Causal Set Theory

A Statistical Perspective

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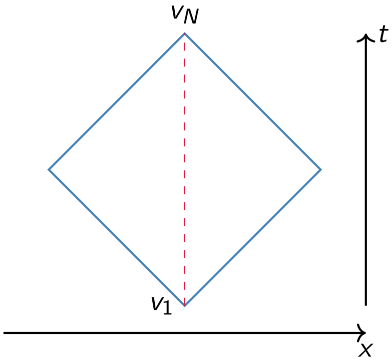
(1+1)D Minkowski Spacetime Setup

- Consider flat $(1+1)$D spacetime with coordinates (t, x)
- Diamond-shaped region with vertices:
  - $(0, 0)$, $(T/2, T/2)$, $(-T/2, T/2)$, $(0, T)$
- Sample N events uniformly at random
- Construct causal set $\mathcal{C} = \{\text{events in diamond}\}$



Partial Order Relations

A causal set $(\mathcal{C}, \prec)$ requires a partial order relation $\prec$ that is:

1. **Reflexive**: $v_x \prec v_x$ for all $v_x \in \mathcal{C}$
2. **Antisymmetric**: If $v_x \prec v_y$ and $v_y \prec v_x$, then $v_x = v_y$
3. **Transitive**: If $v_x \prec v_y$ and $v_y \prec v_z$, then $v_x \prec v_z$
4. **Locally finite**: Finite number of events between any two causally related events

The relation $\prec$ encodes the **causal structure** of spacetime events.

The Coordinate Assignment Problem

Initial Question

Given causal set $(\mathcal{C}, \prec)$, can we assign unique coordinates to events in flat $\mathbb{M}^2$ Minkowski spacetime?

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Well-Defined Question

Given causal set $(\mathcal{C}, \prec)$ with fixed minimal/maximal elements $v_{\mathrm{min}}, v_{\mathrm{max}}$ in flat $\mathbb{M}^2$, can we assign coordinates up to finite precision?

Answer: Yes!

Causal Set Structures

Define important causal structures for event v_x:

$$J^+(v_x) = \{v_y \in \mathcal{C} \mid v_x \prec v_y\} \quad \text{(future light cone)} \tag{1}$$

$$J^-(v_x) = \{v_y \in \mathcal{C} \mid v_y \prec v_x\} \quad \text{(past light cone)} \tag{2}$$

$$L_c(v_x) = J^+(v_x) \cup J^-(v_x) \quad \text{(light cone)} \tag{3}$$

$$[v_x, v_y] = J^+(v_x) \cap J^-(v_y) \quad \text{(Alexandrov interval)} \tag{4}$$

The **Alexandrov interval** $[v_x, v_y]$ contains all events causally between v_x and v_y.

Lorentz Invariance

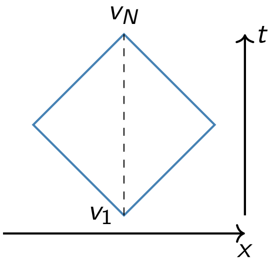


Figure: Original Frame

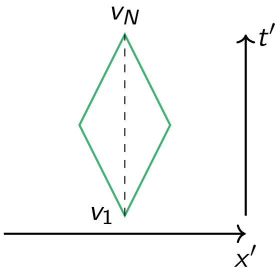
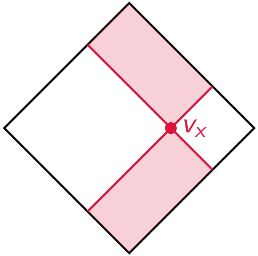


Figure: Boosted Frame

- Proper volume of diamond region is Lorentz invariant
- Uniform (Poisson) sampling maintains Lorentz invariance
- Causal structure preserved under boosts

Accessible Events and Proper Time



Key Insight:

- Fewer events accessible to v_x than to minimal element v_1
- Number of accessible events $\propto$ proper time measure
- Each **step** to nearest future event $=$ unit proper time

Proper Time Construction:

- Each link in causal chain: $t_{\text{link}} = 1$
- Based on accessible event count
- Well-defined in causal set framework

Hasse Diagrams

Minimal graphical representation of partial order

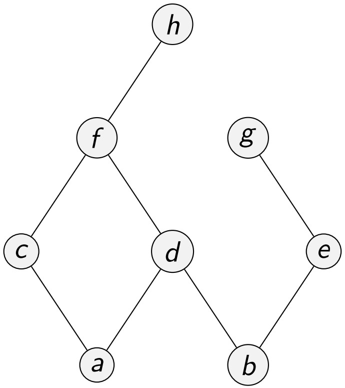


Figure: Hasse diagram of 8 elements

Properties:

- Directed AcyclicGraph (DAG)
- Only direct connections shown
- Transitivity implied
- Non-comparable pairs: (a, b), (c, d), (d, e), etc.

Proper Time:

- Each link $= 1$ unit
- $t_{\text{link}} = 1$
- Natural discretization

Breaking Symmetries

Problem: Causal structure alone is insufficient - need to fix symmetries

1. **Translation invariance**: Fix $v_1 \mapsto (0,0)$ (anchor point)
2. **Parity invariance**: Choose light propagation direction
3. **Lorentz invariance**: Fix $v_N \mapsto (0,T)$ (time coordinate)

Embedding map: $p : \mathcal{C} \to \mathbb{M}^2$ with $p(v_1) = (0,0)$ and $p(v_N) = (0,T)$

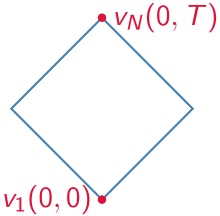


Figure: Skeleton of the embedding

Causal Relations and Proper Time

Causal condition: For $v_x \prec v_y$ in $\mathbb{M}^2$: $x^0 \leq y^0 \quad \text{and} \quad (y^0 - x^0)^2 - (y^1 - x^1)^2 \geq 0$

Proper time for timelike separation: $\tau(y^\mu, x^\mu) = \sqrt{(y^\mu - x^\mu)(y_\mu - x_\mu)}$

Event density: For diamond volume $T^2/2$ with N events: $\rho = \frac{2N}{T^2}$

General dimension: $c_d \cdot T^d \cdot \rho = N$ where $c_2 = 1/2$

Two Definitions of Proper Time

For causally related events $v_x \prec v_y$:

1. **Geodesic-inspired** (τ_G):

$$C(v_x, v_y) = \{\text{chains between } v_x \text{ and } v_y\} \tag{5}$$

$$\tau_G(v_x, v_y) = \max_{\zeta \in C(v_x, v_y)} |\zeta| \tag{6}$$

   Length of maximal chain (longest causal path)

2. **Alexandrov interval-inspired** (τ_A):

$$\tau_A(v_x, v_y) = \left(\frac{|[v_x, v_y]|}{c_d \cdot \rho} \right)^{1/d} \tag{7}$$

   Based on volume of Alexandrov interval

Boundary condition: $\tau_G(v_1, v_N) = \tau_A(v_1, v_N) = T$

Causal Set Geometry

Triangle relations: For $v_x \prec v_y \prec v_z$

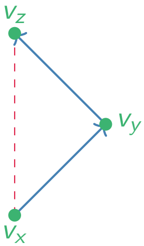


Figure: Three elements of the causet

Causal contraction:

$$\langle v_z - v_x | v_z - v_x \rangle_{\mathcal{C}} := \frac{1}{2}[\tau(v_x, v_z)^2 - \tau(v_x, v_y)^2 - \tau(v_y, v_z)^2] \quad (8)$$

Time coordinate: $t(v_x) = \frac{T}{2} + \frac{\tau(v_1, v_x)^2 - \tau(v_x, v_N)^2}{2T}$

Spatial Coordinate Reconstruction

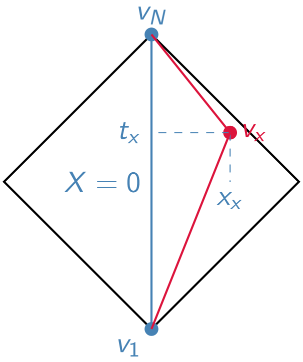


Figure: Pythogorean construction

Spatial coordinates from:

$$\left(\frac{T}{2} - t(v_x)\right)^2 + x(v_x)^2 = -\tau(v_x, v_N)^2 \tag{9}$$

$$\left(\frac{T}{2} + t(v_x)\right)^2 + x(v_x)^2 = -\tau(v_1, v_x)^2 \tag{10}$$

Parity resolution:

- Two solutions: $\pm x(v_x)$
- Symmetric distribution about $X = 0$
- Assign based on density balance

Antichains and Spacelike Separation

Antichain: Set of mutually incomparable elements
For events v_x, v_y, define:

$$\mathfrak{p}(v_x, v_y) = \{v_z \in J^-(v_x) \cap J^+(v_y) \mid \max(t(v_z))\} \tag{11}$$

$$\mathfrak{f}(v_x, v_y) = \{v_z \in J^+(v_x) \cap J^-(v_y) \mid \min(t(v_z))\} \tag{12}$$

Spacelike separation measure: $I(v_x, v_y) := \tau(\mathfrak{p}(v_x, v_y), \mathfrak{f}(v_x, v_y))$

- Proper time when $v_x \prec v_y$
- Spacelike separation when incomparable
- Enables complete coordinate reconstruction

Maximal Chain Assignment of Coordinates

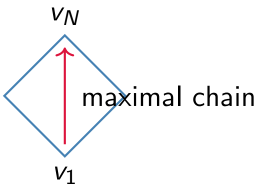
- **Goal:** Assign coordinates to causal set events using maximal chains
- **Method:** Uniformly distribute maximal chain on line from v_1 to v_N
- **Expectation:** Unique maximal chain exists (choose one if multiple)
- **Symmetry:** Event distribution symmetric about $X = 0$ line

Iterative Process

1. Assign time coordinate to maximal chain
2. Move to next pair of maximal chains
3. Continue until reaching maximally spacelike separated events

Dimensional Analysis

- In flat spacetime with d spatial dimensions: expect d next-maximal chains
- **Natural interpretation:** Number of next-maximal chains = spatial dimensions
- **Key question:** Can we determine embedding dimension from causal set?



Tolerance in Chain Length

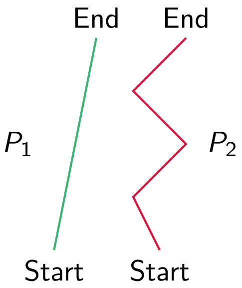
- **Problem:** Maximal chain length may not be unique
- **Solution:** Introduce tolerance ϵ for chain length deviation
- **Physical interpretation:**
  - May relate to system physics
  - In quantum systems: connection to uncertainty principle

Mathematical Formulation

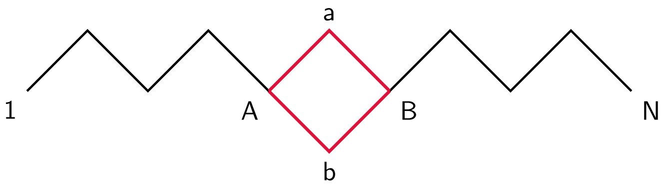
Allow chains with length $L_{\text{max}} - \epsilon \leq L \leq L_{\text{max}}$

Path Shape Problem

- Two paths P_1, P_2 with same length but different shapes:
  - P_1: Nearly straight, small X-deviations
  - P_2: Zig-zag, large X-deviations
- **Question:** Which path to choose as maximal chain?
- Current method ignores path shape
- Overlapping points create ambiguity



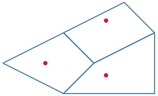
Local Substructure Challenge



- **Problem:** Different paths lead to intermixing of events *a* and *b*
- Coordinate assignment becomes ambiguous in plaquette regions
- This feature is inherent to the theoretical framework

Expected Regions: Voronoi Tessellation

- **Question:** What region does each event occupy across simulations?
- **Answer:** Use Voronoi tessellation
- Each region contains points closer to given event than any other
- Provides statistical expectation for event locations



Voronoi cells

Hexagonal Structure Analysis

Using Euler's formula in 2D: $F + V - E = 2$

- Each vertex connects to 3 edges (Delaunay triangulation)
- Each edge shared by 2 vertices
- In large E limit: $E = 3F$

Result

- Average edges per Voronoi cell: 6
- **Expected shape:** Hexagonal regions in large density limit

Cell Size

- Diamond area: $T^2/2$
- Number of events: N
- Average hexagon area: $\frac{T^2}{2N}$ (determined by event density)