of primary fields $[\phi_s]$ (i.e., their conformal dimensions $h_s, \bar{h}_s$).
2. The structure constants $C_{kl}^{s\{\vec{k}\}\{\bar{\vec{k}}\}}$ that appear in the OPE.

We assume the primary fields are normalized such that their two-point function is

$$\langle \phi_k(w, \overline{w}) \phi_l(z, \overline{z}) \rangle = \frac{\delta_{kl}}{(z-w)^{2h_k} (\overline{z} - \overline{w})^{2\overline{h}_l}}. \tag{3.34}$$

This normalization implies that the Verma modules built on different primary fields are orthogonal. The OPE of two primary fields ϕ_k and ϕ_l can be written as a sum over all conformal families $[\phi_s]$, including all primary and descendant fields:

$$\begin{aligned} \phi_k(z, \overline{z}) \phi_l(0,0) = \sum_{[\phi_s]} \sum_{\{\vec{k}\}} \sum_{\{\bar{\vec{k}}\}} & C_{kl}^{s\{\vec{k}\}\{\bar{\vec{k}}\}} z^{h_s - h_k - h_l + |\vec{k}|} \\ & \times \overline{z}^{\overline{h}_s - \overline{h}_k - \overline{h}_l + |\bar{\vec{k}}|} \phi_s^{\{\vec{k}\}\{\bar{\vec{k}}\}}(0,0), \end{aligned} \tag{3.35}$$

where $|\vec{k}| = \sum_i k_i$ is the level of the descendant. The structure constants $C_{kl}^{s\{\vec{k}\}\{\bar{\vec{k}}\}}$ contain all the dynamical information of the theory.

3.3.1 Three-Point Function and β Coefficients

The structure constants for the primary fields (i.e., level 0, $\{\vec{k}\} = \{\}$) are directly related to the coefficients of the three-point function, $C_{skl} \equiv \langle \phi_s \phi_k \phi_l \rangle$. By computing the three-point function in two ways, first using the OPE (3.35) and then using the state-operator map, we find:

$$C_{kl}^{s\{\}\{\}} = C_{kl}^{s} = C_{skl}. \tag{3.36}$$

Furthermore, the structure constants for the descendant fields can be factored out:

$$C_{kl}^{s\{\vec{k}\}\{\bar{\vec{k}}\}} = C_{kl}^{s} \beta_{kl}^{s\{\vec{k}\}} \overline{\beta}_{kl}^{s\{\bar{\vec{k}}\}}. \tag{3.37}$$

The crucial insight is that the coefficients $\beta_{kl}^{s\{\vec{k}\}}$ are universal functions, completely determined by the Virasoro algebra. They depend only on the central charge c and the conformal dimensions of the primaries (h_k, h_l, h_s). They are independent of the specific dynamics of the theory, which are encoded entirely in the C_{skl}. By definition, the level 0 coefficient is $\beta_{kl}^{s\{\}} = 1$.

The problem of solving the CFT is thus reduced to:

1. Finding the spectrum of primary fields $[\phi_s]$ (the $h_s, \bar{h}_s$).

2. Determining the three-point structure constants C_{skl}.

The β coefficients can be computed recursively by acting with L_n operators on the OPE. For simplicity, let's consider chiral primaries ϕ_k, ϕ_l with $h_k = h_l = h$. The OPE is

$$\phi_k(z)\phi_l(0) = \sum_s C^s_{kl} z^{h_s - 2h} X_s(z), \tag{3.38}$$

where $X_s(z)|0\rangle = \sum_{N=0}^{\infty} z^N |N; h_s\rangle$ generates the states in the Verma module of ϕ_s. By acting with L_n on both sides and comparing powers of z, one can derive a recursion relation for the states $|N; h_s\rangle$. This procedure allows for the calculation of the β coefficients level by level.

Level 1 At level $N = 1$, the state is $|1; h_s\rangle = \beta^{s\{1\}}_{kl} L_{-1}|h_s\rangle$. Applying L_1 and using the recursion relations gives $2h_s\beta^{s\{1\}}_{kl} = h_s$, which fixes the coefficient:

$$\beta^{s\{1\}}_{kl} = \frac{1}{2} \quad (\text{for } h_s \neq 0). \tag{3.39}$$

Level 2 At level $N = 2$, the state is $|2, h_s\rangle = \beta^{s\{2\}}_{kl} L_{-2}|h_s\rangle + \beta^{s\{1,1\}}_{kl} L_{-1}L_{-1}|h_s\rangle$. By acting with L_1 and L_2 and comparing with the recursion relation, we obtain a system of two linear equations for the two β coefficients:

$$\begin{aligned} 3\beta^{s\{2\}}_{kl} + (4h_s + 2)\beta^{s\{1,1\}}_{kl} &= \frac{1}{2}(h_s + 1), \\ (4h_s + \tfrac{c}{2})\beta^{s\{2\}}_{kl} + 6h_s\beta^{s\{1,1\}}_{kl} &= h_s + h. \end{aligned} \tag{3.40}$$

Solving this system gives the explicit universal coefficients $\beta^{s\{2\}}_{kl}$ and $\beta^{s\{1,1\}}_{kl}$ in terms of c, h, h_s. This process can, in principle, be continued to any level.

3.3.2 Conformal Blocks

With the structure of the operator algebra understood, we can revisit the four-point function. Using global $SL(2, \mathbb{C})$ invariance, we can fix three of the points, say $z_1 \to \infty, z_2 \to 1, z_4 \to 0$. The fourth point z_3 is then mapped to the conformally invariant cross-ratio $\eta = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}$. Let's consider the correlator

$$\begin{aligned} G^{lk}_{mn}(\eta, \overline{\eta}) &= \lim_{z,\overline{z} \to \infty} z^{2h_k} \overline{z}^{2\overline{h}_k} \langle \phi_k(z, \overline{z})\phi_l(1, 1)\phi_m(\eta, \overline{\eta})\phi_n(0, 0)\rangle \\ &= \langle h_k, \overline{h}_k|\phi_l(1, 1)\phi_m(\eta, \overline{\eta})|h_n, \overline{h}_n\rangle. \end{aligned} \tag{3.41}$$

We can analyze this by inserting the OPE for $\phi_m(\eta, \overline{\eta})\phi_n(0, 0)$:

$$\begin{aligned} G^{lk}_{mn}(\eta, \overline{\eta}) = \sum_{[\phi_p]} \sum_{\vec{k}, \overline{\vec{k}}} & C^{p\{\vec{k}\}\{\overline{\vec{k}}\}}_{mn} \eta^{h_p - h_m - h_n + |\vec{k}|} \overline{\eta}^{\overline{h}_p - \overline{h}_m - \overline{h}_n + |\overline{\vec{k}}|} \\ & \times \langle h_k, \overline{h}_k|\phi_l(1, 1)L_{-\{\vec{k}\}}\overline{L}_{-\{\overline{\vec{k}}\}}|h_p, \overline{h}_p\rangle. \end{aligned} \tag{3.42}$$

Using (3.37) and the fact that $\langle h_k|\phi_l(1)|h_p\rangle = C_{klp}$, the correlator decomposes into a sum over primary fields $[\phi_p]$ that are exchanged in the OPE:

$$G_{mn}^{lk}(\eta,\overline{\eta}) = \sum_{[\phi_p]} C_{mn}^{p} C_{pkl} \mathcal{F}_{mn}^{lk}(p|\eta) \overline{\mathcal{F}}_{mn}^{lk}(p|\overline{\eta}). \tag{3.43}$$

The functions $\mathcal{F}$ and $\overline{\mathcal{F}}$ are called conformal blocks. They are the kinematically-fixed contributions from a single conformal family $[\phi_p]$ (and all its descendants) being exchanged. The chiral conformal block $\mathcal{F}$ is given by

$$\mathcal{F}_{mn}^{lk}(p|\eta) = \eta^{h_p-h_m-h_n} \sum_{\{\vec{k}\}} \eta^{|\vec{k}|} \beta_{mn}^{p\{\vec{k}\}} \frac{\langle h_k|\phi_l(1) L_{-\{\vec{k}\}}|h_p\rangle}{\langle h_k|\phi_l(1)|h_p\rangle}. \tag{3.44}$$

The conformal blocks are universal: they depend only on the central charge c and the conformal dimensions of the four external fields and the internal field p, but not on the three-point structure constants C_{ijk}. All the non-universal, dynamical information of the specific CFT is contained in the structure constants C_{ijk}.

3.3.3 Crossing Symmetry and the Conformal Bootstrap

The four-point function $\langle\phi_k(z_1)\phi_l(z_2)\phi_m(z_3)\phi_n(z_4)\rangle$ is a single-valued, analytic function. However, our decomposition (3.43) depended on an arbitrary choice: we performed the OPE on $\phi_m\phi_n$ first. This is called the s-channel expansion.

We could have just as easily performed the OPE on $\phi_l\phi_m$ first (the t-channel) or $\phi_l\phi_n$ first (the u-channel). Each choice leads to a different expansion of the same four-point function in terms of different conformal blocks and different cross-ratios. For example, let's use the notation $G_{klmn} = \langle\phi_k(z_1)\phi_l(z_2)\phi_m(z_3)\phi_n(z_4)\rangle$.

- **s-channel:** Fuse $(k,l) \to p$ and $(m,n) \to p$. The cross-ratio is $\eta = \frac{z_{12}z_{34}}{z_{13}z_{24}}$.

$$G_{klmn} = \sum_p C_{kl}^p C_{mn}^p \mathcal{F}_s(p|\eta)\overline{\mathcal{F}}_s(p|\overline{\eta})$$

- **t-channel:** Fuse $(k,n) \to p$ and $(l,m) \to p$. The cross-ratio is $1-\eta = \frac{z_{14}z_{23}}{z_{13}z_{24}}$.

$$G_{klmn} = \sum_p C_{kn}^p C_{lm}^p \mathcal{F}_t(p|1-\eta)\overline{\mathcal{F}}_t(p|1-\overline{\eta})$$

Since all these expansions must be equal, we arrive at a powerful consistency condition called crossing symmetry:

$$\sum_p C_{kl}^p C_{mn}^p \mathcal{F}_s(p|\eta)\overline{\mathcal{F}}_s(p|\overline{\eta}) = \sum_p C_{kn}^p C_{lm}^p \mathcal{F}_t(p|1-\eta)\overline{\mathcal{F}}_t(p|1-\overline{\eta}). \tag{3.45}$$

This is a functional equation that places strong constraints on the allowed CFT data (h_p and C_{ijk}).

This forms the basis of the conformal bootstrap program: instead of starting from a Lagrangian, one assumes a spectrum of primary fields $[\phi_p]$ and a set of structure constants

C_{ijk}. One then demands that this data satisfies the crossing symmetry equations (3.45) for all possible four-point functions. This set of constraints is so restrictive that in many cases, it is sufficient to uniquely determine the CFT data, and thus "solve" the theory. This is particularly successful for minimal models, where the finite number of primary fields makes the problem tractable.

Refer to Appendix D for the quantization of the free boson and free fermion on a cylinder. The solution is very instructive and has applications of all our work done so far.

4 Representation Theory of the Virasoro Algebra

The Verma modules provide a good playground for our analysis of the representation of the Virasoro algebra. We must note however that the representations they furnish may be *reducible*. This means that there may exist a submodule which transforms within itself, and that this submodule is generated by a highest-weight state which is itself a descendant of some other highest-weight state.

4.1 Singular Vectors

A *singular vector* or *null state* or *null vector* is a descendant state $|\chi\rangle$ such that $L_n |\chi\rangle = 0$ for all positive n, i.e. it is a highest-weight state. A singular vector and its descendants are orthogonal to each other (because their level is different) and to the entire Verma module. To prove this, one requires to take the cHermitian onjugate of the definition of descendant states and observe that since $|\chi\rangle$ is a highest-weight state, the inner product vanishes. In particular, the singular vector and all its descendants have vanishing norm.

Through the state-operator map, it is counterintuitively seen that fields corresponding to singular vectors are simultaneously primary and secondary.

A Verma module is reducible if there exists a singular vector $|\chi\rangle$ at some level. This vector generates the submodule V_χ of $V(c,h)$.

If there exist singular vectors $|\chi_i\rangle$ in a Verma module, then we can quotient out the submodules generated by them to obtain an *irreducible* Verma module, denoted by $M(c,h) = V(c,h)/\sim$ where $|\phi\rangle \sim |\phi\rangle + |\psi\rangle$ such that $|\psi\rangle \in \oplus_i V_{\chi_i}$. This means that we have quotiented out states of zero norm, just like we do in Gupta-Bleuler or in BRST when choosing the physical Hilbert space.

One obvious characterization of the representation would be their dimension. We define this by using a generating function,

$$\begin{aligned}\chi_{(c,h)}(\tau) &= \mathrm{Tr}\, q^{L_0 - c/24} \qquad (q \equiv e^{2\pi i \tau}) \\ &= q^{h-c/24} \sum_{n=0}^{\infty} \dim(h+n)\, q^n\end{aligned} \tag{4.1}$$

where $\dim(h+n)$ is the dimension of the Verma module at level n. The factor $q^{-c/24}$ is relevant for a discussion on modular invariance but we do not go into there.

4.2 Unitarity Constraints on h and c

There are representations of the Virasoro algebra that admit states with negative norm. This can be seen by considering a primary field with conformal dimension h and talking about its Verma module. We call such states *ghosts* and these are often non-physical. These make the theory non-unitary.

However, we should note that non-unitary does not non-physical. Open quantum systems are manifestly non-unitary. We will restrict to unitary CFTs in this article, though it should be noted that many examples of non-unitary CFTs are of particular importance in physics at the boundary of condensed matter systems.

4.2.1 Lower Level Calculation

Level 0. $\langle h|h\rangle = 1$ because of normalization.

Level 1. $\langle h|L_1 L_{-1}|h\rangle = 2h\,\langle h|h\rangle = 2h$.

This implies that theories with $h < 0$ admit negative norm states at the first level. We get a singular vector at this level if $h = 0$.

Level 2. The calculation here becomes non-trivial because $p(2) = 2$. We can construct a matrix of inner products of the states. Let's construct $M_{ij} = \langle i|j\rangle$ of inner product of basis states at that level. Since this matrix is Hermitian, we can talk about real eigenvalues.

Iff there exist negative eigenvalues, then there exist negative norm states. Iff zero is an eigenvalue, then the Verma module is reducible. For a generic (possibly reducible) Verma module, these matrices are computed to be[26]

$$\begin{aligned} M^{(0)} &= 1 \\ M^{(1)} &= 2h \\ M^{(2)} &= \begin{pmatrix} 4h(2h+1) & 6h \\ 6h & 4h + c/2 \end{pmatrix} \end{aligned} \tag{4.2}$$

The representation is non-unitary whenever determinant or trace of such matrices is negative. We call these the *Gram* matrices. The determinants are called the *Kac* determinants.

We compute that the trace of this matrix is

$$\operatorname{tr} M^{(2)}(c,h) = 8h(h+1) + \frac{1}{2}c. \tag{4.3}$$

$$\det M^{(2)}(c,h) = 32,(h - h_{1,1})(h - h_{1,2})(h - h_{2,1}). \tag{4.4}$$

Each zero of this determinant corresponds to a singular vector. The first one,

$$h_{1,1} = 0, \tag{4.5}$$

corresponds to the singular vector we have already found. The other two are

$$h_{1,2} = \frac{1}{16}\left(5 - c - \sqrt{(1-c)(25-c)}\right), \tag{4.6}$$

[26] $M^{(0)}$ and $M^{(1)}$ have already been computed.

$$h_{2,1} = \frac{1}{16}\left(5 - c + \sqrt{(1-c)(25-c)}\right). \tag{4.7}$$

There are two singular vectors at level 2, denoted $|\chi_{1,2}\rangle$ and $|\chi_{2,1}\rangle$, associated respectively to $h = h_{1,2}$ and $h = h_{2,1}$.

A First Glance at Unitary Representations We have singular vectors at $h = h_{1,1} = 0$ and $h = h_{1,2}, , h_{2,1}$, which satisfy

$$1 = \left(\frac{4}{3}h + \frac{2}{3}\right)\left(-\frac{4}{3}h - \frac{c}{6} + \frac{3}{2}\right). \tag{4.8}$$

4.2.2 Kac Determinants

The Kac determinants are the determinants of the Gram matrices and display useful properties.

1. A singular vector $|\chi\rangle$ at level K yields null states at level $N > K$:

   $$L_{-\vec{k}}|\chi\rangle, \qquad |\vec{k}| = N - K.$$

   Each singular vector at level K therefore yields $p(N-K)$ null states at level N.

2. The order in h of entries of the Gram matrix $M^{(N)}(c,h)$ is determined as follows. Since each element in $L^\dagger_{-\vec{k}} = L_{\vec{k}}$ "commutes to L_0",

   $$\mathrm{ord}_h\big(\langle h|L^\dagger_{-\vec{k}}L_{-\vec{k}}|h\rangle\big) = \mathrm{length}(\vec{k}). \tag{4.9}$$

   For $\vec{k}' \neq \vec{k}$,

   $$\mathrm{ord}_h\big(\langle h|L^\dagger_{-\vec{k}'}L_{-\vec{k}}|h\rangle\big) < \mathrm{length}(\vec{k}). \tag{4.10}$$

   Thus the diagonal terms give the leading contribution in h, and one finds

   $$\mathrm{ord}_h\big(\det M^{(N)}(c,h)\big) = \sum_{\substack{\vec{k} \\ |\vec{k}|=N}} \mathrm{length}(\vec{k}) = \sum_{\substack{r,s\geq 1 \\ rs\leq N}} p(N - rs). \tag{4.11}$$

3. At level N we have

   $$\#\{(r,s) \mid r,s \geq 1,\ rs = N\}$$

   new singular vectors labelled by $h_{r,s}$.

We can write the general formula for the Kac determinants as

$$\det M^{(N)}(c,h) = \alpha_N \prod_{r,s\geq 1\ rs\leq N} (h - h_{r,s}(c))^{p(N-rs)}, \tag{4.12}$$

where α_N is a non-vanishing constant independent of h and c.

The values of $h_{r,s}$ and c for minimal models are given by

$$h_{r,s}(m) = \frac{((m+1)r - ms)^2 - 1}{4m(m+1)}, \tag{4.13}$$

$$c(m) = 1 - \frac{6}{m(m+1)}. \tag{4.14}$$

$$m_{\pm} = -\frac{1}{2} \pm \sqrt{\frac{25-c}{1-c}}. \tag{4.15}$$

$$h_{r,s}(m_+) = h_{s,r}(m_-). \tag{4.16}$$

For real m we take the convention $m \in \mathbb{R}_{\geq 0}$.

4.3 Unitary Representations of the Virasoro Algebra

We make the following observations.

1. For level 1,
$$\det M^{(1)}(c,h) \geq 0 \quad \Longrightarrow \quad h \geq 0. \tag{4.17}$$

2. For $n \geq 1$,
$$\langle h|L_{-n}^{\dagger} L_{-n}|h\rangle = 2nh + \frac{c}{12} n(n^2-1) > 0, \tag{4.18}$$
which implies $c \geq 0$.

3. Consider the Kac determinants in the range $1 < c < 25$. Then $m_{\pm} \notin \mathbb{R}$, which implies $h_{r,s} \notin \mathbb{R}$ for $r \neq s$. If $r = s$ then $h_{r,r} < 0$ for $r > 1$. For $c \geq 25$ one finds $h_{r,s} < 0$. Taken together, these facts imply that $\det M^{(k)}$ is non-vanishing and positive definite for $c > 1$ and $h \geq 0$.

4. In the region $0 \leq c \leq 1$, $h \geq 0$ we have
$$96\, h_{r,s} + 4(1-c) = \left(\sqrt{1-c}\,(r+s) \pm \sqrt{25-c}\,(r-s)\right)^2 \geq 0. \tag{4.19}$$
Near $c = 1$, set $c = 1 - 6\epsilon$. For $r \neq s$,
$$h_{r,s}(\epsilon) = \frac{1}{4}(r-s)^2 \pm \frac{1}{4}(r^2-s^2)\sqrt{\epsilon} + O(\epsilon). \tag{4.20}$$
For $r = s$,
$$h_{r,r} = \frac{1-c}{24}\,(r^2-1). \tag{4.21}$$

Diagrammatically, this information refines the known structure of unitary and non-unitary representation regions.

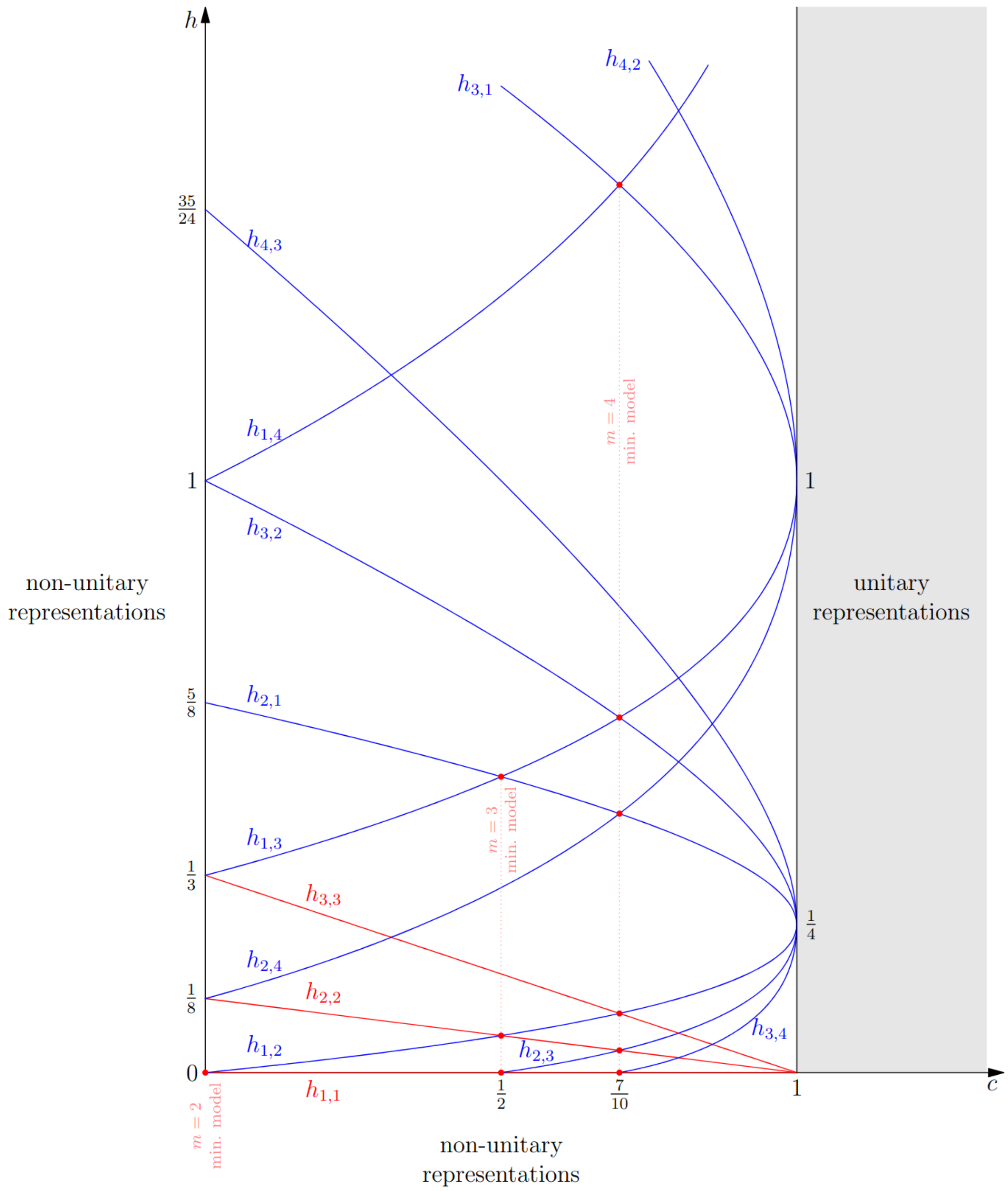


Figure 1. The $h - c$ plot depicting regions of unitary and non-unitary representations. Points marked red will be discussed in coming sections.

4.4 Unitarity in the Region $0 \leq c < 1$

For $0 \leq c < 1, , h \geq 0$:

- A generic point in this region gives rise to non-unitary representations.

- Friedan, Qiu, and Shenker [12] found isolated unitary representations at the first intersections of singular-vector curves. These "first intersections" occur precisely at the Kac values with integer $m \geq 2$ and

$$1 \leq r < m, \qquad 1 \leq s < r.$$

Note that

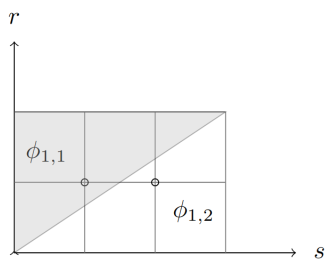
$$h_{r,s}(m) = h_{m-r,,m+1-s}(m), \qquad m \in \mathbb{Z}_{\geq 2}, \tag{4.22}$$

and we denote the associated conformal families by

$$[\phi_{r,s}] \equiv [\phi_{m-r,m+1-s}].$$

4.5 Examples of Conformal Diagrams

For $m = 2$ we have $c = 0$ and $h_{1,2} = 0$; the diagram reduces to the vacuum family.

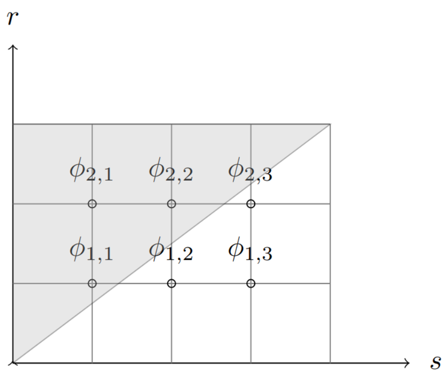


For $m = 3$ we have $c = \frac{1}{2}$. The Kac diagram entries include

$$\phi_{1,1} \ \phi_{1,2} \ \phi_{1,3} \ \phi_{2,1} \ \phi_{2,2} \ \phi_{2,3}$$

with the identifications and dimensions

$$\begin{aligned} [\phi_{1,1}] &= [\phi_{2,3}], & h_{1,1} &= 0 \\ [\phi_{2,1}] &= [\phi_{1,3}], & h_{2,1} &= \tfrac{1}{2} \\ [\phi_{2,2}] &= [\phi_{1,2}], & h_{2,2} &= \tfrac{1}{16} \end{aligned} \tag{4.23}$$



Goal. For a given $c(m)$, construct unitary CFTs from the conformal families associated with the unitary Virasoro representations at that value of $c(m)$. These CFTs are the unitary minimal models.

5 Minimal Models

As introduced in 1, two-dimensional statistical models at their critical point are described by a CFT. A major goal is to classify these theories. A significant breakthrough in this direction was the "minimal model" program, initiated by Belavin, Polyakov, and Zamolodchikov (BPZ). These models are characterized by having a finite number of primary fields and are exactly solvable due to the powerful constraints of their (finite) operator algebra.

They are constructed from the representation theory of the Virasoro algebra, which, as we saw in 3, can be reducible for specific values of the central charge c and conformal dimension h. The existence of null states (singular vectors) in the Verma modules is the key mechanism that allows the operator algebra to truncate to a finite set of fields, leading to a consistent, solvable theory.

In the following we study in detail the models introduced above. Since the spectrum of primary operators for these models fall into vanishing curves $h = h_{r,s}$, the corresponding Verma modules will have null states. For a primary operator with $h = h_{r,s}$, this happens at level $r \times s$. The existence of null vectors imposes strong constraints on the structure of correlators and the operator algebra. In this article we focus on a simple example which shows the main ingredients.

Our main references are [13, 14].

5.1 Correlators and OPE in Unitary Minimal Models

Let us study a simple example of a reducible Verma module. Consider a primary $|h\rangle$ and the following descendant at level two:

$$|\chi\rangle = \big(L_{-2} + \eta L_{-1}^2\big)\,|h\rangle. \tag{5.1}$$

We want to choose η and h so that $|\chi\rangle$ is null. It will suffice to require $L_1|\chi\rangle = L_2|\chi\rangle = 0$, as $L_n|\chi\rangle = 0$ for $n > 2$ then follows from the Virasoro algebra. We obtain

$$L_1|\chi\rangle = \big(3 + 2\eta + 4h\eta\big)L_{-1}|h\rangle, \tag{5.2}$$

$$L_2|\chi\rangle = \big(\tfrac{c}{2} + 4h + 6h\eta\big)\,|h\rangle. \tag{5.3}$$

Hence $|\chi\rangle$ is null provided

$$\eta = -\frac{3}{2(2h+1)}, \tag{5.4}$$

and

$$h = \frac{1}{16}\Big(5 - c \pm \sqrt{(c-1)(c-25)}\Big). \tag{5.5}$$

As expected, the condition on h is either $h = h_{1,2}$ or $h = h_{2,1}$. Let us denote by $\phi(z)$ the field corresponding to the primary operator. To $|\chi\rangle$ we can associate a descendant null field $\chi(z)$ given by

$$\chi(z) = \Big(L_{-2} - \frac{3}{2(2h+1)}L_{-1}^2\Big)\phi(z) = \Big(\,L^{(2)}\phi(z)\, - \frac{3}{2(2h+1)}\partial_z^2\phi(z)\Big), \tag{5.6}$$

The null state is orthogonal to all states in the theory. In terms of correlators this translates into

$$\langle \chi(z)\, \phi_1(z_1) \cdots \phi_n(z_n) \rangle = 0. \tag{5.7}$$

However, according to the mode expansion (see prior definitions) this implies

$$\Big(L_{-2} - \frac{3}{2(2h+1)} L_{-1}^2\Big) \langle \phi(z)\, \phi_1(z_1) \cdots \phi_n(z_n) \rangle = 0, \tag{5.8}$$

where the operators L_n act as differential operators on correlators. More explicitly,

$$\sum_{i=1}^{n} \Big[\frac{h_i}{(z-z_i)^2} + \frac{1}{z-z_i} \frac{\partial}{\partial z_i} \Big] - \frac{3}{2(2h+1)} \frac{\partial^2}{\partial z^2} \Big\} \langle \phi(z)\, \phi_1(z_1) \cdots \phi_n(z_n) \rangle = 0. \tag{5.9}$$

This is a powerful equation. For two-point functions it adds no new information. Consider its effect on a three-point function of $\phi(z)$ with two other primaries:

$$\langle \phi(z)\, \phi_1(z_1)\, \phi_2(z_2) \rangle = C_{h,h_1,h_2}\, (z-z_1)^{h+h_1-h_2} (z_1-z_2)^{h_1+h_2-h} (z-z_2)^{h+h_2-h_1}, \tag{5.10}$$

where we have focused on the holomorphic dependence only and C_{h,h_1,h_2} is a constant not fixed by global conformal invariance. As discussed previously, it appears in the operator algebra, and is the coefficient with which the primary ϕ_2 appears in the OPE $\phi \times \phi_1$. Applying (5.11) to this three-point function we see that C_{h,h_1,h_2} vanishes unless the following constraint holds:

$$2(2h+1)(h+2h_2-h_1) = 3(h-h_1+h_2)(h-h_1+h_2+1). \tag{5.11}$$

This can be seen as a quadratic equation for h_2. We then conclude that in the OPE of the degenerate field $\phi(z)$ with another primary there are only two conformal families (i.e. two primaries plus all their descendants). Assume now that $\phi_1(z)$ is one of the primaries of a minimal model, so that $h_1 = h_{r,s}$. Then

$$h = h_{1,2}, \quad h_1 = h_{r,s} \quad \Longrightarrow \quad h_2 = h_{r,s-1} \text{ or } h_2 = h_{r,s+1}, \tag{5.12}$$

$$h = h_{2,1}, \quad h_1 = h_{r,s} \quad \Longrightarrow \quad h_2 = h_{r-1,s} \text{ or } h_2 = h_{r+1,s}. \tag{5.13}$$

Denoting by $\phi_{(r,s)}$ the field corresponding to $|h_{r,s}\rangle$, we can write these relations symbolically as

$$\phi_{(1,2)} \times \phi_{(r,s)} = \phi_{(r,s-1)} + \phi_{(r,s+1)}, \tag{5.14}$$

$$\phi_{(2,1)} \times \phi_{(r,s)} = \phi_{(r-1,s)} + \phi_{(r+1,s)}. \tag{5.15}$$

These are examples of fusion rules: conditions under which a given conformal family occurs in the short-distance product of two conformal fields. Note there are implicit OPE coefficients which may vanish. One can derive similar fusion rules for more general OPEs $\phi_{r,s} \times \phi_{r',s'}$. A crucial property, already manifest in (5.16), is that the conformal families $[\phi_{(r,s)}]$ associated with reducible modules form a closed set under the operator algebra. The final result for minimal models is a finite set of conformal families which close under fusion; this property allows minimal models to be consistent CFTs.

Let us make two remarks. First, the field $\phi_{1,1}(z)$ has a level-one null descendant. But at level one the only descendant is $L_{-1}\phi_{1,1}$, so $\partial_z\phi_{1,1}(z) = 0$ inside any correlator. We hence identify $\phi_{1,1}(z)$ with the identity operator $\mathbb{I}$; this fits with $h_{1,1} = 0$, since $\phi_{1,1}(0)|0\rangle = \mathbb{I}|0\rangle = |0\rangle$. The fusion relation involving the identity operator is of course

$$\phi_{(r,s)} \times \phi_{(1,1)} = \phi_{(r,s)}. \tag{5.16}$$

Finally, when writing fusion relations it sometimes happens that the conformal families on the r.h.s. fall outside the range defining a minimal model. It is convenient to take conformal families $\phi_{(r,s)}$ with $1 \le r \le m-1$ and $1 \le s \le m$ with the identification

$$\phi_{(r,s)} \equiv \phi_{(m-r,\, m+1-s)}. \tag{5.17}$$

So far our discussion was restricted to the holomorphic sector. The Hilbert space of a physical theory is constructed from tensor products of holomorphic and anti-holomorphic modules. In the diagonal choice we associate to each holomorphic module $M(c, h_{r,s})$ the corresponding anti-holomorphic module $\overline{M}(c, h_{r,s})$ (with $\bar{c} = c$). The Hilbert space of the theory then takes the form

$$\mathcal{H} = \bigoplus_{r,s} M(c, h_{r,s}) \otimes \overline{M}(c, h_{r,s}). \tag{5.18}$$

5.2 Example: The Ising Model

The simplest non-trivial unitary CFT is the critical Ising model. In addition to the identity it contains two fields, $\sigma(z,\bar{z})$ and $\epsilon(z,\bar{z})$, of conformal dimensions

$$(h,\bar{h})_\sigma = \left(\tfrac{1}{16}, \tfrac{1}{16}\right), \qquad (h,\bar{h})_\epsilon = \left(\tfrac{1}{2}, \tfrac{1}{2}\right). \tag{5.19}$$

This identifies the CFT with the unitary minimal model with $m = 3$. Focusing on the holomorphic part, we can make the following identifications:

$$\mathbb{I} \equiv \phi_{(1,1)} = \phi_{(2,3)}, \qquad \phi_{(2,2)} = \phi_{(1,2)}, \qquad \epsilon \equiv \phi_{(2,1)} = \phi_{(1,3)}. \tag{5.20}$$

Then the fusion rules (5.16) lead to

$$\sigma \times \sigma = \mathbb{I} + \epsilon, \tag{5.21}$$

$$\sigma \times \epsilon = \sigma, \tag{5.22}$$

$$\epsilon \times \epsilon = \mathbb{I}. \tag{5.23}$$

To illustrate the power of (5.11), consider the four-point correlator of identical fields $\sigma(z,\bar{z})$. Focusing only on the holomorphic dependence, conformal invariance allows us to write

$$\langle \sigma(z_1)\sigma(z_2)\sigma(z_3)\sigma(z_4)\rangle = g(\eta)\, z_{12}^{-2h} z_{34}^{-2h}, \qquad \eta = \frac{z_{12}z_{34}}{z_{13}z_{24}}, \tag{5.24}$$

with $h = \frac{1}{16}$. The equation (5.11) in this case reads

$$\sum_{i=2}^{4}\left[\frac{h}{(z_1 - z_i)^2} + \frac{1}{z_1 - z_i}\frac{\partial}{\partial z_i}\right] - \frac{3}{2(2h+1)}\frac{\partial^2}{\partial z^2}\Big\}\, \langle \sigma(z_1)\sigma(z_2)\sigma(z_3)\sigma(z_4)\rangle = 0. \tag{5.25}$$

This implies a differential equation for $g(\eta)$. Precisely,

$$g''(\eta) + \frac{4\eta + 4\eta h - 8h + 2}{3\eta - 3\eta^2}\, g'(\eta) - \frac{2h(2h+1)}{3(1-\eta)^2}\, g(\eta) = 0. \tag{5.26}$$

With $h = \frac{1}{16}$, this has two linearly independent solutions

$$g_{\pm}(\eta) = \frac{\sqrt{1 \pm \sqrt{\eta}}}{(1-\eta)^{1/8}}. \tag{5.27}$$

5.2.1 Building the Full Solution

Reinserting the anti-holomorphic dependence, the full correlator is

$$\langle \sigma(z_1, \bar{z}_1)\sigma(z_2, \bar{z}_2)\sigma(z_3, \bar{z}_3)\sigma(z_4, \bar{z}_4)\rangle = g(\eta, \bar{\eta})\, |z_{12}|^{-4h} |z_{34}|^{-4h}. \tag{5.28}$$

Our previous result implies

$$\begin{aligned} g(\eta, \bar{\eta}) = {} & \kappa_{++}\, g_+(\eta) g_+(\bar{\eta}) + \kappa_{+-}\, g_+(\eta) g_-(\bar{\eta}) \\ & + \kappa_{-+}\, g_-(\eta) g_+(\bar{\eta}) + \kappa_{--}\, g_-(\eta) g_-(\bar{\eta}), \end{aligned} \tag{5.29}$$

so that we have fixed the correlator up to four constants. Which further requirements restrict these constants?

5.2.2 Constraints: Single-valuedness and Crossing

Single-valuedness: On the real section $\bar{\eta} = \eta^*$, $g(\eta, \bar{\eta})$ should be single-valued as we move η around the complex plane. The non-trivial points are $\eta = 0, 1$. Define monodromy transformations:

$$M_0\big(g(\eta, \bar{\eta})\big) = \lim_{t \to 1} g(\eta e^{2\pi i t}, \bar{\eta} e^{2\pi i t}), \tag{5.30}$$

$$M_1\big(g(\eta, \bar{\eta})\big) = \lim_{t \to 1} g(1 + (\eta - 1)e^{2\pi i t}, 1 + (\bar{\eta} - 1)e^{2\pi i t}). \tag{5.31}$$

$g(\eta, \bar{\eta})$ must be invariant under both transformations.

Crossing relations: since the operators are identical, the correlator should be invariant under exchanges of any two of them. Under $1 \leftrightarrow 2$ one has $\eta \mapsto \eta/(\eta - 1)$ and

$$g(\eta, \bar{\eta}) = g\Big(\frac{\eta}{\eta - 1}, \frac{\bar{\eta}}{\bar{\eta} - 1}\Big). \tag{5.32}$$

Under $1 \leftrightarrow 3$ one has $\eta \mapsto 1 - \eta$ and

$$g(\eta, \bar{\eta}) = \left(\frac{\eta}{1 - \eta}\right)^{2h} g(1 - \eta, 1 - \bar{\eta}). \tag{5.33}$$

Consistency with the OPE: the lowest dimension field in the OPE is the identity field, so that

$$\sigma(z_1, \bar{z}_1)\sigma(z_2, \bar{z}_2) = \frac{1}{|z_{12}|^{1/4}} + \cdots, \tag{5.34}$$

with exact coefficient one (using canonical normalization and $C_{\mathbb{I}} = 1$). This fixes the small-$\eta, \bar{\eta}$ behaviour:

$$g(\eta, \bar{\eta}) = 1 + \cdots. \tag{5.35}$$

5.2.3 Monodromy Action

The monodromy around $\eta = 0$ acts on the square-root factors as

$$M_0: \qquad \sqrt{1+\sqrt{\eta}} \mapsto \sqrt{1-\sqrt{\eta}}, \qquad \sqrt{1-\sqrt{\eta}} \mapsto \sqrt{1+\sqrt{\eta}}. \tag{5.36}$$

Hence at the level of the basis of solutions

$$M_0: \; g_+ \leftrightarrow g_-. \tag{5.37}$$

And similarly for M_1 one finds a related action which, together with single-valuedness and crossing invariance, fixes the combination up to an overall constant. Matching the small-η behaviour determines the constants, yielding

$$g(\eta,\bar{\eta}) = \frac{|1+\sqrt{\eta}| + |1-\sqrt{\eta}|}{2\,|1-\eta|^{1/4}}. \tag{5.38}$$

5.2.4 Conformal Block Decomposition

We choose normalization so that

$$g(\eta,\bar{\eta}) = \sum_p C_p^2\, \mathcal{F}(p|\eta)\, \overline{\mathcal{F}}(p|\bar{\eta}), \tag{5.39}$$

where

$$\mathcal{F}(p|\eta) = \eta^{h_p}(1 + a\eta + \cdots), \tag{5.40}$$

starts with η^{h_p} and differs from earlier conventions by an overall power.

The explicit answer decomposes as

$$g(\eta,\bar{\eta}) = \mathcal{F}(0|\eta)\overline{\mathcal{F}}(0|\bar{\eta}) + C_\epsilon^2\, \mathcal{F}(\tfrac{1}{2}|\eta)\overline{\mathcal{F}}(\tfrac{1}{2}|\bar{\eta}), \tag{5.41}$$

where $C_\epsilon = \frac{1}{2}$ and

$$\mathcal{F}(0|\eta) = \frac{\sqrt{1+\sqrt{\eta}} + \sqrt{1-\sqrt{\eta}}}{2(1-\eta)^{1/8}} = \eta^0\Big(1 + \frac{\eta}{64} + \cdots\Big), \tag{5.42}$$

$$\mathcal{F}(\tfrac{1}{2}|\eta) = \frac{\sqrt{1+\sqrt{\eta}} - \sqrt{1-\sqrt{\eta}}}{(1-\eta)^{1/8}} = \eta^{1/2}\Big(1 + \frac{\eta}{4} + \cdots\Big). \tag{5.43}$$

These correspond to the conformal blocks of the identity operator and the ϵ operator, consistent with the fusion rule $\sigma \times \sigma = \mathbb{I} + \epsilon$.

5.2.5 Crossing and Block Transformations

Under $\eta \mapsto 1-\eta$ the conformal blocks transform linearly among themselves:

$$\begin{pmatrix} \mathcal{F}(0|1-\eta) \\ \mathcal{F}(\frac{1}{2}|1-\eta) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \mathcal{F}(0|\eta) \\ \mathcal{F}(\frac{1}{2}|\eta) \end{pmatrix}. \tag{5.44}$$

This finite transformation property of the blocks under $\eta \mapsto 1-\eta$ is specific to these two-dimensional minimal models.

Finally, though one could start with an arbitrary OPE coefficient C_ϵ, crossing symmetry and the explicit transformation of blocks fix it; thus dynamical quantities such as C_ϵ are determined by consistency conditions (the conformal bootstrap).

5.3 Summary of Results

Finally, we end this article by reproducing a table in [12]. It illustrates some solutions of minimal models.

model	x	h	$\bar{h}$	$h_{p,q}(c)$
Ising	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{16}$	$m=3$ $\begin{array}{cc} & \frac{1}{16} \\ 0 & \frac{1}{2} \end{array}$
	1	$\frac{1}{2}$	$\frac{1}{2}$	
	$\frac{1}{2}$	$\frac{1}{2}$	0	
tri-critical 3-state Potts	$\frac{2}{21}$	$\frac{1}{21}$	$\frac{1}{21}$	$m=6$ $\begin{array}{ccccc} & & & & \frac{1}{7} \\ & & & \frac{5}{56} & \frac{5}{7} \\ & & \frac{1}{21} & \frac{33}{56} & \frac{12}{7} \\ & \frac{1}{56} & \frac{10}{21} & \frac{85}{56} & \frac{22}{7} \\ 0 & \frac{3}{8} & \frac{4}{3} & \frac{23}{8} & 5 \end{array}$
	$\frac{2}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	
	$\frac{20}{21}$	$\frac{10}{21}$	$\frac{10}{21}$	
	$\frac{10}{7}$	$\frac{5}{7}$	$\frac{5}{7}$	
	$\frac{17}{7}$	$\frac{12}{7}$	$\frac{5}{7}$	
	$\frac{8}{3}$	$\frac{4}{3}$	$\frac{4}{3}$	
	$\frac{23}{7}$	$\frac{22}{7}$	$\frac{1}{7}$	
	5	5	0	
tri critical Ising	$\frac{3}{40}$	$\frac{3}{80}$	$\frac{3}{80}$	$m=4$ $\begin{array}{ccc} & & \frac{1}{10} \\ & \frac{3}{80} & \frac{3}{5} \\ 0 & \frac{7}{16} & \frac{3}{2} \end{array}$
	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{10}$	
	$\frac{7}{8}$	$\frac{7}{16}$	$\frac{7}{16}$	
	$\frac{6}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	
	3	$\frac{3}{2}$	$\frac{3}{2}$	
3-state Potts	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$m=5$ $\begin{array}{cccc} & & & \frac{1}{8} \\ & & \frac{1}{15} & \frac{2}{3} \\ & \frac{1}{40} & \frac{21}{40} & \frac{13}{8} \\ 0 & \frac{2}{5} & \frac{7}{5} & 3 \end{array}$
	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	
	$\frac{4}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	
	$\frac{9}{5}$	$\frac{7}{5}$	$\frac{2}{5}$	
	$\frac{14}{5}$	$\frac{7}{5}$	$\frac{7}{5}$	
	3	3	0	

Figure 2. "TABLE I. Comparison with known scaling dimensions. The $h_{p,q}$ from [(4.19)] are listed with p running horizontally from 1 to $m-1$ and q running vertica1ly from 1 to p. The spins $h - \bar{h}$ are consistent with what is known. Some operators have alternative interpretations as derivatives." – as in the paper.

A Physical Meaning of the Central Charge

In this appendix we explain the physical interpretation of the central charge in two-dimensional conformal field theory (CFT). The starting point is the Virasoro algebra generated by the modes L_n of the holomorphic component of the stress tensor,

$$T(z) = \sum_{n \in \mathbb{Z}} z^{-n-2} L_n. \tag{A.1}$$

The operator product expansion (OPE) of the stress tensor with itself is

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \text{regular}, \tag{A.2}$$

which implies the Virasoro commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0}. \tag{A.3}$$

The presence of the central term proportional to c means that although the classical algebra of infinitesimal conformal transformations in two dimensions is infinite-dimensional without an anomaly, the quantum theory develops an obstruction: the stress tensor is not a primary operator under conformal transformations at the quantum level. This is the origin of the central extension.

Physically, the central charge measures several quantities:

(i) Degrees of freedom

For unitary CFTs, the central charge is a measure of the number of local degrees of freedom. The c-theorem states that along RG flows from a UV fixed point to an IR fixed point,

$$c_{\mathrm{UV}} > c_{\mathrm{IR}}, \tag{A.4}$$

thus establishing c as a monotonic measure of effective degrees of freedom.

(ii) Casimir energy on the cylinder

Mapping the complex plane to the cylinder via $z = e^{\tau + i\sigma}$, the vacuum energy becomes

$$E_0 = -\frac{\pi c}{6L}, \tag{A.5}$$

where L is the circumference of the spatial circle. Thus, c directly controls the Casimir energy.

(iii) Weyl anomaly

The expectation value of the trace of the stress tensor under a background Weyl transformation is proportional to the central charge:

$$\langle T^{\mu}{}_{\mu} \rangle = -\frac{c}{12} R, \tag{A.6}$$

where R is the scalar curvature of the background metric. The derivation of this relation is given in Appendix B.

Thus the central charge is simultaneously a measure of algebraic quantum anomaly, geometric response to curvature, and RG-monotone count of effective degrees of freedom.

B The Trace Anomaly

A classically conformal field theory in two dimensions has a traceless stress tensor, $T^{\mu}{}_{\mu} = 0$. Quantum mechanically, a conformal anomaly appears because the path integral measure is not invariant under Weyl transformations.

Let the metric transform under a local Weyl rescaling $g_{\mu\nu} \to e^{2\omega(x)} g_{\mu\nu}$. The variation of the effective action $W[g] = -\ln Z[g]$ is

$$\delta_\omega W[g] = \frac{1}{2} \int d^2x \sqrt{g}\, \langle T^{\mu\nu} \rangle\, \delta_\omega g_{\mu\nu} = \int d^2x \sqrt{g}\, \omega(x)\, \langle T^{\mu}{}_{\mu} \rangle. \tag{B.1}$$

In a two-dimensional CFT the anomalous variation of the effective action is fixed by locality and diffeomorphism invariance to take the form

$$\delta_\omega W[g] = -\frac{c}{24\pi} \int d^2x \sqrt{g}\, \omega\, R. \tag{B.2}$$

Comparing (B.1) and (B.2) yields the trace anomaly:

$$\langle T^{\mu}{}_{\mu} \rangle = -\frac{c}{24\pi} R. \tag{B.3}$$

In complex coordinates with metric $ds^2 = e^{2\phi(z,\bar{z})} dz\, d\bar{z}$, the trace is related to the non-holomorphicity of the stress tensor. One finds

$$\partial_{\bar{z}} T(z) = -\frac{c}{12}\, \partial_z^3 \phi(z, \bar{z}), \tag{B.4}$$

showing that curvature obstructs holomorphicity of T.

A related identity is the Schwarzian transformation law of the stress tensor. Under a conformal map $z \to f(z)$,

$$T(z) = f'(z)^2\, T(f(z)) + \frac{c}{12}\{f, z\}, \tag{B.5}$$

where the Schwarzian derivative is

$$\{f, z\} = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2. \tag{B.6}$$

The Schwarzian term is precisely the finite version of the trace anomaly.

Thus the trace anomaly is the origin of the central charge appearing in the transformation of the stress tensor and encodes the coupling of the theory to background curvature.

C Fusion Rules for Minimal Models

Rational CFTs, and in particular minimal models, have a finite number of primary fields whose operator product expansions close among themselves. Fusion rules encode the decomposition of the OPE of two primaries into irreducible Virasoro representations.

Minimal models $\mathcal{M}(p, p')$ with coprime integers $p > p' > 1$ have central charge

$$c = 1 - 6\frac{(p - p')^2}{pp'}. \tag{C.1}$$

Primary fields are labeled by Kac indices (r, s) with $1 \le r \le p' - 1$ and $1 \le s \le p - 1$, and the conformal weights are

$$h_{r,s} = \frac{(pr - p's)^2 - (p - p')^2}{4pp'}. \tag{C.2}$$

The fusion rules follow from the Verlinde formula. Let S be the modular S-matrix of the characters $\chi_{r,s}$. The fusion coefficient $N_{(r_1,s_1),(r_2,s_2)}{}^{(r_3,s_3)}$ is

$$N_{12}{}^{3} = \sum_{\rho} \frac{S_{1\rho}\, S_{2\rho}\, S^{*}_{3\rho}}{S_{(1,1)\rho}}. \tag{C.3}$$

For minimal models this yields simple selection rules:

$$(r_1, s_1) \times (r_2, s_2) = \sum_{\substack{r=|r_1-r_2|+1 \\ r \equiv r_1+r_2+1 \pmod 2}}^{\min(r_1+r_2-1,\, 2p'-r_1-r_2-1)} \; \sum_{\substack{s=|s_1-s_2|+1 \\ s \equiv s_1+s_2+1 \pmod 2}}^{\min(s_1+s_2-1,\, 2p-s_1-s_2-1)} (r, s). \tag{C.4}$$

The sums proceed in steps of 2 because of the parity constraint.

For example, in the Ising model $\mathcal{M}(4,3)$ with primaries $\mathbf{1}=(1,1)$, $\sigma=(1,2)$, $\varepsilon=(2,1)$, the nontrivial fusion rules are

$$\begin{aligned}\sigma\times\sigma &= \mathbf{1}+\varepsilon,\\ \sigma\times\varepsilon &= \sigma,\\ \varepsilon\times\varepsilon &= \mathbf{1}.\end{aligned} \tag{C.5}$$

Fusion rules determine the spectrum of intermediate channels in correlation functions and encode the algebraic structure of primary fields in rational CFT.

D Quantization of Free Boson and Free Fermion on a Cylinder

D.1 Canonical Quantization

We consider a two-dimensional Euclidean CFT on a cylinder with coordinates

$$w=\tau+i\sigma, \qquad \sigma\sim\sigma+2\pi, \tag{D.1}$$

related to the complex plane coordinate z by the exponential map (Sec. 2.1):

$$z=e^{w}. \tag{D.2}$$

Under this map, primary operators transform according to Eq. (2.1.15).

D.1.1 Free Boson

The action on the cylinder is obtained from Sec. 2.3.1 (free boson):

$$S=\frac{1}{8\pi}\int d\tau d\sigma\ (\partial_\mu X)(\partial^\mu X). \tag{D.3}$$

The canonical momentum is

$$\Pi(\tau,\sigma)=\frac{\delta S}{\delta(\partial_\tau X)}=\frac{1}{4\pi}\partial_\tau X. \tag{D.4}$$

Canonical quantization imposes

$$[X(\tau,\sigma),\Pi(\tau,\sigma')]=i\delta(\sigma-\sigma'). \tag{D.5}$$

Using Eq. (D.4), this gives

$$[X(\sigma),\partial_\tau X(\sigma')]=4\pi i\delta(\sigma-\sigma'). \tag{D.6}$$

Expanding the field consistent with the cylinder periodicity

$$X(\tau,\sigma)=x_0+p\tau+i\sum_{n\neq 0}\frac{1}{n}\left(\alpha_n e^{-in(\tau+\sigma)}+\tilde{\alpha}_n e^{-in(\tau-\sigma)}\right), \tag{D.7}$$

and substituting into the canonical commutator yields the oscillator algebra (using Sec. 3.1 on normal ordering and Sec. 3.2 on Hilbert space structure):

$$[\alpha_m,\alpha_n]=m\delta_{m+n,0}, \qquad [\tilde{\alpha}_m,\tilde{\alpha}_n]=m\delta_{m+n,0}, \tag{D.8}$$

and left/right movers commute

$$[\alpha_m, \tilde{\alpha}_n] = 0. \tag{D.9}$$

The stress tensor is obtained from Noether's theorem (Sec. 1.1):

$$T(z) = -\frac{1}{2} :\partial X \partial X:, \qquad \tilde{T}(\bar{z}) = -\frac{1}{2} :\bar{\partial} X \bar{\partial} X: . \tag{D.10}$$

The corresponding Virasoro generators are defined as usual:

$$L_n = \frac{1}{2} \sum_{m \in \mathbb{Z}} :\alpha_{n-m} \alpha_m:, \tag{D.11}$$

and similarly for $\tilde{L}_n$. Using the oscillator algebra (D.8) and the prescription of Sec. 3.1, one derives (using standard contour manipulation in Sec. 2.4)

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{1}{12} m(m^2 - 1) \delta_{m+n,0}. \tag{D.12}$$

D.1.2 Free Fermion

The Euclidean action for a free Majorana fermion (Sec. 2.3.2) is

$$S = \frac{1}{2\pi} \int d\tau d\sigma \left(\psi \partial_{\bar{z}} \psi + \tilde{\psi} \partial_z \tilde{\psi} \right). \tag{D.13}$$

The canonical anticommutator on equal-τ slices is

$$\{\psi(\sigma), \psi(\sigma')\} = 2\pi \delta(\sigma - \sigma'), \tag{D.14}$$

and similarly for $\tilde{\psi}$.

Depending on boundary conditions (treated fully in Sec. D.4) the mode expansion is

$$\psi(\tau, \sigma) = \sum_{r \in \mathbb{Z}+\nu} \psi_r e^{-ir(\tau+\sigma)}, \qquad \tilde{\psi}(\tau, \sigma) = \sum_{r \in \mathbb{Z}+\nu} \tilde{\psi}_r e^{-ir(\tau-\sigma)}, \tag{D.15}$$

where $\nu = 1/2$ (NS) or $\nu = 0$ (R). Eq. (D.14) gives

$$\{\psi_r, \psi_s\} = \delta_{r+s,0}, \qquad \{\tilde{\psi}_r, \tilde{\psi}_s\} = \delta_{r+s,0}. \tag{D.16}$$

The stress tensor is (Sec. 2.4)

$$T(z) = -\frac{1}{2} :\psi \partial \psi: . \tag{D.17}$$

Using the above anticommutators gives the fermionic Virasoro algebra with $c = \frac{1}{2}$.

D.2 Vertex Operators

From Sec. 2.1 and Sec. 2.3.1, free boson OPE is

$$X(z)X(w) \sim -\ln(z-w). \tag{D.18}$$

This gives for the derivative

$$\partial X(z)X(w) \sim -\frac{1}{z-w}. \tag{D.19}$$

We define the holomorphic vertex operator of momentum α:

$$V_\alpha(z) =: e^{i\alpha X(z)} : . \tag{D.20}$$

Using the Baker–Campbell–Hausdorff identity and Eq. (D.18):

$$e^A e^B =: e^{A+B} : e^{\langle AB \rangle} \tag{D.21}$$

gives the OPE

$$V_\alpha(z)V_\beta(w) \sim (z-w)^{\alpha\beta} : e^{i(\alpha+\beta)X(w)} : . \tag{D.22}$$

Using the Ward identity from Sec. 2.2 and stress tensor (D.10),

$$T(z)V_\alpha(w) \sim \frac{\alpha^2/2}{(z-w)^2}V_\alpha(w) + \frac{1}{z-w}\partial_w V_\alpha(w). \tag{D.23}$$

Thus the conformal weight is

$$h(\alpha) = \frac{\alpha^2}{2}. \tag{D.24}$$

D.3 Fock Space

The bosonic Hilbert space follows the construction in Sec. 3.2: Define a highest weight state $|p\rangle$ such that

$$\alpha_0|p\rangle = p|p\rangle, \qquad \alpha_{n>0}|p\rangle = 0. \tag{D.25}$$

Acting with α_{-n}, $n > 0$ builds the Fock space

$$\mathcal{H}_p = \text{span}\{\alpha_{-n_1}\cdots\alpha_{-n_k}|p\rangle\}. \tag{D.26}$$

The Virasoro generator L_0 from Eq. (D.11) gives the energy:

$$L_0|p\rangle = \left(\frac{p^2}{2} + a\right)|p\rangle, \tag{D.27}$$

where a is the normal-ordering constant. Using zeta regularization (Sec. 3.1),

$$\sum_{n=1}^{\infty} n = -\frac{1}{12}, \tag{D.28}$$

we find for a holomorphic boson

$$a = -\frac{1}{24}. \tag{D.29}$$

The fermionic Fock space depends on NS/R moding: In the NS sector, the vacuum satisfies

$$\psi_{r>0}|0\rangle_{NS} = 0. \tag{D.30}$$

In the R sector zero modes appear; their algebra will be derived in the next subsection.

D.4 Ramond vs Neveu–Schwarz Boundary Conditions

For a fermion on the cylinder, boundary conditions are

$$\psi(\tau, \sigma + 2\pi) = \pm\psi(\tau, \sigma). \tag{D.31}$$

The $-$ sign (antiperiodic) defines the NS sector; the $+$ sign (periodic) defines the R sector.

Neveu–Schwarz Sector

Antiperiodicity gives

$$\psi(\sigma + 2\pi) = -\psi(\sigma) \quad \Rightarrow \quad r \in \mathbb{Z} + \frac{1}{2}. \tag{D.32}$$

There are no zero modes, and the vacuum obeys (D.30). The normal-ordering constant from fermionic zeta function is

$$a_{NS} = +\frac{1}{48}. \tag{D.33}$$

Ramond Sector

Periodicity gives

$$\psi(\sigma + 2\pi) = \psi(\sigma) \quad \Rightarrow \quad r \in \mathbb{Z}. \tag{D.34}$$

In particular, ψ_0 exists. From (D.16),

$$\{\psi_0, \psi_0\} = 1. \tag{D.35}$$

Thus ψ_0 generates a one-dimensional Clifford algebra. Therefore the R sector vacuum is two-fold degenerate:

$$\psi_0|+\rangle_R = |-\rangle_R, \qquad \psi_0|-\rangle_R = |+\rangle_R. \tag{D.36}$$

Using Sec. 2.3 (bosonization), write $\psi = e^{iH}$ with

$$H(z)H(w) \sim -\ln(z - w), \tag{D.37}$$

so the spin fields are

$$S_\pm(z) =: e^{\pm\frac{i}{2}H(z)} : . \tag{D.38}$$

Using Eq. (D.24), their conformal dimension is

$$h(S_\pm) = \frac{1}{2}\left(\frac{1}{2}\right)^2 = \frac{1}{8}. \tag{D.39}$$

These create the R vacua from the NS vacuum:

$$|\pm\rangle_R = S_\pm(0)|0\rangle_{NS}. \tag{D.40}$$

Normal-ordering constant in the R sector is

$$a_R = 0. \tag{D.41}$$

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