Problem 1

Consider the Lagrangian

$$L = -\frac{a[\tau]}{2}t'[\tau]^2 + \frac{b[\tau]}{2}r'[\tau]^2 + \frac{r^2[\tau]}{2}\theta'[\tau]^2 + \frac{(r[\tau]\sin(\theta[\tau]))^2}{2}\phi'[\tau]^2$$

- Calculate non-vanishing Christoffel symbols by comparing geodesic equations and Euler-Lagrange equations.
- Now calculate non-vanishing components of the Ricci tensor $R_{\mu\nu}$.
- From Einstein equations show that $a = 1/b$.

Solution:

Clearly, scaling the Lagrangian by 2 will not change the equations of motion. In the following page, I have used the online Mathematica notebook 'Einstein Field Equations' and modified it to include the given Lagrangian. The notebook calculates the Christoffel symbols and the Ricci tensor components, among other things.

Using the results of the notebook, one can relate, and confirm, that Christoffel symbols computed by comparing the geodesic and Euler-Lagrange equations are the same as the ones calculated using the metric.

Finally, I have supplied the notebook with the initial conditon that $a = 1/b$ and from it, the notebook has calculated that the Einstein tensor is zero. This is a confirmation of the fact that the Einstein equations are satisfied for $\Lambda = 0$ and $T_{\mu\nu} = 0$.

Note that the equations of motion become zero under two assumptions:

1. $a = 1/b$. This is not an assumption as much as a result from $G_{11} = 0$ (that shows that $\partial_r ab = 0$).
2. $\partial_t a = 0$ and $\partial_t b = 0$

These assumptions correspond physically to the fact that this metric is independent of time, and spherically symmetric because $T_{\mu\nu}$ is zero.

These give the solution:

$$a(r) = 1 + \frac{C}{r - r_0} \tag{1}$$

where C and r_0 are constants, and $r_0 \geq 0$. I would expect $r_0 \to 0$ and $C \to -2M$ such that equation (1) reduces to the Schwarzschild metric, due to Birkoff's theorem.

The notebook pdf is attached on the next page.

In[1]:=
```
Clear["Global`*"];
(*Version 0.2.1*)
```

Einstein Field Equations

Input Section

In[2]:=
```
Coordinates = {t[τ], r[τ], θ[τ], ϕ[τ]};
co[μ_] := Coordinates⟦μ⟧; (*To simplify the Code*)
Dim = Length[Coordinates];(*Dimention of Space-Time*)
PoM = τ; (*Parameter of Motion*)
(* ω[r]=1-2GM/r;
v[r]=1/ω[r]; *)
Metric(*gμν*) = {{-a[t[τ], r[τ]], 0, 0, 0},
                 {0, b[t[τ], r[τ]], 0, 0},
                 {0, 0, r[τ]^2, 0},
                 {0, 0, 0, r[τ]^2 Sin[θ[τ]]^2}};
SETensor(*Tμν*) = {{0, 0, 0, 0},
                 {0, 0, 0, 0},
                 {0, 0, 0, 0},
                 {0, 0, 0, 0}};
MetricInverse = Simplify[Inverse[Metric]];
pdConv[f_] := TraditionalForm[
  f /. Derivative[inds__][g_][vars__] :>
    Apply[Defer[D[g[vars], ##]] &,
     Transpose[{{vars}, {inds}}] /.
      {{var_, 0} :> Sequence[], {var_, 1} :> {var}}]]
DCoordinates = {t'[τ], r'[τ], θ'[τ], ϕ'[τ]};
```

Christoffel symbols (2_{nd} kind) ($\Gamma^{\beta}{}_{\mu\nu}$)

In[11]:=
```
Christoffel[β_, μ_, ν_] :=
 Christoffel[β, μ, ν] = Simplify[1/2 Sum[MetricInverse⟦β, α⟧ (D[Metric⟦α, μ⟧, co[ν]] +
      D[Metric⟦α, ν⟧, co[μ]] - D[Metric⟦μ, ν⟧, co[α]]), {α, Dim}]]
```

Riemann curvature tensor ($R^{\beta}{}_{\sigma\mu\nu}$) & Ricci tensor ($R_{\mu\nu}$) & Scalar curvature (R)

In[12]:=
```
Riemann[β_, σ_, μ_, ν_] :=
  Riemann[β, σ, μ, ν] = Simplify[D[Christoffel[β, ν, σ], co[μ]] -
     D[Christoffel[β, μ, σ], co[ν]] + Sum[Christoffel[β, μ, λ] × Christoffel[λ, ν, σ] -
       Christoffel[β, ν, λ] × Christoffel[λ, μ, σ], {λ, Dim}]];
Ricci[μ_, ν_] := Ricci[μ, ν] = Simplify[Sum[Riemann[λ, μ, λ, ν], {λ, Dim}]];
RicciScalar = Simplify[
   Sum[Ricci[μ, ν] × MetricInverse[[μ, ν]], {μ, Length[Coordinates]}, {ν, Dim}]];
```

Einstein Tensor

In[15]:=
```
EFE[μ_, ν_] :=
  EFE[μ, ν] = Simplify[Ricci[μ, ν] - 1/2 RicciScalar Metric[[μ, ν]]] + Λ Metric[[μ, ν]] //
    FullSimplify;
Λ = 0;
```

Other Useful Quantities ($R_{\beta\sigma\mu\nu}$, $R^{\beta\sigma\mu\nu}$, Kretschmann scalar)

In[17]:=
```
Riemanndddd[β_, σ_, μ_, ν_] (*R_βσμν*) := Riemanndddd[β, σ, μ, ν] =
   Sum[Riemann[β1, σ, μ, ν] × Metric[[β1, β]], {β1, Dim}] // Simplify;
Riemannuuuu[β_, σ_, μ_, ν_] (*R^βσμν*) := Riemannuuuu[β, σ, μ, ν] =
   Sum[Riemann[β, σ1, μ1, ν1] × MetricInverse[[σ1, σ]] × MetricInverse[[μ1, μ]] ×
      MetricInverse[[ν1, ν]], {σ1, Dim}, {μ1, Dim}, {ν1, Dim}] // Simplify;
Kretschmann = Sum[Riemanndddd[β, σ, μ, ν] × Riemannuuuu[β, σ, μ, ν],
    {β, Dim}, {σ, Dim}, {μ, Dim}, {ν, Dim}] // Simplify;
```

Geodesic Equations

In[20]:=
```
Geo = Array[geo, Dim];
For[λ = 1, λ ≤ Dim, λ++,
 Geo[[λ]] = Simplify[D[co[λ][PoM], {PoM, 2}] +
     Sum[(Christoffel[λ, μ, ν] /. (# → #[PoM] & /@ Coordinates))
       D[co[μ][PoM], PoM] × D[co[ν][PoM], PoM], {μ, Dim}, {ν, Dim}] == 0]
]
```

Euler Lagrange Equations

In[22]:=
```
Needs["VariationalMethods`"];
Euler =
 EulerEquations[DCoordinates.Metric.DCoordinatesᵀ, {t[τ], r[τ], θ[τ], ϕ[τ]}, τ]
```

Out[23]=

$$\{2\,a[t[\tau],r[\tau]]\,t''[\tau]+2\,r'[\tau]\,t'[\tau]\,a^{(0,1)}[t[\tau],r[\tau]]+t'[\tau]^2\,a^{(1,0)}[t[\tau],r[\tau]]+r'[\tau]^2\,b^{(1,0)}[t[\tau],r[\tau]]=0,$$
$$2\,r[\tau]\left(\theta'[\tau]^2+\mathrm{Sin}[\theta[\tau]]^2\,\phi'[\tau]^2\right)-2\,b[t[\tau],r[\tau]]\,r''[\tau]-t'[\tau]^2\,a^{(0,1)}[t[\tau],r[\tau]]-r'[\tau]^2\,b^{(0,1)}[t[\tau],r[\tau]]-2\,r'[\tau]\,t'[\tau]\,b^{(1,0)}[t[\tau],r[\tau]]=0,$$
$$r[\tau]\left(-4\,r'[\tau]\,\theta'[\tau]+r[\tau]\left(\mathrm{Sin}[2\,\theta[\tau]]\,\phi'[\tau]^2-2\,\theta''[\tau]\right)\right)=0,\ -2\,r[\tau]\,\mathrm{Sin}[\theta[\tau]]\,(2\,\mathrm{Sin}[\theta[\tau]]\,r'[\tau]\,\phi'[\tau]+r[\tau]\,(2\,\mathrm{Cos}[\theta[\tau]]\,\theta'[\tau]\,\phi'[\tau]+\mathrm{Sin}[\theta[\tau]]\,\phi''[\tau]))=0\}$$

Results

In[24]:=
```
Grid[{{"Metric Tensor (g_μν) = ", MatrixForm[Metric]}},
 Frame → True, FrameStyle → Blue]
Grid[{{"Inverse Metric Tensor (g^μν) = ", MatrixForm[MetricInverse]}},
 Frame → True, FrameStyle → Blue]
Grid[{{"Stress-Energy Tensor (T_μν) = ", MatrixForm[SETensor]}},
 Frame → True, FrameStyle → Blue]
ChristoffelList = {{"Γ^β_μν", ":", "Christoffel symbols ( 2_nd kind )"}};
ChristoffelPrintAction := AppendTo[ChristoffelList,
   {Subsuperscript["Γ", " " <> ToString[co[μ]] <> ToString[co[ν]], ToString[co[β]]],
    " = ", Christoffel[β, μ, ν] // pdConv}];
For[β = 1, β ≤ Dim, β++,
 For[μ = 1, μ ≤ Dim, μ++,
  For[ν = 1, ν ≤ μ, ν++,
   If[Christoffel[β, μ, ν] ≠ 0,
    ChristoffelPrintAction, , ChristoffelPrintAction]
   ]
  ]
 ]
If[Length@ChristoffelList == 1,
  ChristoffelList[[1]] = {"Γ^β_μν", "=", "Christoffel symbols ( 2_nd kind ) = 0"}];
Grid[ChristoffelList, Frame → True, FrameStyle → Blue]
RiemannList = {{"R^β_σμν", ":", "Riemann curvature tensor"}};
RiemannPrintAction := AppendTo[RiemannList,
   {Subsuperscript["R", " " <> ToString[co[σ]] <> ToString[co[μ]] <> ToString[co[ν]],
     ToString[co[β]]], " = ", Riemann[β, σ, μ, ν] // pdConv}];
For[β = 1, β ≤ Dim, β++,
 For[σ = 1, σ ≤ Dim, σ++,
  For[μ = 1, μ ≤ Dim, μ++,
   For[ν = 1, ν ≤ Dim, ν++,
```

```
    If[Riemann[β, σ, μ, ν] ≠ 0,
     RiemannPrintAction, , RiemannPrintAction]
   ]
  ]
 ]
]
If[Length@RiemannList == 1,
  RiemannList[[1]] = {"R^β_σμν", "=", "Riemann curvature tensor = 0"}];
Grid[RiemannList, Frame → True, FrameStyle → Blue]
RicciList = {{"R_μν", ":", "Ricci tensor"}};
RicciPrintAction :=
  AppendTo[RicciList, {Subscript["R", ToString[co[μ]] <> ToString[co[ν]]],
    " = ", Ricci[μ, ν] // pdConv}];
For[μ = 1, μ ≤ Dim, μ++,
 For[ν = 1, ν ≤ μ, ν++,
  If[Ricci[μ, ν] ≠ 0,
   RicciPrintAction, , RicciPrintAction]
 ]
]
If[Length@RicciList == 1, RicciList[[1]] = {"R_μν", "=", "Ricci tensor = 0"}];
Grid[RicciList, Frame → True, FrameStyle → Blue]
Grid[{{"Scalar curvature (R) = ", RicciScalar // pdConv}},
 Frame → True, FrameStyle → Blue]
Grid[{{"Kretschmann (K) = ", Kretschmann // pdConv}},
 Frame → True, FrameStyle → Blue]
EFEList = {{"G_μν", "+ Λ", "g_μν", "=", "Einstein Field Equations", "= (8 π G)/c^4 T_μν"}};
EFEPrintAction :=
  AppendTo[EFEList, {Subscript["G", ToString[co[μ]] <> ToString[co[ν]]],
    "+ Λ ", Subscript["g", ToString[co[μ]] <> ToString[co[ν]]], " = ",
    {{EFE[μ, ν] // pdConv, " = (8 π G)/c^4 ", SETensor[[μ, ν]] // pdConv}} // Grid,
    Subscript[" = (8 π G)/c^4 T", ToString[co[μ]] <> ToString[co[ν]]]}];
For[μ = 1, μ ≤ Dim, μ++,
 For[ν = 1, ν ≤ μ, ν++,
  If[EFE[μ, ν] ≠ 0,
   EFEPrintAction, , EFEPrintAction]
 ]
]
If[Length@EFEList == 1, EFEList[[1]] = {"G_μν", "+ Λ", "g_μν", "=", 0, "= (8 π G)/c^4 T_μν"}];
Grid[EFEList, Frame → True, FrameStyle → Blue]

GeoList = {{"Geo^λ", ":", ""}};
GeoPrintAction := AppendTo[GeoList,
   {Superscript["Geo", " " <> ToString[co[λ]]], " ≡ ", Geo[[λ]] // pdConv}];
For[λ = 1, λ ≤ Dim, λ++,
 If[Geo[[λ]] ≠ 0,
```

```
    GeoPrintAction, , GeoPrintAction]
]
Grid[GeoList, Frame → True, FrameStyle → Blue]

EulerList = {{"Euler^λ", ":", ""}};
EulerPrintAction := AppendTo[EulerList,
    {Superscript["Euler", " " <> ToString[co[λ]]], " ≡ ", Euler[[λ]] // pdConv}];
For[λ = 1, λ ≤ Dim, λ++,
  If[Euler[[λ]] ≠ 0,
    EulerPrintAction, , EulerPrintAction]
]
Grid[EulerList, Frame → True, FrameStyle → Blue]
```

Out[24]=

Metric Tensor $(g_{\mu\nu})$ = $\begin{pmatrix} -a[t[\tau], r[\tau]] & 0 & 0 & 0 \\ 0 & b[t[\tau], r[\tau]] & 0 & 0 \\ 0 & 0 & r[\tau]^2 & 0 \\ 0 & 0 & 0 & r[\tau]^2 \mathrm{Sin}[\theta[\tau]]^2 \end{pmatrix}$

Out[25]=

Inverse Metric Tensor $(g^{\mu\nu})$ = $\begin{pmatrix} -\frac{1}{a[t[\tau],r[\tau]]} & 0 & 0 & 0 \\ 0 & \frac{1}{b[t[\tau],r[\tau]]} & 0 & 0 \\ 0 & 0 & \frac{1}{r[\tau]^2} & 0 \\ 0 & 0 & 0 & \frac{\mathrm{Csc}[\theta[\tau]]^2}{r[\tau]^2} \end{pmatrix}$

Out[26]=

Stress-Energy Tensor $(T_{\mu\nu})$ = $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Out[31]=

$\Gamma^{\beta}{}_{\mu\nu}$: Christoffel symbols (2_{nd} kind)

$$\Gamma^{t[\tau]}{}_{t[\tau]t[\tau]} = \frac{\frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}}{2\,a(t(\tau),r(\tau))}$$

$$\Gamma^{t[\tau]}{}_{r[\tau]t[\tau]} = \frac{\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}}{2\,a(t(\tau),r(\tau))}$$

$$\Gamma^{t[\tau]}{}_{r[\tau]r[\tau]} = \frac{\frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)}}{2\,a(t(\tau),r(\tau))}$$

$$\Gamma^{r[\tau]}{}_{t[\tau]t[\tau]} = \frac{\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}}{2\,b(t(\tau),r(\tau))}$$

$$\Gamma^{r[\tau]}{}_{r[\tau]t[\tau]} = \frac{\frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)}}{2\,b(t(\tau),r(\tau))}$$

$$\Gamma^{r[\tau]}{}_{r[\tau]r[\tau]} = \frac{\frac{\partial b(t(\tau),r(\tau))}{\partial r(\tau)}}{2\,b(t(\tau),r(\tau))}$$

$$\Gamma^{r[\tau]}{}_{\theta[\tau]\theta[\tau]} = -\frac{r(\tau)}{b(t(\tau),r(\tau))}$$

$$\Gamma^{r[\tau]}{}_{\phi[\tau]\phi[\tau]} = -\frac{r(\tau)\sin^2(\theta(\tau))}{b(t(\tau),r(\tau))}$$

$$\Gamma^{\theta[\tau]}{}_{\theta[\tau]r[\tau]} = \frac{1}{r(\tau)}$$

$$\Gamma^{\theta[\tau]}{}_{\phi[\tau]\phi[\tau]} = \sin(\theta(\tau))\,(-\cos(\theta(\tau)))$$

$$\Gamma^{\phi[\tau]}{}_{\phi[\tau]r[\tau]} = \frac{1}{r(\tau)}$$

$$\Gamma^{\phi[\tau]}{}_{\phi[\tau]\theta[\tau]} = \cot(\theta(\tau))$$

Out[36]=

$R^{\beta}{}_{\sigma\mu\nu}$: Riemann curvature tensor

$$R^{t[\tau]}{}_{r[\tau]t[\tau]r[\tau]} = \frac{1}{4\,a(t(\tau),r(\tau))^2\,b(t(\tau),r(\tau))}\Bigg(b(t(\tau),r(\tau))\left(-2\,a(t(\tau),r(\tau))\left(\frac{\partial^2 a(t(\tau),r(\tau))}{\partial r(\tau)^2}-\frac{\partial^2 b(t(\tau),r(\tau))}{\partial t(\tau)^2}\right)-\frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}\,\frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)}+\left(\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}\right)^2\right)+a(t(\tau),r(\tau))\left(\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}\,\frac{\partial b(t(\tau),r(\tau))}{\partial r(\tau)}-\left(\frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)}\right)^2\right)\Bigg)$$

$$R^{t[\tau]}{}_{r[\tau]r[\tau]t[\tau]} = \frac{1}{4\,a(t(\tau),r(\tau))^2\,b(t(\tau),r(\tau))}\Bigg(b(t(\tau),r(\tau))\left(2\,a(t(\tau),r(\tau))\left(\frac{\partial^2 a(t(\tau),r(\tau))}{\partial r(\tau)^2}-\frac{\partial^2 b(t(\tau),r(\tau))}{\partial t(\tau)^2}\right)+\frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}\,\frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)}-\left(\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}\right)^2\right)+a(t(\tau),r(\tau))\left(\left(\frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)}\right)^2-\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}\,\frac{\partial b(t(\tau),r(\tau))}{\partial r(\tau)}\right)\Bigg)$$

$$R^{t[\tau]}{}_{\theta[\tau]t[\tau]\theta[\tau]} = -\frac{r(\tau)\,\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}}{2\,a(t(\tau),r(\tau))\,b(t(\tau),r(\tau))}$$

$$R^{t[\tau]}{}_{\theta[\tau]r[\tau]\theta[\tau]} = -\frac{r(\tau)\,\frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)}}{2\,a(t(\tau),r(\tau))\,b(t(\tau),r(\tau))}$$

$$R^{t[\tau]}{}_{\theta[\tau]\theta[\tau]t[\tau]} = \frac{r(\tau)\,\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}}{2\,a(t(\tau),r(\tau))\,b(t(\tau),r(\tau))}$$

$$R^{t[\tau]}{}_{\theta[\tau]\theta[\tau]r[\tau]} = \frac{r(\tau)\,\frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)}}{2\,a(t(\tau),r(\tau))\,b(t(\tau),r(\tau))}$$

$$R^{t[\tau]}{}_{\phi[\tau]t[\tau]\phi[\tau]} = -\frac{r(\tau)\sin^2(\theta(\tau))\,\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}}{2\,a(t(\tau),r(\tau))\,b(t(\tau),r(\tau))}$$

$$R^{t[\tau]}{}_{\phi[\tau]r[\tau]\phi[\tau]} = -\frac{r(\tau)\sin^2(\theta(\tau))\,\frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)}}{2\,a(t(\tau),r(\tau))\,b(t(\tau),r(\tau))}$$

$$R^{t[\tau]}{}_{\phi[\tau]\phi[\tau]t[\tau]} = \frac{r(\tau)\sin^2(\theta(\tau))\,\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}}{2\,a(t(\tau),r(\tau))\,b(t(\tau),r(\tau))}$$

$$R^{t[\tau]}{}_{\phi[\tau]\phi[\tau]r[\tau]} = \frac{r(\tau)\sin^2(\theta(\tau))\,\frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)}}{2\,a(t(\tau),r(\tau))\,b(t(\tau),r(\tau))}$$

$$R^{r[\tau]}{}_{t[\tau]t[\tau]r[\tau]} = \frac{1}{4\,a(t(\tau),r(\tau))\,b(t(\tau),r(\tau))^2}\Bigg(b(t(\tau),r(\tau))\left(-2\,a(t(\tau),r(\tau))\left(\frac{\partial^2 a(t(\tau),r(\tau))}{\partial r(\tau)^2}-\frac{\partial^2 b(t(\tau),r(\tau))}{\partial t(\tau)^2}\right)-\frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}\,\frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)}+\left(\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}\right)^2\right)+a(t(\tau),r(\tau))\left(\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}\,\frac{\partial b(t(\tau),r(\tau))}{\partial r(\tau)}-\left(\frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)}\right)^2\right)\Bigg)$$

$$R^{r[\tau]}{}_{t[\tau]r[\tau]t[\tau]} = \frac{1}{4\,a(t(\tau),r(\tau))\,b(t(\tau),r(\tau))^2}\Bigg(b(t(\tau),r(\tau))\left(2\,a(t(\tau),r(\tau))\left(\frac{\partial^2 a(t(\tau),r(\tau))}{\partial r(\tau)^2}-\frac{\partial^2 b(t(\tau),r(\tau))}{\partial t(\tau)^2}\right)+\frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}\,\frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)}-\left(\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}\right)^2\right)+a(t(\tau),r(\tau))\left(\left(\frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)}\right)^2-\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}\,\frac{\partial b(t(\tau),r(\tau))}{\partial r(\tau)}\right)\Bigg)$$

$$R^{r[\tau]}{}_{\theta[\tau]t[\tau]\theta[\tau]} = \frac{r(\tau)\,\frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)}}{2\,b(t(\tau),r(\tau))^2}$$

$$R^{r[\tau]}{}_{\theta[\tau]r[\tau]\theta[\tau]} = \frac{r(\tau)\,\frac{\partial b(t(\tau),r(\tau))}{\partial r(\tau)}}{2\,b(t(\tau),r(\tau))^2}$$

$$R^{r[\tau]}{}_{\theta[\tau]\theta[\tau]t[\tau]} = -\frac{r(\tau)\,\frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)}}{2\,b(t(\tau),r(\tau))^2}$$

$$R^{r[\tau]}{}_{\theta[\tau]\theta[\tau]r[\tau]} = -\frac{r(\tau)\,\frac{\partial b(t(\tau),r(\tau))}{\partial r(\tau)}}{2\,b(t(\tau),r(\tau))^2}$$

$$R^{r[\tau]}{}_{\phi[\tau]t[\tau]\phi[\tau]} = \frac{r(\tau)\,\sin^2(\theta(\tau))\,\frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)}}{2\,b(t(\tau),r(\tau))^2}$$

$$R^{r[\tau]}{}_{\phi[\tau]r[\tau]\phi[\tau]} = \frac{r(\tau)\,\sin^2(\theta(\tau))\,\frac{\partial b(t(\tau),r(\tau))}{\partial r(\tau)}}{2\,b(t(\tau),r(\tau))^2}$$

$$R^{r[\tau]}{}_{\phi[\tau]\phi[\tau]t[\tau]} = -\frac{r(\tau)\,\sin^2(\theta(\tau))\,\frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)}}{2\,b(t(\tau),r(\tau))^2}$$

$$R^{r[\tau]}{}_{\phi[\tau]\phi[\tau]r[\tau]} = -\frac{r(\tau)\,\sin^2(\theta(\tau))\,\frac{\partial b(t(\tau),r(\tau))}{\partial r(\tau)}}{2\,b(t(\tau),r(\tau))^2}$$

$$R^{\theta[\tau]}{}_{t[\tau]t[\tau]\theta[\tau]} = -\frac{\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}}{2\,r(\tau)\,b(t(\tau),r(\tau))}$$

$$R^{\theta[\tau]}{}_{t[\tau]r[\tau]\theta[\tau]} = -\frac{\frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)}}{2\,r(\tau)\,b(t(\tau),r(\tau))}$$

$$R^{\theta[\tau]}{}_{t[\tau]\theta[\tau]t[\tau]} = \frac{\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}}{2\,r(\tau)\,b(t(\tau),r(\tau))}$$

$$R^{\theta[\tau]}{}_{t[\tau]\theta[\tau]r[\tau]} = \frac{\frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)}}{2\,r(\tau)\,b(t(\tau),r(\tau))}$$

$$R^{\theta[\tau]}{}_{r[\tau]t[\tau]\theta[\tau]} = -\frac{\frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)}}{2\,r(\tau)\,b(t(\tau),r(\tau))}$$

$$R^{\theta[\tau]}{}_{r[\tau]r[\tau]\theta[\tau]} = -\frac{\frac{\partial b(t(\tau),r(\tau))}{\partial r(\tau)}}{2\,r(\tau)\,b(t(\tau),r(\tau))}$$

$$R^{\theta[\tau]}{}_{r[\tau]\theta[\tau]t[\tau]} = \frac{\frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)}}{2\,r(\tau)\,b(t(\tau),r(\tau))}$$

$$R^{\theta[\tau]}{}_{r[\tau]\theta[\tau]r[\tau]} = \frac{\frac{\partial b(t(\tau),r(\tau))}{\partial r(\tau)}}{2\,r(\tau)\,b(t(\tau),r(\tau))}$$

$$R^{\theta[\tau]}{}_{\phi[\tau]\theta[\tau]\phi[\tau]} = \sin^2(\theta(\tau))\left(1-\frac{1}{b(t(\tau),r(\tau))}\right)$$

$$R^{\theta[\tau]}{}_{\phi[\tau]\phi[\tau]\theta[\tau]} = \sin^2(\theta(\tau))\left(\frac{1}{b(t(\tau),r(\tau))}-1\right)$$

$$R^{\phi[\tau]}{}_{t[\tau]t[\tau]\phi[\tau]} = -\frac{\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}}{2\,r(\tau)\,b(t(\tau),r(\tau))}$$

$$R^{\phi[\tau]}{}_{t[\tau]r[\tau]\phi[\tau]} = -\frac{\frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)}}{2\,r(\tau)\,b(t(\tau),r(\tau))}$$

$$R^{\phi[\tau]}{}_{t[\tau]\phi[\tau]t[\tau]} = \frac{\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}}{2\,r(\tau)\,b(t(\tau),r(\tau))}$$

$$R^{\phi[\tau]}{}_{t[\tau]\phi[\tau]r[\tau]} = \frac{\frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)}}{2\,r(\tau)\,b(t(\tau),r(\tau))}$$

$$R^{\phi[\tau]}{}_{r[\tau]t[\tau]\phi[\tau]} = -\frac{\frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)}}{2\,r(\tau)\,b(t(\tau),r(\tau))}$$

$$R^{\phi[\tau]}{}_{r[\tau]r[\tau]\phi[\tau]} = -\frac{\frac{\partial b(t(\tau),r(\tau))}{\partial r(\tau)}}{2\,r(\tau)\,b(t(\tau),r(\tau))}$$

$$R^{\phi[\tau]}{}_{r[\tau]\phi[\tau]t[\tau]} = \frac{\frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)}}{2\,r(\tau)\,b(t(\tau),r(\tau))}$$

$$R^{\phi[\tau]}{}_{r[\tau]\phi[\tau]r[\tau]} = \frac{\frac{\partial b(t(\tau),r(\tau))}{\partial r(\tau)}}{2\,r(\tau)\,b(t(\tau),r(\tau))}$$

$$R^{\phi[\tau]}{}_{\theta[\tau]\theta[\tau]\phi[\tau]} = \frac{1}{b(t(\tau),r(\tau))} - 1$$

$$R^{\phi[\tau]}{}_{\theta[\tau]\phi[\tau]\theta[\tau]} = 1 - \frac{1}{b(t(\tau),r(\tau))}$$

Out[41]=

$R_{\mu\nu}$: Ricci tensor

$$R_{t[\tau]t[\tau]} = \frac{1}{4\,b(t(\tau),r(\tau))^2}\Bigg(b(t(\tau),r(\tau))\left(2\left(\frac{\partial^2 a(t(\tau),r(\tau))}{\partial r(\tau)^2} - \frac{\partial^2 b(t(\tau),r(\tau))}{\partial t(\tau)^2}\right) + \frac{\frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}\frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)} - \left(\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}\right)^2}{a(t(\tau),r(\tau))} + \frac{4\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}}{r(\tau)}\right) - \frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}\frac{\partial b(t(\tau),r(\tau))}{\partial r(\tau)} + \left(\frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)}\right)^2\Bigg)$$

$$R_{r[\tau]t[\tau]} = \frac{\frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)}}{r(\tau)\,b(t(\tau),r(\tau))}$$

$$R_{r[\tau]r[\tau]} = \frac{1}{4\,b(t(\tau),r(\tau))}\Bigg(\frac{1}{a(t(\tau),r(\tau))^2}\Bigg(b(t(\tau),r(\tau))\left(-2\,a(t(\tau),r(\tau))\left(\frac{\partial^2 a(t(\tau),r(\tau))}{\partial r(\tau)^2} - \frac{\partial^2 b(t(\tau),r(\tau))}{\partial t(\tau)^2}\right) - \frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}\frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)} + \left(\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}\right)^2\right) + a(t(\tau),r(\tau))\left(\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}\frac{\partial b(t(\tau),r(\tau))}{\partial r(\tau)} - \left(\frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)}\right)^2\right)\Bigg) + \frac{4\frac{\partial b(t(\tau),r(\tau))}{\partial r(\tau)}}{r(\tau)}\Bigg)$$

$$R_{\theta[\tau]\theta[\tau]} = \frac{1}{2}\left(-\frac{\frac{r(\tau)\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}}{a(t(\tau),r(\tau))} + 2}{b(t(\tau),r(\tau))} + \frac{r(\tau)\frac{\partial b(t(\tau),r(\tau))}{\partial r(\tau)}}{b(t(\tau),r(\tau))^2} + 2\right)$$

$$R_{\phi[\tau]\phi[\tau]} = \Bigg(\sin^2(\theta(\tau))\Bigg(a(t(\tau),r(\tau))\left(2\,b(t(\tau),r(\tau))^2 - 2\,b(t(\tau),r(\tau)) + r(\tau)\frac{\partial b(t(\tau),r(\tau))}{\partial r(\tau)}\right) - r(\tau)\,b(t(\tau),r(\tau))\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}\Bigg)\Bigg) \Big/ \left(2\,a(t(\tau),r(\tau))\,b(t(\tau),r(\tau))^2\right)$$

Out[42]=

$$\text{Scalar curvature (R)} = \frac{1}{2\,r(\tau)^2\,a(t(\tau),r(\tau))^2\,b(t(\tau),r(\tau))^2}\Bigg(r(\tau)\,a(t(\tau),r(\tau))\Bigg(r(\tau)\left(\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}\frac{\partial b(t(\tau),r(\tau))}{\partial r(\tau)} - \left(\frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)}\right)^2\right) - 2\,b(t(\tau),r(\tau))\left(r(\tau)\left(\frac{\partial^2 a(t(\tau),r(\tau))}{\partial r(\tau)^2} - \frac{\partial^2 b(t(\tau),r(\tau))}{\partial t(\tau)^2}\right) + 2\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}\right)\Bigg) + 4\,a(t(\tau),r(\tau))^2\left(b(t(\tau),r(\tau))^2 - b(t(\tau),r(\tau)) + r(\tau)\frac{\partial b(t(\tau),r(\tau))}{\partial r(\tau)}\right) + r(\tau)^2\,b(t(\tau),r(\tau))\left(\left(\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}\right)^2 - \frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}\frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)}\right)\Bigg)$$

Out[43]=

$$
\begin{aligned}
\text{Kretschmann (K)} = {} & \frac{1}{4\, r(\tau)^4\, a(t(\tau),r(\tau))^4\, b(t(\tau),r(\tau))^4} \Bigg(a(t(\tau), r(\tau))^2 \\
& \left(-4\, r(\tau)^4\, b(t(\tau), r(\tau)) \left(\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)} \frac{\partial b(t(\tau),r(\tau))}{\partial r(\tau)} - \left(\frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)} \right)^2 \right) \right. \\
& \quad \left(\frac{\partial^2 a(t(\tau),r(\tau))}{\partial r(\tau)^2} - \frac{\partial^2 b(t(\tau),r(\tau))}{\partial t(\tau)^2} \right) + 4\, b(t(\tau), r(\tau))^2 \\
& \quad \left(r(\tau)^4 \left(\frac{\partial^2 a(t(\tau),r(\tau))}{\partial r(\tau)^2} - \frac{\partial^2 b(t(\tau),r(\tau))}{\partial t(\tau)^2} \right)^2 + 2\, r(\tau)^2 \left(\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)} \right)^2 \right) + \\
& \quad \left. r(\tau)^4 \left(\left(\frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)} \right)^2 - \frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)} \frac{\partial b(t(\tau),r(\tau))}{\partial r(\tau)} \right)^2 \right) - \\
& 2\, r(\tau)^4\, a(t(\tau), r(\tau))\, b(t(\tau), r(\tau)) \\
& \left(\left(\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)} \right)^2 - \frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)} \frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)} \right) \\
& \left(2\, b(t(\tau), r(\tau)) \left(\frac{\partial^2 a(t(\tau),r(\tau))}{\partial r(\tau)^2} - \frac{\partial^2 b(t(\tau),r(\tau))}{\partial t(\tau)^2} \right) - \right. \\
& \quad \left. \frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)} \frac{\partial b(t(\tau),r(\tau))}{\partial r(\tau)} + \left(\frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)} \right)^2 \right) + \\
& 8\, a(t(\tau), r(\tau))^4 \left(2\, b(t(\tau), r(\tau))^4 - 4\, b(t(\tau), r(\tau))^3 + \right. \\
& \quad \left. 2\, b(t(\tau), r(\tau))^2 + r(\tau)^2 \left(\frac{\partial b(t(\tau),r(\tau))}{\partial r(\tau)} \right)^2 \right) - \\
& 16\, r(\tau)^2\, a(t(\tau), r(\tau))^3\, b(t(\tau), r(\tau)) \left(\frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)} \right)^2 + \\
& r(\tau)^4\, b(t(\tau), r(\tau))^2 \left(\left(\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)} \right)^2 - \frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)} \frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)} \right)^2 \Bigg)
\end{aligned}
$$

Out[48]=

$$G_{\mu\nu} + \Lambda \; g_{\mu\nu} = \text{Einstein Field Equations} = \frac{8\pi G}{c^4}\, T_{\mu\nu}$$

$$G_{t[\tau]t[\tau]} + \Lambda \; g_{t[\tau]t[\tau]} = \left(a(t(\tau), r(\tau)) \left((b(t(\tau), r(\tau)) - 1)\, b(t(\tau), r(\tau)) + r(\tau) \frac{\partial b(t(\tau),r(\tau))}{\partial r(\tau)} \right) \right) \Big/ \left(r(\tau)^2\, b(t(\tau), r(\tau))^2 \right) = \frac{8\pi G}{c^4}\, 0 = \frac{8\pi G}{c^4}\, T_{t[\tau]t[\tau]}$$

$$G_{r[\tau]t[\tau]} + \Lambda \; g_{r[\tau]t[\tau]} = \frac{\frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)}}{r(\tau)\, b(t(\tau),r(\tau))} = \frac{8\pi G}{c^4}\, 0 = \frac{8\pi G}{c^4}\, T_{r[\tau]t[\tau]}$$

$$G_{r[\tau]r[\tau]} + \Lambda \; g_{r[\tau]r[\tau]} = \frac{\frac{r(\tau)\, \frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}}{a(t(\tau),r(\tau))} - b(t(\tau),r(\tau)) + 1}{r(\tau)^2} = \frac{8\pi G}{c^4}\, 0 = \frac{8\pi G}{c^4}\, T_{r[\tau]r[\tau]}$$

$$G_{\theta[\tau]\theta[\tau]} + \Lambda \; g_{\theta[\tau]\theta[\tau]} = \frac{1}{4\, a(t(\tau),r(\tau))^2\, b(t(\tau),r(\tau))^2}\, r(\tau) \left(a(t(\tau), r(\tau)) \left(2\, b(t(\tau), r(\tau)) \left(r(\tau) \left(\frac{\partial^2 a(t(\tau),r(\tau))}{\partial r(\tau)^2} - \frac{\partial^2 b(t(\tau),r(\tau))}{\partial t(\tau)^2} \right) + \frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)} \right) + r(\tau) \left(\left(\frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)} \right)^2 - \frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)} \frac{\partial b(t(\tau),r(\tau))}{\partial r(\tau)} \right) \right) - 2\, a(t(\tau), r(\tau))^2 \frac{\partial b(t(\tau),r(\tau))}{\partial r(\tau)} + r(\tau)\, b(t(\tau), r(\tau)) \left(\frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)} \frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)} - \left(\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)} \right)^2 \right) \right) = \frac{8\pi G}{c^4}\, 0 = \frac{8\pi G}{c^4}\, T_{\theta[\tau]\theta[\tau]}$$

$$G_{\phi[\tau]\phi[\tau]} + \Lambda \; g_{\phi[\tau]\phi[\tau]} = \frac{1}{4\, a(t(\tau),r(\tau))^2\, b(t(\tau),r(\tau))^2}\, r(\tau) \sin^2(\theta(\tau)) \left(a(t(\tau), r(\tau)) \left(2\, b(t(\tau), r(\tau)) \left(r(\tau) \left(\frac{\partial^2 a(t(\tau),r(\tau))}{\partial r(\tau)^2} - \frac{\partial^2 b(t(\tau),r(\tau))}{\partial t(\tau)^2} \right) + \frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)} \right) + r(\tau) \left(\left(\frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)} \right)^2 - \frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)} \frac{\partial b(t(\tau),r(\tau))}{\partial r(\tau)} \right) \right) - 2\, a(t(\tau), r(\tau))^2 \frac{\partial b(t(\tau),r(\tau))}{\partial r(\tau)} + r(\tau)\, b(t(\tau), r(\tau)) \left(\frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)} \frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)} - \left(\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)} \right)^2 \right) \right) = \frac{8\pi G}{c^4}\, 0 = \frac{8\pi G}{c^4}\, T_{\phi[\tau]\phi[\tau]}$$

Out[52]=

Geo^{λ} :

$$\mathrm{Geo}^{\,t[\tau]} \equiv \frac{\left(\frac{\partial r(\tau)(\tau)}{\partial\tau}+\frac{\partial r(\tau)}{\partial\tau}(\tau)\right)^2 \frac{\partial b(t(\tau)(\tau),r(\tau)(\tau))}{\partial t(\tau)(\tau)}}{2\,a(t(\tau)(\tau),r(\tau)(\tau))} + \frac{\left(\frac{\partial r(\tau)(\tau)}{\partial\tau}+\frac{\partial r(\tau)}{\partial\tau}(\tau)\right)\left(\frac{\partial t(\tau)(\tau)}{\partial\tau}+\frac{\partial t(\tau)}{\partial\tau}(\tau)\right) \frac{\partial a(t(\tau)(\tau),r(\tau)(\tau))}{\partial r(\tau)(\tau)}}{a(t(\tau)(\tau),r(\tau)(\tau))} + \frac{\left(\frac{\partial t(\tau)(\tau)}{\partial\tau}+\frac{\partial t(\tau)}{\partial\tau}(\tau)\right)^2 \frac{\partial a(t(\tau)(\tau),r(\tau)(\tau))}{\partial t(\tau)(\tau)}}{2\,a(t(\tau)(\tau),r(\tau)(\tau))} + 2\,\frac{\partial t'(\tau)(\tau)}{\partial\tau} + \frac{\partial^2 t(\tau)(\tau)}{\partial\tau^2} + \frac{\partial^2 t(\tau)}{\partial\tau^2}(\tau) = 0$$

$$\mathrm{Geo}^{\,r[\tau]} \equiv \frac{1}{2\,b(t(\tau)(\tau),r(\tau)(\tau))} \Bigg(\left(\frac{\partial t(\tau)(\tau)}{\partial\tau}+\frac{\partial t(\tau)}{\partial\tau}(\tau)\right)^2 \frac{\partial a(t(\tau)(\tau),r(\tau)(\tau))}{\partial r(\tau)(\tau)} + 4\,\frac{\partial r'(\tau)(\tau)}{\partial\tau}\,b(t(\tau)(\tau),r(\tau)(\tau)) + 2\,\frac{\partial^2 r(\tau)(\tau)}{\partial\tau^2}\,b(t(\tau)(\tau),r(\tau)(\tau)) + 2\,\frac{\partial^2 r(\tau)}{\partial\tau^2}(\tau)\,b(t(\tau)(\tau),r(\tau)(\tau)) + \left(\frac{\partial r(\tau)(\tau)}{\partial\tau}+\frac{\partial r(\tau)}{\partial\tau}(\tau)\right)^2 \frac{\partial b(t(\tau)(\tau),r(\tau)(\tau))}{\partial r(\tau)(\tau)} + 2\left(\frac{\partial r(\tau)(\tau)}{\partial\tau}+\frac{\partial r(\tau)}{\partial\tau}(\tau)\right)\left(\frac{\partial t(\tau)(\tau)}{\partial\tau}+\frac{\partial t(\tau)}{\partial\tau}(\tau)\right)\frac{\partial b(t(\tau)(\tau),r(\tau)(\tau))}{\partial t(\tau)(\tau)} - 2\,r(\tau)(\tau)\,\sin^2(\Theta(\tau)(\tau))\left(\frac{\partial\phi(\tau)(\tau)}{\partial\tau}+\frac{\partial\phi(\tau)}{\partial\tau}(\tau)\right)^2 - 2\,r(\tau)(\tau)\left(\frac{\partial\Theta(\tau)(\tau)}{\partial\tau}+\frac{\partial\Theta(\tau)}{\partial\tau}(\tau)\right)^2\Bigg) = 0$$

$$\mathrm{Geo}^{\,\Theta[\tau]} \equiv 2\,\frac{\partial\Theta'(\tau)(\tau)}{\partial\tau} + \frac{\partial^2\Theta(\tau)(\tau)}{\partial\tau^2} + \frac{\partial^2\Theta(\tau)}{\partial\tau^2}(\tau) + \frac{2\left(\frac{\partial\Theta(\tau)(\tau)}{\partial\tau}+\frac{\partial\Theta(\tau)}{\partial\tau}(\tau)\right)\left(\frac{\partial r(\tau)(\tau)}{\partial\tau}+\frac{\partial r(\tau)}{\partial\tau}(\tau)\right)}{r(\tau)(\tau)} = \frac{1}{2}\sin(2\,\Theta(\tau)(\tau))\left(\frac{\partial\phi(\tau)(\tau)}{\partial\tau}+\frac{\partial\phi(\tau)}{\partial\tau}(\tau)\right)^2$$

$$\mathrm{Geo}^{\,\phi[\tau]} \equiv 2\left(\frac{\partial\Theta(\tau)(\tau)}{\partial\tau}+\frac{\partial\Theta(\tau)}{\partial\tau}(\tau)\right)\cot(\Theta(\tau)(\tau))\left(\frac{\partial\phi(\tau)(\tau)}{\partial\tau}+\frac{\partial\phi(\tau)}{\partial\tau}(\tau)\right) + \frac{2\left(\frac{\partial r(\tau)(\tau)}{\partial\tau}+\frac{\partial r(\tau)}{\partial\tau}(\tau)\right)\left(\frac{\partial\phi(\tau)(\tau)}{\partial\tau}+\frac{\partial\phi(\tau)}{\partial\tau}(\tau)\right)}{r(\tau)(\tau)} + \frac{\partial^2\phi(\tau)(\tau)}{\partial\tau^2} + \frac{\partial^2\phi(\tau)}{\partial\tau^2}(\tau) + 2\,\frac{\partial\phi'(\tau)(\tau)}{\partial\tau} = 0$$

Out[56]=

Euler^{λ} :

$$\mathrm{Euler}^{\,t[\tau]} \equiv 2\,\frac{\partial^2 t(\tau)}{\partial\tau^2}\,a(t(\tau),r(\tau)) + 2\,\frac{\partial r(\tau)}{\partial\tau}\,\frac{\partial t(\tau)}{\partial\tau}\,\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)} + \left(\frac{\partial t(\tau)}{\partial\tau}\right)^2 \frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)} + \left(\frac{\partial r(\tau)}{\partial\tau}\right)^2 \frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)} = 0$$

$$\mathrm{Euler}^{\,r[\tau]} \equiv -\left(\frac{\partial t(\tau)}{\partial\tau}\right)^2 \frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)} - 2\,\frac{\partial^2 r(\tau)}{\partial\tau^2}\,b(t(\tau),r(\tau)) + \left(\frac{\partial r(\tau)}{\partial\tau}\right)^2\left(-\frac{\partial b(t(\tau),r(\tau))}{\partial r(\tau)}\right) - 2\,\frac{\partial r(\tau)}{\partial\tau}\,\frac{\partial t(\tau)}{\partial\tau}\,\frac{\partial b(t(\tau),r(\tau))}{\partial t(\tau)} + 2\,r(\tau)\left(\sin^2(\Theta(\tau))\left(\frac{\partial\phi(\tau)}{\partial\tau}\right)^2 + \left(\frac{\partial\Theta(\tau)}{\partial\tau}\right)^2\right) = 0$$

$$\mathrm{Euler}^{\,\Theta[\tau]} \equiv r(\tau)\left(r(\tau)\left(\sin(2\,\Theta(\tau))\left(\frac{\partial\phi(\tau)}{\partial\tau}\right)^2 - 2\,\frac{\partial^2\Theta(\tau)}{\partial\tau^2}\right) - 4\,\frac{\partial\Theta(\tau)}{\partial\tau}\,\frac{\partial r(\tau)}{\partial\tau}\right) = 0$$

$$\mathrm{Euler}^{\,\phi[\tau]} \equiv -2\,r(\tau)\,\sin(\Theta(\tau))\left(r(\tau)\left(\sin(\Theta(\tau))\,\frac{\partial^2\phi(\tau)}{\partial\tau^2} + 2\,\frac{\partial\Theta(\tau)}{\partial\tau}\,\cos(\Theta(\tau))\,\frac{\partial\phi(\tau)}{\partial\tau}\right) + 2\,\sin(\Theta(\tau))\,\frac{\partial r(\tau)}{\partial\tau}\,\frac{\partial\phi(\tau)}{\partial\tau}\right) = 0$$

In[1]:=
```
Clear["Global`*"];
(*Version 0.2.1*)
```

Einstein Field Equations

Input Section

In[2]:=
```
Coordinates = {t[τ], r[τ], θ[τ], ϕ[τ]};
co[μ_] := Coordinates[[μ]]; (*To simplify the Code*)
Dim = Length[Coordinates];(*Dimention of Space-Time*)
PoM = τ; (*Parameter of Motion*)
(* ω[r]=1-2GM/r;
v[r]=1/ω[r]; *)
Metric(*g_μν*) = {{-a[t[τ], r[τ]], 0, 0, 0},
                  {0, (a[t[τ], r[τ]])^-1, 0, 0},
                  {0, 0, r[τ]^2, 0},
                  {0, 0, 0, r[τ]^2 Sin[θ[τ]]^2}};
SETensor(*T_μν*) = {{0, 0, 0, 0},
                    {0, 0, 0, 0},
                    {0, 0, 0, 0},
                    {0, 0, 0, 0}};
MetricInverse = Simplify[Inverse[Metric]];
pdConv[f_] := TraditionalForm[
  f /. Derivative[inds__][g_][vars__] :>
    Apply[Defer[D[g[vars], ##]] &,
     Transpose[{{vars}, {inds}}] /.
      {{var_, 0} :> Sequence[], {var_, 1} :> {var}}]]
DCoordinates = {t'[τ], r'[τ], θ'[τ], ϕ'[τ]};
```

Christoffel symbols (2_{nd} kind) ($\Gamma^{\beta}{}_{\mu\nu}$)

In[11]:=
```
Christoffel[β_, μ_, ν_] :=
 Christoffel[β, μ, ν] = Simplify[1/2 Sum[MetricInverse[[β, α]] (D[Metric[[α, μ]], co[ν]] +
      D[Metric[[α, ν]], co[μ]] - D[Metric[[μ, ν]], co[α]]), {α, Dim}]]
```

Riemann curvature tensor ($R^{\beta}{}_{\sigma\mu\nu}$) & Ricci tensor ($R_{\mu\nu}$) & Scalar curvature (R)

In[12]:=
```
Riemann[β_, σ_, μ_, ν_] :=
  Riemann[β, σ, μ, ν] = Simplify[D[Christoffel[β, ν, σ], co[μ]] -
     D[Christoffel[β, μ, σ], co[ν]] + Sum[Christoffel[β, μ, λ] × Christoffel[λ, ν, σ] -
       Christoffel[β, ν, λ] × Christoffel[λ, μ, σ], {λ, Dim}]];
Ricci[μ_, ν_] := Ricci[μ, ν] = Simplify[Sum[Riemann[λ, μ, λ, ν], {λ, Dim}]];
RicciScalar = Simplify[
   Sum[Ricci[μ, ν] × MetricInverse⟦μ, ν⟧, {μ, Length[Coordinates]}, {ν, Dim}]];
```

Einstein Tensor

In[15]:=
```
EFE[μ_, ν_] :=
  EFE[μ, ν] = Simplify[Ricci[μ, ν] - 1/2 RicciScalar Metric⟦μ, ν⟧] + Λ Metric⟦μ, ν⟧ //
    FullSimplify;
Λ = 0;
```

Other Useful Quantities ($R_{\beta\sigma\mu\nu}$, $R^{\beta\sigma\mu\nu}$, Kretschmann scalar)

In[17]:=
```
Riemanndddd[β_, σ_, μ_, ν_] (*R_βσμν*) := Riemanndddd[β, σ, μ, ν] =
   Sum[Riemann[β1, σ, μ, ν] × Metric⟦β1, β⟧, {β1, Dim}] // Simplify;
Riemannuuuu[β_, σ_, μ_, ν_] (*R^βσμν*) := Riemannuuuu[β, σ, μ, ν] =
   Sum[Riemann[β, σ1, μ1, ν1] × MetricInverse⟦σ1, σ⟧ × MetricInverse⟦μ1, μ⟧ ×
      MetricInverse⟦ν1, ν⟧, {σ1, Dim}, {μ1, Dim}, {ν1, Dim}] // Simplify;
Kretschmann = Sum[Riemanndddd[β, σ, μ, ν] × Riemannuuuu[β, σ, μ, ν],
    {β, Dim}, {σ, Dim}, {μ, Dim}, {ν, Dim}] // Simplify;
```

Geodesic Equations

In[20]:=
```
Geo = Array[geo, Dim];
For[λ = 1, λ ≤ Dim, λ++,
 Geo⟦λ⟧ = Simplify[D[co[λ][PoM], {PoM, 2}] +
     Sum[(Christoffel[λ, μ, ν] /. (# → #[PoM] & /@ Coordinates))
       D[co[μ][PoM], PoM] × D[co[ν][PoM], PoM], {μ, Dim}, {ν, Dim}] == 0]
]
```

Euler Lagrange Equations

In[22]:=
```
Needs["VariationalMethods`"];
Euler =
 EulerEquations[DCoordinates.Metric.DCoordinatesᵀ, {t[τ], r[τ], θ[τ], ϕ[τ]}, τ]
```

Out[23]=

$$\left\{2\,a[t[\tau],r[\tau]]\,t''[\tau]-\frac{r'[\tau]^2\,a^{(1,0)}[t[\tau],r[\tau]]}{a[t[\tau],r[\tau]]^2}+t'[\tau]\left(2\,r'[\tau]\,a^{(0,1)}[t[\tau],r[\tau]]+t'[\tau]\,a^{(1,0)}[t[\tau],r[\tau]]\right)==0,\right.$$
$$2\,r[\tau]\left(\theta'[\tau]^2+\mathrm{Sin}[\theta[\tau]]^2\,\phi'[\tau]^2\right)-\frac{2\,r''[\tau]}{a[t[\tau],r[\tau]]}-t'[\tau]^2\,a^{(0,1)}[t[\tau],r[\tau]]+\frac{r'[\tau]\left(r'[\tau]\,a^{(0,1)}[t[\tau],r[\tau]]+2\,t'[\tau]\,a^{(1,0)}[t[\tau],r[\tau]]\right)}{a[t[\tau],r[\tau]]^2}==0,$$
$$r[\tau]\left(-4\,r'[\tau]\,\theta'[\tau]+r[\tau]\left(\mathrm{Sin}[2\,\theta[\tau]]\,\phi'[\tau]^2-2\,\theta''[\tau]\right)\right)==0,\ -2\,r[\tau]\,\mathrm{Sin}[\theta[\tau]]$$
$$\left.(2\,\mathrm{Sin}[\theta[\tau]]\,r'[\tau]\,\phi'[\tau]+r[\tau]\,(2\,\mathrm{Cos}[\theta[\tau]]\,\theta'[\tau]\,\phi'[\tau]+\mathrm{Sin}[\theta[\tau]]\,\phi''[\tau]))==0\right\}$$

Results

In[24]:=
```
Grid[{{"Metric Tensor (g_μν) = ", MatrixForm[Metric]}},
 Frame → True, FrameStyle → Blue]
Grid[{{"Inverse Metric Tensor (g^μν) = ", MatrixForm[MetricInverse]}},
 Frame → True, FrameStyle → Blue]
Grid[{{"Stress-Energy Tensor (T_μν) = ", MatrixForm[SETensor]}},
 Frame → True, FrameStyle → Blue]
ChristoffelList = {{"Γ^β_μν", ":", "Christoffel symbols ( 2_nd kind )"}};
ChristoffelPrintAction := AppendTo[ChristoffelList,
   {Subsuperscript["Γ", " " <> ToString[co[μ]] <> ToString[co[ν]], ToString[co[β]]],
    " = ", Christoffel[β, μ, ν] // pdConv}];
For[β = 1, β ≤ Dim, β++,
 For[μ = 1, μ ≤ Dim, μ++,
  For[ν = 1, ν ≤ μ, ν++,
   If[Christoffel[β, μ, ν] ≠ 0,
    ChristoffelPrintAction, , ChristoffelPrintAction]
  ]
 ]
]
If[Length@ChristoffelList == 1,
  ChristoffelList[[1]] = {"Γ^β_μν", "=", "Christoffel symbols ( 2_nd kind ) = 0"}];
Grid[ChristoffelList, Frame → True, FrameStyle → Blue]
RiemannList = {{"R^β_σμν", ":", "Riemann curvature tensor"}};
RiemannPrintAction := AppendTo[RiemannList,
   {Subsuperscript["R", " " <> ToString[co[σ]] <> ToString[co[μ]] <> ToString[co[ν]],
     ToString[co[β]]], " = ", Riemann[β, σ, μ, ν] // pdConv}];
```

```
For[β = 1, β ≤ Dim, β++,
 For[σ = 1, σ ≤ Dim, σ++,
  For[μ = 1, μ ≤ Dim, μ++,
   For[ν = 1, ν ≤ Dim, ν++,
    If[Riemann[β, σ, μ, ν] ≠ 0,
     RiemannPrintAction, , RiemannPrintAction]
   ]
  ]
 ]
]
If[Length@RiemannList == 1,
  RiemannList[[1]] = {"R^β_σμν", "=", "Riemann curvature tensor = 0"}];
Grid[RiemannList, Frame → True, FrameStyle → Blue]
RicciList = {{"R_μν", ":", "Ricci tensor"}};
RicciPrintAction :=
  AppendTo[RicciList, {Subscript["R", ToString[co[μ]] <> ToString[co[ν]]],
    " = ", Ricci[μ, ν] // pdConv}];
For[μ = 1, μ ≤ Dim, μ++,
 For[ν = 1, ν ≤ μ, ν++,
  If[Ricci[μ, ν] ≠ 0,
   RicciPrintAction, , RicciPrintAction]
 ]
]
If[Length@RicciList == 1, RicciList[[1]] = {"R_μν", "=", "Ricci tensor = 0"}];
Grid[RicciList, Frame → True, FrameStyle → Blue]
Grid[{{"Scalar curvature (R) = ", RicciScalar // pdConv}},
 Frame → True, FrameStyle → Blue]
Grid[{{"Kretschmann (K) = ", Kretschmann // pdConv}},
 Frame → True, FrameStyle → Blue]
EFEList = {{"G_μν", "+ Λ", "g_μν", "=", "Einstein Field Equations", "= (8 π G)/c^4 T_μν"}};
EFEPrintAction :=
  AppendTo[EFEList, {Subscript["G", ToString[co[μ]] <> ToString[co[ν]]],
    "+ Λ ", Subscript["g", ToString[co[μ]] <> ToString[co[ν]]], " = ",
    {{EFE[μ, ν] // pdConv, " = (8 π G)/c^4 ", SETensor[[μ, ν]] // pdConv}} // Grid,
    Subscript[" = (8 π G)/c^4 T", ToString[co[μ]] <> ToString[co[ν]]]}];
For[μ = 1, μ ≤ Dim, μ++,
 For[ν = 1, ν ≤ μ, ν++,
  If[EFE[μ, ν] ≠ 0,
   EFEPrintAction, , EFEPrintAction]
 ]
]
If[Length@EFEList == 1, EFEList[[1]] = {"G_μν", "+ Λ", "g_μν", "=", 0, "= (8 π G)/c^4 T_μν"}];
Grid[EFEList, Frame → True, FrameStyle → Blue]

GeoList = {{"Geo^λ", ":", ""}};
```

```
GeoPrintAction := AppendTo[GeoList,
    {Superscript["Geo", " " <> ToString[co[λ]]], " ≡ ", Geo⟦λ⟧ // pdConv}];
For[λ = 1, λ ≤ Dim, λ++,
 If[Geo⟦λ⟧ ≠ 0,
  GeoPrintAction, , GeoPrintAction]
]
Grid[GeoList, Frame → True, FrameStyle → Blue]

EulerList = {{"Euler^λ", ":", ""}};
EulerPrintAction := AppendTo[EulerList,
    {Superscript["Euler", " " <> ToString[co[λ]]], " ≡ ", Euler⟦λ⟧ // pdConv}];
For[λ = 1, λ ≤ Dim, λ++,
 If[Euler⟦λ⟧ ≠ 0,
  EulerPrintAction, , EulerPrintAction]
]
Grid[EulerList, Frame → True, FrameStyle → Blue]
```

Out[24]=

Metric Tensor $(g_{\mu\nu})$ = $\begin{pmatrix} -a[t[\tau], r[\tau]] & 0 & 0 & 0 \\ 0 & \frac{1}{a[t[\tau],r[\tau]]} & 0 & 0 \\ 0 & 0 & r[\tau]^2 & 0 \\ 0 & 0 & 0 & r[\tau]^2 \mathrm{Sin}[\theta[\tau]]^2 \end{pmatrix}$

Out[25]=

Inverse Metric Tensor $(g^{\mu\nu})$ = $\begin{pmatrix} -\frac{1}{a[t[\tau],r[\tau]]} & 0 & 0 & 0 \\ 0 & a[t[\tau], r[\tau]] & 0 & 0 \\ 0 & 0 & \frac{1}{r[\tau]^2} & 0 \\ 0 & 0 & 0 & \frac{\mathrm{Csc}[\theta[\tau]]^2}{r[\tau]^2} \end{pmatrix}$

Out[26]=

Stress-Energy Tensor $(T_{\mu\nu})$ = $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Out[31]=

$$\Gamma^{\beta}{}_{\mu\nu} \quad : \quad \text{Christoffel symbols } (2_{\text{nd}} \text{ kind})$$

$$\Gamma^{t[\tau]}{}_{t[\tau]t[\tau]} = \frac{\frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}}{2\,a(t(\tau),r(\tau))}$$

$$\Gamma^{t[\tau]}{}_{r[\tau]t[\tau]} = \frac{\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}}{2\,a(t(\tau),r(\tau))}$$

$$\Gamma^{t[\tau]}{}_{r[\tau]r[\tau]} = -\frac{\frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}}{2\,a(t(\tau),r(\tau))^3}$$

$$\Gamma^{r[\tau]}{}_{t[\tau]t[\tau]} = \frac{1}{2}\,a(t(\tau),r(\tau))\,\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}$$

$$\Gamma^{r[\tau]}{}_{r[\tau]t[\tau]} = -\frac{\frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}}{2\,a(t(\tau),r(\tau))}$$

$$\Gamma^{r[\tau]}{}_{r[\tau]r[\tau]} = -\frac{\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}}{2\,a(t(\tau),r(\tau))}$$

$$\Gamma^{r[\tau]}{}_{\Theta[\tau]\Theta[\tau]} = r(\tau)\,(-a(t(\tau),r(\tau)))$$

$$\Gamma^{r[\tau]}{}_{\phi[\tau]\phi[\tau]} = r(\tau)\,\sin^2(\Theta(\tau))\,(-a(t(\tau),r(\tau)))$$

$$\Gamma^{\Theta[\tau]}{}_{\Theta[\tau]r[\tau]} = \frac{1}{r(\tau)}$$

$$\Gamma^{\Theta[\tau]}{}_{\phi[\tau]\phi[\tau]} = \sin(\Theta(\tau))\,(-\cos(\Theta(\tau)))$$

$$\Gamma^{\phi[\tau]}{}_{\phi[\tau]r[\tau]} = \frac{1}{r(\tau)}$$

$$\Gamma^{\phi[\tau]}{}_{\phi[\tau]\Theta[\tau]} = \cot(\Theta(\tau))$$

Out[36]=

$$R^{\beta}{}_{\sigma\mu\nu} \quad : \quad \text{Riemann curvature tensor}$$

$$R^{t[\tau]}{}_{r[\tau]t[\tau]r[\tau]} = -\frac{a(t(\tau),r(\tau))^3\,\frac{\partial^2 a(t(\tau),r(\tau))}{\partial r(\tau)^2} + a(t(\tau),r(\tau))\,\frac{\partial^2 a(t(\tau),r(\tau))}{\partial t(\tau)^2} - 2\left(\frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}\right)^2}{2\,a(t(\tau),r(\tau))^4}$$

$$R^{t[\tau]}{}_{r[\tau]r[\tau]t[\tau]} = \frac{a(t(\tau),r(\tau))^3\,\frac{\partial^2 a(t(\tau),r(\tau))}{\partial r(\tau)^2} + a(t(\tau),r(\tau))\,\frac{\partial^2 a(t(\tau),r(\tau))}{\partial t(\tau)^2} - 2\left(\frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}\right)^2}{2\,a(t(\tau),r(\tau))^4}$$

$$R^{t[\tau]}{}_{\Theta[\tau]t[\tau]\Theta[\tau]} = -\frac{1}{2}\,r(\tau)\,\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}$$

$$R^{t[\tau]}{}_{\Theta[\tau]r[\tau]\Theta[\tau]} = \frac{r(\tau)\,\frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}}{2\,a(t(\tau),r(\tau))^2}$$

$$R^{t[\tau]}{}_{\Theta[\tau]\Theta[\tau]t[\tau]} = \frac{1}{2}\,r(\tau)\,\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}$$

$$R^{t[\tau]}{}_{\Theta[\tau]\Theta[\tau]r[\tau]} = -\frac{r(\tau)\,\frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}}{2\,a(t(\tau),r(\tau))^2}$$

$$R^{t[\tau]}{}_{\phi[\tau]t[\tau]\phi[\tau]} = -\frac{1}{2}\,r(\tau)\,\sin^2(\Theta(\tau))\,\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}$$

$$R^{t[\tau]}{}_{\phi[\tau]r[\tau]\phi[\tau]} = \frac{r(\tau)\,\sin^2(\Theta(\tau))\,\frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}}{2\,a(t(\tau),r(\tau))^2}$$

$$R^{t[\tau]}{}_{\phi[\tau]\phi[\tau]t[\tau]} = \frac{1}{2}\,r(\tau)\,\sin^2(\Theta(\tau))\,\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}$$

$$R^{t[\tau]}{}_{\phi[\tau]\phi[\tau]r[\tau]} = -\frac{r(\tau)\,\sin^2(\Theta(\tau))\,\frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}}{2\,a(t(\tau),r(\tau))^2}$$

$$R^{r[\tau]}{}_{t[\tau]t[\tau]r[\tau]} = -\frac{a(t(\tau),r(\tau))^3\,\frac{\partial^2 a(t(\tau),r(\tau))}{\partial r(\tau)^2} + a(t(\tau),r(\tau))\,\frac{\partial^2 a(t(\tau),r(\tau))}{\partial t(\tau)^2} - 2\left(\frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}\right)^2}{2\,a(t(\tau),r(\tau))^2}$$

$$R^{r[\tau]}{}_{t[\tau]r[\tau]t[\tau]} = \frac{a(t(\tau),r(\tau))^3\,\frac{\partial^2 a(t(\tau),r(\tau))}{\partial r(\tau)^2} + a(t(\tau),r(\tau))\,\frac{\partial^2 a(t(\tau),r(\tau))}{\partial t(\tau)^2} - 2\left(\frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}\right)^2}{2\,a(t(\tau),r(\tau))^2}$$

$$R^{r[\tau]}{}_{\Theta[\tau]t[\tau]\Theta[\tau]} = -\frac{1}{2}\,r(\tau)\,\frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}$$

$$R^{r[\tau]}{}_{\Theta[\tau]r[\tau]\Theta[\tau]} = -\frac{1}{2}\,r(\tau)\,\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}$$

$$R^{r[\tau]}{}_{\Theta[\tau]\Theta[\tau]t[\tau]} = \frac{1}{2}\,r(\tau)\,\frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}$$

$$R^{r[\tau]}{}_{\Theta[\tau]\Theta[\tau]r[\tau]} = \frac{1}{2}\,r(\tau)\,\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}$$

$$R^{r[\tau]}{}_{\phi[\tau]t[\tau]\phi[\tau]} = -\frac{1}{2}\,r(\tau)\,\sin^2(\Theta(\tau))\,\frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}$$

$$R^{r[\tau]}{}_{\phi[\tau]r[\tau]\phi[\tau]} = -\frac{1}{2}\, r(\tau)\, \sin^2(\Theta(\tau))\, \frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}$$

$$R^{r[\tau]}{}_{\phi[\tau]\phi[\tau]t[\tau]} = \frac{1}{2}\, r(\tau)\, \sin^2(\Theta(\tau))\, \frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}$$

$$R^{r[\tau]}{}_{\phi[\tau]\phi[\tau]r[\tau]} = \frac{1}{2}\, r(\tau)\, \sin^2(\Theta(\tau))\, \frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}$$

$$R^{\Theta[\tau]}{}_{t[\tau]t[\tau]\Theta[\tau]} = -\frac{a(t(\tau),r(\tau))\, \frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}}{2\, r(\tau)}$$

$$R^{\Theta[\tau]}{}_{t[\tau]r[\tau]\Theta[\tau]} = \frac{\frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}}{2\, r(\tau)\, a(t(\tau),r(\tau))}$$

$$R^{\Theta[\tau]}{}_{t[\tau]\Theta[\tau]t[\tau]} = \frac{a(t(\tau),r(\tau))\, \frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}}{2\, r(\tau)}$$

$$R^{\Theta[\tau]}{}_{t[\tau]\Theta[\tau]r[\tau]} = -\frac{\frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}}{2\, r(\tau)\, a(t(\tau),r(\tau))}$$

$$R^{\Theta[\tau]}{}_{r[\tau]t[\tau]\Theta[\tau]} = \frac{\frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}}{2\, r(\tau)\, a(t(\tau),r(\tau))}$$

$$R^{\Theta[\tau]}{}_{r[\tau]r[\tau]\Theta[\tau]} = \frac{\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}}{2\, r(\tau)\, a(t(\tau),r(\tau))}$$

$$R^{\Theta[\tau]}{}_{r[\tau]\Theta[\tau]t[\tau]} = -\frac{\frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}}{2\, r(\tau)\, a(t(\tau),r(\tau))}$$

$$R^{\Theta[\tau]}{}_{r[\tau]\Theta[\tau]r[\tau]} = -\frac{\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}}{2\, r(\tau)\, a(t(\tau),r(\tau))}$$

$$R^{\Theta[\tau]}{}_{\phi[\tau]\Theta[\tau]\phi[\tau]} = -\left(\sin^2(\Theta(\tau))\, (a(t(\tau),\, r(\tau)) - 1)\right)$$

$$R^{\Theta[\tau]}{}_{\phi[\tau]\phi[\tau]\Theta[\tau]} = \sin^2(\Theta(\tau))\, (a(t(\tau),\, r(\tau)) - 1)$$

$$R^{\phi[\tau]}{}_{t[\tau]t[\tau]\phi[\tau]} = -\frac{a(t(\tau),r(\tau))\, \frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}}{2\, r(\tau)}$$

$$R^{\phi[\tau]}{}_{t[\tau]r[\tau]\phi[\tau]} = \frac{\frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}}{2\, r(\tau)\, a(t(\tau),r(\tau))}$$

$$R^{\phi[\tau]}{}_{t[\tau]\phi[\tau]t[\tau]} = \frac{a(t(\tau),r(\tau))\, \frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}}{2\, r(\tau)}$$

$$R^{\phi[\tau]}{}_{t[\tau]\phi[\tau]r[\tau]} = -\frac{\frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}}{2\, r(\tau)\, a(t(\tau),r(\tau))}$$

$$R^{\phi[\tau]}{}_{r[\tau]t[\tau]\phi[\tau]} = \frac{\frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}}{2\, r(\tau)\, a(t(\tau),r(\tau))}$$

$$R^{\phi[\tau]}{}_{r[\tau]r[\tau]\phi[\tau]} = \frac{\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}}{2\, r(\tau)\, a(t(\tau),r(\tau))}$$

$$R^{\phi[\tau]}{}_{r[\tau]\phi[\tau]t[\tau]} = -\frac{\frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}}{2\, r(\tau)\, a(t(\tau),r(\tau))}$$

$$R^{\phi[\tau]}{}_{r[\tau]\phi[\tau]r[\tau]} = -\frac{\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}}{2\, r(\tau)\, a(t(\tau),r(\tau))}$$

$$R^{\phi[\tau]}{}_{\Theta[\tau]\Theta[\tau]\phi[\tau]} = a(t(\tau),\, r(\tau)) - 1$$

$$R^{\phi[\tau]}{}_{\Theta[\tau]\phi[\tau]\Theta[\tau]} = 1 - a(t(\tau),\, r(\tau))$$

Out[41]=

$R_{\mu\nu}$: Ricci tensor

$$R_{t[\tau]t[\tau]} = \frac{\frac{\partial^2 a(t(\tau),r(\tau))}{\partial t(\tau)^2}}{2\, a(t(\tau),r(\tau))} + a(t(\tau),\, r(\tau)) \left(\frac{1}{2}\, \frac{\partial^2 a(t(\tau),r(\tau))}{\partial r(\tau)^2} + \frac{\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}}{r(\tau)}\right) - \frac{\left(\frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}\right)^2}{a(t(\tau),r(\tau))^2}$$

$$R_{r[\tau]t[\tau]} = -\frac{\frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}}{r(\tau)\, a(t(\tau),r(\tau))}$$

$$R_{r[\tau]r[\tau]} = \frac{a(t(\tau),r(\tau))^3 \left(-\frac{\partial^2 a(t(\tau),r(\tau))}{\partial r(\tau)^2} - \frac{2\, \frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}}{r(\tau)}\right) - a(t(\tau),r(\tau))\, \frac{\partial^2 a(t(\tau),r(\tau))}{\partial t(\tau)^2} + 2 \left(\frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}\right)^2}{2\, a(t(\tau),r(\tau))^4}$$

$$R_{\Theta[\tau]\Theta[\tau]} = -a(t(\tau),\, r(\tau)) - r(\tau)\, \frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)} + 1$$

$$R_{\phi[\tau]\phi[\tau]} = -\left(\sin^2(\Theta(\tau)) \left(a(t(\tau),\, r(\tau)) + r(\tau)\, \frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)} - 1\right)\right)$$

Out[42]=

$$\text{Scalar curvature } (R) = \frac{2-2\,a(t(\tau),r(\tau))}{r(\tau)^2} - \frac{\partial^2 a(t(\tau),r(\tau))}{\partial r(\tau)^2} + \frac{2\left(\frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}\right)^2 - a(t(\tau),r(\tau))\,\frac{\partial^2 a(t(\tau),r(\tau))}{\partial t(\tau)^2}}{a(t(\tau),r(\tau))^3} - \frac{4\,\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}}{r(\tau)}$$

Out[43]=

$$\text{Kretschmann } (K) = \frac{1}{r(\tau)^4\,a(t(\tau),r(\tau))^6}\Bigg(4\,(a(t(\tau),r(\tau))-1)^2\,a(t(\tau),r(\tau))^6 + r(\tau)^4\left(a(t(\tau),r(\tau))^3\,\frac{\partial^2 a(t(\tau),r(\tau))}{\partial r(\tau)^2} + a(t(\tau),r(\tau))\,\frac{\partial^2 a(t(\tau),r(\tau))}{\partial t(\tau)^2} - 2\left(\frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}\right)^2\right)^2 + 4\,r(\tau)^2\,a(t(\tau),r(\tau))^6\left(\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}\right)^2 - 4\,r(\tau)^2\,a(t(\tau),r(\tau))^4\left(\frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}\right)^2\Bigg)$$

Out[48]=

$$G_{\mu\nu} + \Lambda\; g_{\mu\nu} = \text{Einstein Field Equations} = \frac{8\pi G}{c^4}\,T_{\mu\nu}$$

$$G_{t[\tau]t[\tau]} + \Lambda\; g_{t[\tau]t[\tau]} = -\frac{a(t(\tau),r(\tau))\left(a(t(\tau),r(\tau)) + r(\tau)\,\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)} - 1\right)}{r(\tau)^2} = \frac{8\pi G}{c^4}\;0 = \frac{8\pi G}{c^4}\,T_{t[\tau]t[\tau]}$$

$$G_{r[\tau]t[\tau]} + \Lambda\; g_{r[\tau]t[\tau]} = -\frac{\frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}}{r(\tau)\,a(t(\tau),r(\tau))} = \frac{8\pi G}{c^4}\;0 = \frac{8\pi G}{c^4}\,T_{r[\tau]t[\tau]}$$

$$G_{r[\tau]r[\tau]} + \Lambda\; g_{r[\tau]r[\tau]} = \frac{a(t(\tau),r(\tau)) + r(\tau)\,\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)} - 1}{r(\tau)^2\,a(t(\tau),r(\tau))} = \frac{8\pi G}{c^4}\;0 = \frac{8\pi G}{c^4}\,T_{r[\tau]r[\tau]}$$

$$G_{\theta[\tau]\theta[\tau]} + \Lambda\; g_{\theta[\tau]\theta[\tau]} = \frac{1}{2}\,r(\tau)\left(r(\tau)\left(\frac{\partial^2 a(t(\tau),r(\tau))}{\partial r(\tau)^2} + \left(a(t(\tau),r(\tau))\,\frac{\partial^2 a(t(\tau),r(\tau))}{\partial t(\tau)^2} - 2\left(\frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}\right)^2\right)\Big/ a(t(\tau),r(\tau))^3\right) + 2\,\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}\right) = \frac{8\pi G}{c^4}\;0 = \frac{8\pi G}{c^4}\,T_{\theta[\tau]\theta[\tau]}$$

$$G_{\phi[\tau]\phi[\tau]} + \Lambda\; g_{\phi[\tau]\phi[\tau]} = \frac{1}{2\,a(t(\tau),r(\tau))^3}\,r(\tau)\,\sin^2(\theta(\tau))\left(a(t(\tau),r(\tau))^3\left(r(\tau)\,\frac{\partial^2 a(t(\tau),r(\tau))}{\partial r(\tau)^2} + 2\,\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}\right) + r(\tau)\,a(t(\tau),r(\tau))\,\frac{\partial^2 a(t(\tau),r(\tau))}{\partial t(\tau)^2} - 2\,r(\tau)\left(\frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}\right)^2\right) = \frac{8\pi G}{c^4}\;0 = \frac{8\pi G}{c^4}\,T_{\phi[\tau]\phi[\tau]}$$

Out[52]=

Geo^{λ} :

$$\text{Geo}^{t[\tau]} \equiv \frac{\left(\frac{\partial t(\tau)(\tau)}{\partial\tau}+\frac{\partial t(\tau)}{\partial\tau}(\tau)\right)^2 \frac{\partial a(t(\tau)(\tau),r(\tau)(\tau))}{\partial t(\tau)(\tau)}}{2\,a(t(\tau)(\tau),r(\tau)(\tau))}+\frac{\left(\frac{\partial r(\tau)(\tau)}{\partial\tau}+\frac{\partial r(\tau)}{\partial\tau}(\tau)\right)\left(\frac{\partial t(\tau)(\tau)}{\partial\tau}+\frac{\partial t(\tau)}{\partial\tau}(\tau)\right)\frac{\partial a(t(\tau)(\tau),r(\tau)(\tau))}{\partial r(\tau)(\tau)}}{a(t(\tau)(\tau),r(\tau)(\tau))}+ 2\,\frac{\partial t'(\tau)(\tau)}{\partial\tau}+\frac{\partial^2 t(\tau)(\tau)}{\partial\tau^2}+\frac{\partial^2 t(\tau)}{\partial\tau^2}(\tau)=\frac{\left(\frac{\partial r(\tau)(\tau)}{\partial\tau}+\frac{\partial r(\tau)}{\partial\tau}(\tau)\right)^2\frac{\partial a(t(\tau)(\tau),r(\tau)(\tau))}{\partial t(\tau)(\tau)}}{2\,a(t(\tau)(\tau),r(\tau)(\tau))^3}$$

$$\text{Geo}^{r[\tau]} \equiv \frac{1}{2}\left(\frac{\partial t(\tau)(\tau)}{\partial\tau}+\frac{\partial t(\tau)}{\partial\tau}(\tau)\right)^2 a(t(\tau)(\tau),\, r(\tau)(\tau))\,\frac{\partial a(t(\tau)(\tau),r(\tau)(\tau))}{\partial r(\tau)(\tau)}+ 2\,\frac{\partial r'(\tau)(\tau)}{\partial\tau}+\frac{\partial^2 r(\tau)(\tau)}{\partial\tau^2}+\frac{\partial^2 r(\tau)}{\partial\tau^2}(\tau)=\frac{1}{2\,a(t(\tau)(\tau),r(\tau)(\tau))} \Bigg(2\,r(\tau)(\tau)\sin^2(\theta(\tau)(\tau))\left(\frac{\partial\phi(\tau)(\tau)}{\partial\tau}+\frac{\partial\phi(\tau)}{\partial\tau}(\tau)\right)^2 a(t(\tau)(\tau),\, r(\tau)(\tau))^2+ 2\,r(\tau)(\tau)\left(\frac{\partial\theta(\tau)(\tau)}{\partial\tau}+\frac{\partial\theta(\tau)}{\partial\tau}(\tau)\right)^2 a(t(\tau)(\tau),\, r(\tau)(\tau))^2+ \left(\frac{\partial r(\tau)(\tau)}{\partial\tau}+\frac{\partial r(\tau)}{\partial\tau}(\tau)\right)^2\frac{\partial a(t(\tau)(\tau),r(\tau)(\tau))}{\partial r(\tau)(\tau)}+ 2\left(\frac{\partial r(\tau)(\tau)}{\partial\tau}+\frac{\partial r(\tau)}{\partial\tau}(\tau)\right)\left(\frac{\partial t(\tau)(\tau)}{\partial\tau}+\frac{\partial t(\tau)}{\partial\tau}(\tau)\right)\frac{\partial a(t(\tau)(\tau),r(\tau)(\tau))}{\partial t(\tau)(\tau)}\Bigg)$$

$$\text{Geo}^{\theta[\tau]} \equiv 2\,\frac{\partial\theta'(\tau)(\tau)}{\partial\tau}+\frac{\partial^2\theta(\tau)(\tau)}{\partial\tau^2}+\frac{\partial^2\theta(\tau)}{\partial\tau^2}(\tau)+\frac{2\left(\frac{\partial\theta(\tau)(\tau)}{\partial\tau}+\frac{\partial\theta(\tau)}{\partial\tau}(\tau)\right)\left(\frac{\partial r(\tau)(\tau)}{\partial\tau}+\frac{\partial r(\tau)}{\partial\tau}(\tau)\right)}{r(\tau)(\tau)}= \frac{1}{2}\sin(2\,\theta(\tau)(\tau))\left(\frac{\partial\phi(\tau)(\tau)}{\partial\tau}+\frac{\partial\phi(\tau)}{\partial\tau}(\tau)\right)^2$$

$$\text{Geo}^{\phi[\tau]} \equiv 2\left(\frac{\partial\theta(\tau)(\tau)}{\partial\tau}+\frac{\partial\theta(\tau)}{\partial\tau}(\tau)\right)\cot(\theta(\tau)(\tau))\left(\frac{\partial\phi(\tau)(\tau)}{\partial\tau}+\frac{\partial\phi(\tau)}{\partial\tau}(\tau)\right)+ \frac{2\left(\frac{\partial r(\tau)(\tau)}{\partial\tau}+\frac{\partial r(\tau)}{\partial\tau}(\tau)\right)\left(\frac{\partial\phi(\tau)(\tau)}{\partial\tau}+\frac{\partial\phi(\tau)}{\partial\tau}(\tau)\right)}{r(\tau)(\tau)}+\frac{\partial^2\phi(\tau)(\tau)}{\partial\tau^2}+\frac{\partial^2\phi(\tau)}{\partial\tau^2}(\tau)+2\,\frac{\partial\phi'(\tau)(\tau)}{\partial\tau}=0$$

Out[56]=

Euler^{λ} :

$$\text{Euler}^{t[\tau]} \equiv 2\,\frac{\partial^2 t(\tau)}{\partial\tau^2}\,a(t(\tau),\, r(\tau))-\frac{\left(\frac{\partial r(\tau)}{\partial\tau}\right)^2\frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}}{a(t(\tau),r(\tau))^2}+ \frac{\partial t(\tau)}{\partial\tau}\left(2\,\frac{\partial r(\tau)}{\partial\tau}\,\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}+\frac{\partial t(\tau)}{\partial\tau}\,\frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}\right)=0$$

$$\text{Euler}^{r[\tau]} \equiv -\frac{2\,\frac{\partial^2 r(\tau)}{\partial\tau^2}}{a(t(\tau),r(\tau))}+\left(\frac{\partial t(\tau)}{\partial\tau}\right)^2\left(-\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}\right)+\frac{\frac{\partial r(\tau)}{\partial\tau}\left(\frac{\partial r(\tau)}{\partial\tau}\,\frac{\partial a(t(\tau),r(\tau))}{\partial r(\tau)}+2\,\frac{\partial t(\tau)}{\partial\tau}\,\frac{\partial a(t(\tau),r(\tau))}{\partial t(\tau)}\right)}{a(t(\tau),r(\tau))^2}+ 2\,r(\tau)\left(\sin^2(\theta(\tau))\left(\frac{\partial\phi(\tau)}{\partial\tau}\right)^2+\left(\frac{\partial\theta(\tau)}{\partial\tau}\right)^2\right)=0$$

$$\text{Euler}^{\theta[\tau]} \equiv r(\tau)\left(r(\tau)\left(\sin(2\,\theta(\tau))\left(\frac{\partial\phi(\tau)}{\partial\tau}\right)^2-2\,\frac{\partial^2\theta(\tau)}{\partial\tau^2}\right)-4\,\frac{\partial\theta(\tau)}{\partial\tau}\,\frac{\partial r(\tau)}{\partial\tau}\right)=0$$

$$\text{Euler}^{\phi[\tau]} \equiv -2\,r(\tau)\sin(\theta(\tau))\left(r(\tau)\left(\sin(\theta(\tau))\,\frac{\partial^2\phi(\tau)}{\partial\tau^2}+2\,\frac{\partial\theta(\tau)}{\partial\tau}\cos(\theta(\tau))\,\frac{\partial\phi(\tau)}{\partial\tau}\right)+ 2\sin(\theta(\tau))\,\frac{\partial r(\tau)}{\partial\tau}\,\frac{\partial\phi(\tau)}{\partial\tau}\right)=0$$

Problem 2

Consider the following metric

$$ds^2 = -dt^2 + \left(\frac{A/3}{r-t}\right)^{2/3} dr^2 + \left(\frac{9A}{8}(r-t)^2\right)^{2/3} d\Omega^2$$

Choose a proper coordinate system so that the metric becomes Schwarzschild metric.

Solution:

First thing that we want is

$$\left(\frac{9A}{8}(r-t)^2\right)^{2/3} = R^2 \tag{2}$$

that gives us

$$\frac{8R^3}{9A} = (r-t)^2 \tag{3}$$

We also simultaneously want

$$-dt^2 + \left(\frac{A/3}{r-t}\right)^{2/3} dr^2 = -f(R)dT^2 + \frac{1}{f(R)}dR^2 \tag{4}$$

where $f(R) = 1 - 2M/R$.

We have from equation (3) that $\left(\frac{A/3}{r-t}\right)^{2/3} = A/2R$ but we do not yet have the transformations of t and r in terms of T and R. To obtain that, let's write the differential form of equation (3).

$$\left(\frac{2R}{A}\right)^{1/2} dR = dr - dt \tag{5}$$

We can now write

$$ds^2 = -dt^2 + \frac{A}{2R}(dr-dt)^2 + \frac{A}{2R}(2dr\cdot dt - dt^2) + R^2 d\Omega^2 \tag{6}$$

$$= -\left(1 - \frac{A}{2R}\right) dt^2 + dR^2 + \sqrt{\frac{2A}{R}} dR\cdot dt + R^2 d\Omega^2 \tag{7}$$

Assuming $dt = dT + XdR$, and solving for X in (4) and (7) together, we can write

$$dt = dT + \frac{\sqrt{\frac{A}{2R}}}{1 - \frac{A}{2R}} dR \tag{8}$$

Equations (5) and (9) give the complete transformation of the metric to Schwarzschild metric, with $2r_s = A$. Equation (3) is the integral of these transformations and gives information about the boundary conditions of the transformation.

Problem 3

Assume a source with energy momentum tensor $T_{\mu\nu}$.

- Derive the solutions of linearised Einstein equations in terms of Green's function.
- Use the conservation of energy momentum tensor and show that far from the source the spatial components of $\tilde{h}$ are

$$\tilde{h}_{ij} \approx \frac{1}{r}\frac{d^2}{dt^2}\int d^3x' \rho(t-r,x')x'_i x'_j$$

- Now consider two stars each with mass M separated by a distance R orbiting in the x-y plane. Write down the energy density in terms of delta function assuming these stars as point particles. Derive the metric perturbation $\tilde{h}_{ij}$ in this case.

Solution:

(a) The linearised gravitational wave equation, assuming both the source and perturbed metric are small is

$$\Box\overline{h}_{\mu\nu} = -16\pi G T_{\mu\nu} \tag{9}$$

We can solve this by introducing Green's function $\mathcal{G}(x,x')$ such that

$$\Box\mathcal{G}(x,x') = \delta(x-x') \tag{10}$$

This gives

$$\overline{h}_{\mu\nu} = -16\pi G \int d^4x' \; \mathcal{G}(x,x')T_{\mu\nu}(x') \tag{11}$$

We will consider a situation in which matter fields are localised to some spatial region Σ. In this region, there is a time-dependent source of energy and momentum $T_{\mu\nu}(x',t)$, such as two orbiting black holes. Outside of this region, the energy-momentum tensor vanishes: $T_{\mu\nu}(x',t) = 0$ for $x' \in \Sigma$. We want to know what the metric $h_{\mu\nu}$ looks like a long way from the region Σ. The solution to Equation (11) outside of Σ can be given using the Green's function. One can calculate the Green's Function through the Feynman propagator approach on a flat spacetime. This results in

$$\mathcal{G}(x,x') = \frac{-1}{4\pi|x-x'|}\left(|\vec{x}-\vec{x}'| - |x^0 - x'^0|\right)\Theta(x^0 - x'^0) \tag{12}$$

which is perfectly causal due to

$$\Theta(x^0 - x'^0) = \begin{cases} 1 & x^0 > x'^0 \\ 0 & x^0 < x'^0 \end{cases} \tag{13}$$

Putting this in equation (11) gives

$$\overline{h}_{\mu\nu} = 4G \int d^3x' \ \frac{1}{|\vec{x} - \vec{x}'|} T_{\mu\nu}(x^0 - |\vec{x} - \vec{x}'|, \vec{x}') \tag{14}$$

This can be written in terms of retarded time $t_r = t - |\vec{x} - \vec{x}'|$ as

$$\overline{h}_{\mu\nu} = 4G \int d^3x' \ \frac{1}{|\vec{x} - \vec{x}'|} T_{\mu\nu}(t_r, \vec{x}') \tag{15}$$

This solutions satisfies the de Donder gauge $\partial^\mu \overline{h}_{\mu\nu} = 0$ provided that the source satisfies the conservation of energy-momentum tensor $\partial^\mu T_{\mu\nu} = 0$. I change the notation $\vec{x}' = \boldsymbol{x'}$ for ease now.

(b) We denote the size of the region Σ as d. We're interested in what's happening at a point $\boldsymbol{x}$ which is a distance $r = |\boldsymbol{x}|$ away. If $|\boldsymbol{x} - \boldsymbol{x'}| >> d \ \forall \ \boldsymbol{x'} \in \Sigma$ then we can approximate

$$|\boldsymbol{x} - \boldsymbol{x'}| = r - \frac{\boldsymbol{x} \cdot \boldsymbol{x'}}{r} + \cdots \implies \frac{1}{|\boldsymbol{x} - \boldsymbol{x'}|} = \frac{1}{r} + \frac{\boldsymbol{x} \cdot \boldsymbol{x'}}{r^3} + \dots \tag{16}$$

We also have a factor of $|\boldsymbol{x} - \boldsymbol{x'}|$ that sits inside $t_r = t - |\boldsymbol{x} - \boldsymbol{x'}|$. This means that we should also Taylor expand the argument of the energy-momentum tensor

$$T_{\mu\nu}(t_r, \boldsymbol{x'}) = T_{\mu\nu}(t - r + \frac{\boldsymbol{x} \cdot \boldsymbol{x'}}{r} + \dots, \boldsymbol{x'}) \tag{17}$$

Now we would like to further expand out this argument. But, to do that, we need to know something about what the source is doing. We will assume that the motion of matter is non-relativistic, so that the energy momentum tensor doesn't change very much over the time $\tau \approx d$ that it takes light to cross the region Σ. For example, if we have a system comprised of two objects (say, neutron starts or black holes) orbiting each other with characteristic frequency ω then $T_{\mu\nu} \approx e^{-i\omega t}$ and the requirement that the motion is non-relativistic becomes $d << 1/\omega$. Then we can further Taylor expand the current to write

$$T_{\mu\nu}(t_r, \boldsymbol{x'}) = T_{\mu\nu}(t - r, \boldsymbol{x'}) + \dot{T}_{\mu\nu}(t - r, \boldsymbol{x'}) \frac{\boldsymbol{x} \cdot \boldsymbol{x'}}{r} + \dots \tag{18}$$

At leading order in d/r we take the first term from both these expansions to find

$$\overline{h}_{\mu\nu}(\boldsymbol{x}, t) \approx \frac{4G}{r} \int_\Sigma d^3x' \ T_{\mu\nu}(\boldsymbol{x'}, t - r) \tag{19}$$

For $\overline{h}_{00}$ and $\overline{h}_{0i}$, we obtain standard first order conservation laws and no new information. The former carries information about the total energy in the region Σ and the latter carries information about the total momentum, both of which are clearly conserved in our first order limit. Both of these terms are independent of time. In particular, we can choose the total energy to be time-independent and go to a frame in which the total momentum is zero. This leaves us with the spatial components of the metric perturbation.

The spatial components of the metric perturbation are given by

$$\overline{h}_{ij}(\boldsymbol{x}, t) = \frac{4G}{r}\int_{\Sigma} d^3x'\ T_{ij}(\boldsymbol{x'}, t-r) \tag{20}$$

where the RHS is not conserved. We can relate it to the quadropole moment of the source. I state the proof of this without motivating it any more

$$T^{ij} = \partial_k(T^{ik}x^j) - (\partial_k T^{ik})x^j = \partial_k(T^{ik}x^j) + \partial_0 T^{0i}x^j \tag{21}$$

Then, symmetrising over i and j in the last term.

$$T^{0(i}x^{j)} = \frac{1}{2}\partial_k(T^{0k}x^ix^j) - \frac{1}{2}(\partial_k T^{0k})x^ix^j = \frac{1}{2}\partial_k(T^{0k}x^ix^j) + \frac{1}{2}\partial_0 T^{00}x^ix^j \tag{22}$$

which on integration over Σ gives

$$\int_{\Sigma} d^3x'\ T_{ij}(\boldsymbol{x'}, t-r) = \frac{1}{2}\partial_0^2\int_{\Sigma} d^3x'\ T^{00}(\boldsymbol{x'}, t-r)x'_ix'_j \tag{23}$$

We identify the integral in the RHS as the quadropole moment I_{ij} of the source. This gives us the desired result.

$$\overline{h}_{ij}(\boldsymbol{x}, t) \approx \frac{2G}{r}\ddot{I}_{ij}(t-r) \tag{24}$$

This is the same as the required result if we note that the $\rho = 2G \cdot T^{00}$. Note that $t_r = t - r$. This is the physics that we want: if we shake the matter distribution in some way then, once the signal has had time to propagate, this will affect the metric. Because the equations are linear, if the matter shakes at some frequency ω the spacetime will respond by creating waves at parametrically same frequency. (In fact, we'll see a factor of 2 arises in the example of a binary system). We can write higher order equations for $\overline{h}_{00}$ and $\overline{h}_{0i}$ that are time dependent from the de Donder gauge conditions. These are not the original first order terms and have physical interest. However, since that is not asked here, I do not derive them.

(c) Using Newtonian gravity, they orbit with a frequency

$$\omega^2 = \frac{2GM}{R^3} \tag{25}$$

Treating these stars as point particles, and assuming that they revolve around the origin in the x-y plane, we can write the energy density as

$$T^{00}(\boldsymbol{x}, t) = M\delta(z)\left[\delta(x - \frac{R}{2}\cos(\omega t))\delta(y - \frac{R}{2}\sin(\omega t)) + \delta(x + \frac{R}{2}\cos(\omega t))\delta(y + \frac{R}{2}\sin(\omega t))\right] \tag{26}$$

The quadropole moment is computed as

$$I_{ij} = \frac{MR^2}{4}\begin{pmatrix} 1+\cos(2\omega t) & \sin(2\omega t) & 0 \\ \sin(2\omega t) & 1-\cos(2\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{27}$$

Using the result from part (b), we can write the metric perturbation as

$$\overline{h}_{ij} = \frac{2GMR^2\omega^2}{r}\begin{pmatrix} \cos(2\omega(t_r)) & \sin(2\omega(t_r)) & 0 \\ \sin(2\omega(t_r)) & -\cos(2\omega(t_r)) & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{28}$$

We can write this in terms of a rotation matrix as

$$\overline{h}_{ij} = \frac{2GMR^2\omega^2}{r}\mathcal{R}_z(2\omega t_r) \tag{29}$$

Problem 4

Show that in a locally inertial frame for $\tau^{\mu\nu\sigma} = \frac{1}{16\pi}\frac{\partial}{\partial x^\alpha}\left[(-g)(g^{\mu\nu}g^{\alpha\sigma} - g^{\mu\alpha}g^{\nu\sigma})\right]$ we have

$$(-g)T^{\mu\nu} = \frac{\partial}{\partial x^\alpha}\tau^{\mu\nu\alpha}$$

What is the Landau-Lifschitz pseudotensor?

Solution:

A locally intertial frame points to the fact that $\Gamma^\mu_{\alpha\beta} = 0$ but its derivatives may not be zero. This guides us to write $T_{\mu\nu}$ in terms of $\partial\Gamma$. We can then say

$$T^{\mu\nu} = \frac{1}{8\pi G}G^{\mu\nu} = \frac{1}{8\pi G}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right) \tag{30}$$

We can replace $R^{\mu\nu} \to \partial_\sigma\Gamma^\sigma_{\mu\nu} - \partial_\nu\Gamma^\sigma_{\mu\sigma}$ and $\Gamma^\sigma_{\mu\nu} = \frac{1}{2}g^{\sigma\alpha}(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu})$

$$\begin{aligned}(-g)T^{\mu\nu} &= (16\pi G)^{-1}\left[2g^{\mu\kappa}g^{\nu\sigma}g^{\alpha\rho}R_{\alpha\kappa\rho\sigma} - g^{\mu\nu}g^{\kappa\sigma}g^{\alpha\rho}R_{\alpha\kappa\rho\sigma}\right] &(31)\\ &= (16\pi G)^{-1}[2g^{\mu\kappa}g^{\nu\sigma}g^{\alpha\rho} - g^{\mu\nu}g^{\kappa\sigma}g^{\alpha\rho}]\left(\partial_\kappa\partial_\sigma g_{\alpha\rho} + \partial_\alpha\partial_\rho g_{\kappa\sigma} - \partial_\kappa\partial_\alpha g_{\rho\sigma} - \partial_\kappa\partial_\rho g_{\alpha\sigma}\right) &(32)\end{aligned}$$

Now we show that the RHS of (32) reduces. Observe that

$$\partial_\alpha\partial_\beta g = \partial_\alpha\partial_\beta(g^{\mu\nu}g_{\mu\nu}) \tag{33}$$

We can open this and rearrange to

$$g^{\mu\nu}\partial_\alpha\partial_\beta g_{\mu\nu} = \partial_\alpha\partial_\beta g - g_{\mu\nu}\partial_\alpha\partial_\beta g^{\mu\nu} \tag{34}$$

We should also observe that

$$g^{\mu\kappa}\partial_\alpha\partial_\beta g_{\kappa\sigma} + g_{\kappa\sigma}\partial_\alpha\partial_\beta g^{\mu\kappa} = \partial_\alpha\partial_\beta(g \cdot \delta^\kappa_\sigma) \tag{35}$$

Putting these in (32) gives

$$(-g)T^{\mu\nu} = (16\pi G)^{-1}\left[2g^{\mu\kappa}g^{\nu\sigma} - g^{\mu\nu}g^{\kappa\sigma}\right]\left(\partial_\kappa\partial_\sigma g + g^{\alpha\rho}\partial_\alpha\partial_\rho g_{\kappa\sigma} + g_{\rho\sigma}\partial_\kappa\partial_\alpha g^{\alpha\rho}\right) \tag{36}$$

This, on simplification, will give

$$\begin{aligned}(-g)T^{\mu\nu} &= (16\pi G)^{-1}\left(\partial_\kappa\partial_\sigma(g\cdot g^{\mu\kappa}g^{\nu\sigma}) - \partial_\kappa\partial_\sigma(g\cdot g^{\mu\nu}g^{\kappa\sigma})\right) &(37)\\ &= \partial_\sigma\left[\frac{1}{16\pi G}\partial_\kappa\left[(-g)(g^{\mu\nu}g^{\kappa\sigma} - g^{\mu\kappa}g^{\nu\sigma})\right]\right] &(38)\\ &= \partial_\sigma\tau^{\mu\nu\sigma} &(39)\end{aligned}$$

The **Landau-Lifschitz pseudotensor**, a stress-energy-momentum pseudotensor for gravity, when combined with terms for matter (including photons and neutrinos), allows the energy-momentum conservation laws to be extended into general relativity.

Landau and Lifshitz were led by four requirements in their search for a gravitational energy momentum pseudotensor, $t^{\mu\nu}_{LL}$:

1. that it be constructed entirely from the metric tensor, so as to be purely geometrical or gravitational in origin.
2. that it be index symmetric, i.e. $t^{\mu\nu}_{LL} = t^{\nu\mu}_{LL}$, (to conserve angular momentum)
3. that, when added to the stress-energy tensor of matter, $T^{\mu\nu}$, its total 4-divergence vanishes (this is required of any conserved current) so that we have a conserved expression for the total stress-energy-momentum.
4. that it vanish locally in an inertial frame of reference (which requires that it only contains first order and not second or higher order derivatives of the metric). This is because the equivalence principle requires that the gravitational force field, the Christoffel symbols, vanish locally in some frames. If gravitational energy is a function of its force field, as is usual for other forces, then the associated gravitational pseudotensor should also vanish locally.

The unique pseudotensor which satisfies these requirements is

$$t^{\mu\nu}_{LL} = -\frac{1}{8\pi G}G^{\mu\nu} + \frac{1}{(-g)}\partial_\sigma\tau^{\mu\nu\sigma} \tag{40}$$

The local vanishing is apparent in its affine connection version

$$\begin{aligned}t^{\mu\nu}_{LL} + \frac{c^4\Lambda g^{\mu\nu}}{8\pi G} = \frac{c^4}{16\pi G}\Big[&\left(2\Gamma^\sigma_{\alpha\beta}\Gamma^\rho_{\sigma\rho} - \Gamma^\sigma_{\alpha\rho}\Gamma^\rho_{\beta\sigma} - \Gamma^\sigma_{\alpha\sigma}\Gamma^\rho_{\beta\rho}\right)\left(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta}\right) +\\ &\left(\Gamma^\nu_{\alpha\rho}\Gamma^\rho_{\beta\sigma} + \Gamma^\nu_{\beta\sigma}\Gamma^\rho_{\alpha\rho} - \Gamma^\nu_{\sigma\rho}\Gamma^\rho_{\alpha\beta} - \Gamma^\nu_{\alpha\beta}\Gamma^\rho_{\sigma\rho}\right)g^{\mu\alpha}g^{\beta\sigma} +\\ &\left(\Gamma^\mu_{\alpha\rho}\Gamma^\rho_{\beta\sigma} + \Gamma^\mu_{\beta\sigma}\Gamma^\rho_{\alpha\rho} - \Gamma^\mu_{\sigma\rho}\Gamma^\rho_{\alpha\beta} - \Gamma^\mu_{\alpha\beta}\Gamma^\rho_{\sigma\rho}\right)g^{\nu\alpha}g^{\beta\sigma} +\\ &\left(\Gamma^\mu_{\alpha\sigma}\Gamma^\nu_{\beta\rho} - \Gamma^\mu_{\alpha\beta}\Gamma^\nu_{\sigma\rho}\right)g^{\alpha\beta}g^{\sigma\rho}\Big]\end{aligned}$$

Problem 5

Calculate stress tensor for following metric using Mathematica/Python. (Explicitly calculate quantities involved before reaching final solution)

Solution:

In the following, I have again used the 'Einstein Field Equations' notebook (used in Problem 1) to write down the components of the Einstein tensor. In the general non-zero Λ case, the stress energy tensor is diagonal. It is written below.

$$
\begin{array}{lcccl}
G_{\mu\nu} + \Lambda\, g_{\mu\nu} & = & \text{Einstein Field Equations} & = & \frac{8\pi G}{c^4} T_{\mu\nu} \\
G_{tt} + \Lambda\, g_{tt} & = & \frac{3GM^2}{r^2 (r-2GM)^2} + \Lambda \left(\frac{2GM}{r} - 1\right) = \frac{8\pi G}{c^4}\, \mathtt{SETensor[[1, 1]]} & = & \frac{8\pi G}{c^4} T_{tt} \\
G_{rr} + \Lambda\, g_{rr} & = & \frac{GM(4r-5GM)}{r^2 (r-2GM)^2} + \Lambda \left(1 - \frac{2GM}{r}\right) = \frac{8\pi G}{c^4}\, \mathtt{SETensor[[2, 2]]} & = & \frac{8\pi G}{c^4} T_{rr} \\
G_{\theta\theta} + \Lambda\, g_{\theta\theta} & = & \Lambda r^2 \left(1 - \frac{2GM}{r}\right) + \frac{GM(GM-2r)}{(r-2GM)^2} = \frac{8\pi G}{c^4}\, \mathtt{SETensor[[3, 3]]} & = & \frac{8\pi G}{c^4} T_{\theta\theta} \\
G_{\phi\phi} + \Lambda\, g_{\phi\phi} & = & \Lambda r^2 \left(1 - \frac{2GM}{r}\right) \sin^2(\theta) + \frac{GM(GM-2r)\sin^2(\theta)}{(r-2GM)^2} = \frac{8\pi G}{c^4}\, \mathtt{SETensor[[4, 4]]} & = & \frac{8\pi G}{c^4} T_{\phi\phi}
\end{array}
$$

This gives the stress tensor as

$$
T^{\mu\nu} = \frac{c^4}{8\pi G} \begin{pmatrix} \frac{3GM^2}{r^2(r-2GM)^2} & 0 & 0 & 0 \\ 0 & \frac{GM(4r-5GM)}{r^2(r-2GM)^2} & 0 & 0 \\ 0 & 0 & \frac{GM(GM-2r)}{(r-2GM)^2} & 0 \\ 0 & 0 & 0 & \frac{GM(GM-2r)\sin^2(\theta)}{(r-2GM)^2} \end{pmatrix} + \frac{\Lambda c^4}{8\pi G} g^{\mu\nu} \tag{41}
$$

Problem 6

Find the stable circular orbit for a massive and mass-less particle for 4-D Schwarzschild. (You can use mathematica/python.)

Solution:

For a Lagrangian of the form

$$
L = -f(r)^2 \dot{t}^2 + f(r)^{-2} \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2(\theta) \dot{\phi}^2 \tag{42}
$$

we can formulate a general method to solve for the equations of motion.

Firstly, we observe the cyclic coordinates t and ϕ and write down the conserved quantities.

$$
e = -\frac{1}{2}\frac{dL}{d\dot{t}} = f(r)^2 \dot{t} \tag{43}
$$

$$
b = \frac{1}{2}\frac{dL}{d\dot{\phi}} = r^2 \sin^2(\theta) \dot{\phi} \tag{44}
$$

Then, we can WLOG constraint the motion to the equatorial plane $\theta = \pi/2$ and $\dot{\theta} = 0$. The final step now involves restricting the nature of trajectory, timelike or null.

$$L = \varepsilon \tag{45}$$

where $\varepsilon = 0$ for null geodesics and $\varepsilon = -1$ for timelike geodesics. This prescription reduces the equations of motion to

$$\dot{r}^2 + V_{\text{eff}}(r) = e^2 \tag{46}$$

with the effective potential given by $V_{\text{eff}}(r) = \left(\frac{b^2}{r^2} - \varepsilon\right) f(r)^2$

Stable circular orbits exist when $\dot{r} = 0$ and $\dot{V}_{\text{eff}}(r) = 0$ and $\ddot{V}_{\text{eff}}(r) < 0$. We can solve for this from equation (46).

$$\frac{d}{dr}\left(\left(1 - \frac{2M}{r}\right)\left(\frac{b^2}{r^2} - \varepsilon\right)\right) = 0 \tag{47}$$

1. **For massive particle;** $\varepsilon = -1$

```
In[1]:= Solve[D[(B^2/r^2 + 1) (1 - 2 M/r), r] == 0, r]

Out[1]= {{r -> (B^2 - Sqrt[B^4 - 12 B^2 M^2])/(2 M)}, {r -> (B^2 + Sqrt[B^4 - 12 B^2 M^2])/(2 M)}}
```

Qualitatively, since the potential should go to negative infinity at origin and zero at infinity, we can surely say that the radius closer to the origin has the unstable circular orbit. For $B \to \infty$, an unstable orbit exists at $r = 3M$, and the stable orbit scales as B^2/M. Stable orbit exists when the discriminant is zero and both r coincide. This happens for $B = \sqrt{12}M$. and $r = 6M = 3r_s$. From this analysis, we can also say that $3M < r < 6M$ is the region of unstable orbits, and $r > 6M$ is the region of stable orbits. $B < \sqrt{12}M$ does not admit any circular orbits.

2. **For massless particle;** $\varepsilon = 0$

```
In[1]:= Solve[D[(B^2/r^2) (1 - 2 M/r), r] == 0, r]

Out[1]= {{r -> 3 M}}

In[2]:= D[D[(B^2/r^2) (1 - 2 M/r), r], r] /. r -> 3 M

Out[2]= -(2 B^2)/(81 M^4)
```

this admits a stable circular orbit at $r = 3M$. This is a stable orbit for massless particles.

Problem 7

The surface area A of a rotating, electrically charged Kerr-Newman black hole is given by

$$A = 4\pi \left[2M^2 + 2(M^4 - L^2 - M^2Q^2)^{1/2} - Q^2\right]$$

where M is the mass, L is the angular momentum and Q is the charge of the black hole. Show that

$$M = 2TA + 2\Omega L + \Phi Q$$

where T is the effective surface tension, Ω is the angular velocity and Φ is the electromagnetic potential. Find T, Ω and Φ as functions of M, A, L and Q.

Solution:

Invert the given relation for A, M, L, and Q to get one for M.

$$M = \left[\frac{A}{16\pi} + \frac{4\pi L^2}{A} + \frac{Q^2}{2} + \frac{\pi Q^4}{A}\right]^{1/2} \tag{48}$$

and then we write the differential of M

$$dM = TdA + \Omega dL + \Phi dQ \tag{49}$$

Differentiating equation (48), we get

$$T = \frac{1}{M}\left[\frac{1}{32\pi} - \frac{2\pi L^2}{A^2} - \frac{\pi Q^4}{2A^2}\right] \tag{50}$$

$$\Omega = \frac{4\pi L}{MA} \tag{51}$$

$$\Phi = \frac{1}{M}\left[\frac{Q}{2} + \frac{2\pi Q^3}{A}\right] \tag{52}$$

We can get M by adding up $2TA + 2\Omega L + \Phi Q$ and using (48). The result is exactly the same as required.

Problem 8

Write the Schwarzschild metric in terms of incoming and outgoing Eddington-Finkelstein Coordinates.

Solution:

$$ds^2 = 0 \implies \frac{dr}{dt} = \pm\left(1 - \frac{2M}{r}\right) \tag{53}$$

Define

$$dr_\star = \left(1 - \frac{2M}{r}\right) dr \tag{54}$$

which has a solution $r_\star = r + r_s \log\left(\frac{r-r_s}{r_s}\right)$.

Now, by equation (53) and (54), the null geodesics follow $t \pm r_\star =$ constant, where the plus sign corresponds to ingoing geodesics (as t increases, $r_\star$ must decrease) and the negative sign to outgoing geodesics. Next, we introduce a pair of null coordinates $v = t + r_\star$ and $u = t - r_\star$.

- **Ingoing Eddington-Finkelstein coordinates**
  Replace $t \to v - r_\star$
  The new metric is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dv \cdot dr + r^2 d\Omega_2^2 \tag{55}$$

- **Outgoing Eddington-Finkelstein coordinates**
  Replace $t \to u + r_\star$
  The new metric is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du^2 - 2du \cdot dr + r^2 d\Omega_2^2 \tag{56}$$