Hawking Radiation and Unruh Temperature with Recent Phenomenology

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This work is divided into three sections. The first is a review of explicit mathematical derivations of particle creation in static Schwarzschild spacetime. The second generalizes to transient phenomena in Minkowski background spacetimes. The final part covers the recent phenomenology and proposed experiments to detect the Unruh effect, including the use of Berry's phase, and a possible detection of the Unruh effect in the LHC energy scales.

Keywords: Black Holes, Schwarzschild, Minkowski spacetimes, Feynman Propagator, Geometric Optics Approximation, Euclidean Field Theory, Response Functions

Note: All figures in section 2 are taken from [1], in section 3 from [2], in section 4 from Hartman's notes, and in section 5 from [3].

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I. HISTORICAL BACKGROUND

Black Holes have always been toy models for the study of quantum gravity. Studies of their causal structure, macroscopic parameters, and thermodynamics are of great interest. Breakthroughs by Hawking [1], Davies [4] and Unruh [5] show that black holes behave like black body thermal systems.

Before moving to the complicated example of black holes, we try to apply consistent quantum field theory to some arbitrary spacetime and try to draw simple conclusions from it. The main idea of quantum field theory in any background spacetime follows from two principles: that of a classical metric describing the spacetime, and that of matter fields coupled to this metric. The metric is assumed to be a solution of the Einstein field equations, and the matter fields are quantized. The quantization of matter fields is done in a curved spacetime, and the metric is treated classically. This is a semiclassical approximation to quantum gravity. Of course, this is an approximation, and the full theory of quantum gravity is still unknown.

We must test to what extent this approximation holds, and whether it is reasonable to not treat the metric in a quantum mechanical way as well. We assume that Ehrenfest's theorem holds, i.e. in solving the Einstein field equations, the metric is a classical field, and the right-hand side (involving the energy-momentum tensor) is the expectation value of a suitably defined operator for matter fields.

The easiest example to compute is the massless Hermitian scalar field in a Minkowski background. Let the field be $\boldsymbol{\phi}$ and the metric is $\eta_{\mu\nu}$.

$$\Box\boldsymbol{\phi} + m^2\boldsymbol{\phi} = 0 \tag{1}$$

with $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$, $g^{\mu\nu} = \eta^{\mu\nu}$ and $m = 0$ is the Klein-Gordon wave equation followed by the scalar field. The field is quantized by expanding it in terms of creation and annihilation operators, with negative and positive frequency modes respectively.

$$\boldsymbol{\phi} = \sum_i f_i \boldsymbol{a_i} + \overline{f}_i \boldsymbol{a_i}^\dagger \tag{2}$$

where f_i are a complete orthonormal family of complex-valued solutions of $\Box f_i = 0$ which contain only positive frequency modes with respect to the usual Minkowski time coordinate. An important point to note (see Section IV) is that these are positive frequency modes with respect to a specific coordinate system, which points to the fact that the choice of coordinates is important in defining the vacuum state. For example, the usual $-\infty < t < \infty$ Minkowski time coordinate covers the entire flat manifold, but the Rindler wedge covers only one-fourth. As will be shown, the vacuum state in the Rindler wedge is different from the Minkowski vacuum state.

We can now define the vacuum state $|0\rangle$ to be the state from which annihilation of particles is not possible, i.e.

$$\boldsymbol{a_i}|0\rangle = 0 \tag{3}$$

Note that the vacuum state is vacuum for all i.

For a curved spacetime, positive and negative frequency modes (with respect to some coordinate time) lose their invariant physical meaning and depend on the choice of coordinates. Mathematically, we are left with only one condition to apply to the modes: that they form a complete basis for the space of solutions of the wave equation, with their corresponding current being conserved. This can be written as

$$\nabla_\mu J^\mu = 0 \tag{4}$$

or in terms of its pullback to some suitable surface S,

$$\frac{1}{2} i \int_S d\Sigma^\mu (f_i \nabla_\mu \overline{f_j} - \overline{f_j} \nabla_\mu f_i) = \delta_{ij} \tag{5}$$

Equation 5 does not uniquely fix the subspace of the space of solutions spanned by f_i and therefore does not determine the splitting of the operator $\boldsymbol{\phi}$ into creation and annihilation operators.

For an asymptotically flat spacetime, we can resolve this ambiguity by requiring that the modes f_i be positive frequency with respect to the time coordinate at infinity. On the other hand, a spacetime with, say, two flat sections joined with a curved part will have different vacuum states for the two flat sections. It is reasonable to think that such a situation may arise in time-dependent metric fields, such as those during collapse, and we can interpret this change in vacuum states across the curved part as the creation of particles.

Let there be an asymptotically flat spacetime $(\mathcal{M}, g)$ and an observer moving with velocity v^a at a point $p \in \mathcal{M}$. Let B be the least upper bound of $|R_{abcd}|$ in a neighborhood of p and we can now set up a vielbein coordinate system with a local radius of curvature $B^{-1/2}$. Choose f_i to be positive frequency with respect to the time coordinate of the observer in this coordinate system. There is an indeterminacy in f_i and $\overline{f_i}$ to the order of the exponential of some multiple of $-\omega B^{-1/2}$ when $\omega > B^{1/2}$ is the characteristic frequency of the mode. This shows that the indeterminacy between $\boldsymbol{a_i}$ and $\boldsymbol{a_i}^\dagger$ is exponentially small in the limit of high frequencies. For $\omega < B^{1/2}$, the indeterminacy is of the order of unity and leads to the particle number operator having an uncertainty of order unity (actually $\pm 1/2$).

The number of modes per unit volume with frequency between ω and $\omega + d\omega$ is given by the integral over all positive frequencies of $\omega^2 d\omega$ multiplied by ω. The uncertainty in defining this is of order B^2 for $\omega < B^{1/2}$ and exponentially small for $\omega > B^{1/2}$, and we can effectively ignore the latter. One can think of this uncertainty as corresponding to the local energy density of particles created by the gravitational field. We can safely approximate that locally if we have a radius of curvature larger than the Planck length, this creation of particles is negligible. Hence, the semiclassical approximation to quantum gravity is valid for all practical purposes from $t > 10^{-43}$ seconds after the Big Bang.

In this review, I show that even though locally this creation of particles is negligible, it can have a significant effect over longer time scales (such as the age of the universe) and in the presence of strong gravitational fields (such as those near black holes).

Qualitatively, the creation of particles and their emission to infinity by a black hole is equivalent to the emission of thermal radiation by a black body. Since the black hole is now a thermal system, we should be able to apply zeroeth law of thermodynamics to it. There are three possibilities; the trivial case of external temperature being the same as that of the black hole, or it being smaller or larger. The latter case is worth noting, as one would expect that if the temperature of the black hole is higher than the external temperature, it would emit particles and lose mass and hence area. This *feels* like a violation of Area Increase Theorem [6] but can be explained using the four laws of black hole mechanics [7] considering that $\Delta(S_{\text{surrounding}} + A/4) > 0$ instead of $\Delta A > 0$. A generalized second law of black hole mechanics is hence $\Delta(S_{\text{surrounding}} + A/4) \geq 0$, which Beckenstein [8] suggested, without considering emission. Identifying $T_{\text{BH}} = \kappa/2\pi$ as the temperature of the black hole (in geometrical units), Beckenstein-Hawking generalized second law is confirmed.

A. What kind of radiation is emitted by a black hole?

Consider the case when $T_{\text{BH}} = \hbar\kappa/2\pi > T_{\text{CMB}} \cong 3K$. There will be an emission of radiation by this black hole, and a loss of mass. When the mass decreases, they get hotter, and hence radiate faster. It will reach a point when the temperature will exceed the rest mass of particles like electrons and muons. These will now be emitted.

At this the point the black hole will start losing mass rapidly, and will eventually evaporate. When the mass reaches nearly 10^{11} kg, it may happen that the black hole expels all kinds of particles and radiations on a time scale associated with strong interactions, and the black hole will explode, emitting all its rest mass. This is the final stage of the black hole evaporation. Even if the number of types of particles is not too much, it will still radiate all its rest mass in the order of $10^{-28}M^3$ seconds with an energy of order 10^{35} ergs. Note that all this is just a handwavey argument, and the actual process is much more complicated [1, 9].

This however leads us to a very important question, how are charged particles being created and emitted? Surely, their antiparticles must also be created, with a negative energy. A heuristic argument of what is happening is presented next.

B. Particle and Antiparticle Pair Creation

Energy of any particle is defined whenever a time translation Killing vector exists, as it does in the static Schwarzschild solution, but it is defined only locally. The nature of this Killing vector determines the energy of particles with respect to some standard asymptotically flat region, like infinity in the case of the Schwarzschild solution.

A timelike time translation Killing vector prescribes a positive energy, and a spacelike one prescribes a negative energy to particles with timelike momentum vectors. Observe that the event horizon in the Schwarzschild solution is a null hypersurface, and the Killing vector is spacelike inside the horizon. This means that the energy of particles with timelike momentum vectors is negative inside the horizon. It is a known fact that when a null hypersurface is closed in a Lorentzian manifold, then it is also a Killing horizon. From this, we can conclude that outside the event horizon, particles with timelike momentum vectors have positive energy, and can thus escape to infinity.

In this sense, one can explain the creation of particles and antiparticles by the black hole. The antiparticles are just particles with negative energy. The emitted particles constitute part of the thermal radiation. Antiparticles can now be thought to be particles inside a finite, deep potential well in a flat spacetime [10], and consequently one can think of scattering and bound states. Bound states correspond to completely absorbed particles, and scattering corresponds to particles that escape to infinity. A classical analysis of gravitational-electromagnetic entities (a.k.a. *'geons'*) was done by Wheeler [11] and later by Brill and Hartle [12] which works on a similar principle but does not involve quantum mechanics.

It must be emphasized that arguments presented so far have little to no mathematical rigor, and are only explanatory. The details are presented in Sections II and III.

II. EXPLICIT QUANTUM FIELD THEORETIC FORMULATION

A study of scalar quantum field theory in a closed universe of constant curvature was done by Fulling in his 1972 PhD thesis [13], followed by a study on the non-uniqueness of canonical quantization in curved spacetime[14]. In 1975, this was used by Hawking [1] and Davies [4] (see IV) to work out different interpretations of the same concept. We proceed with first discussing Hawking's derivation that focuses on the specific case of a black hole. This involves the scattering of a massless scalar field off a black hole and its interpretation as particle creation.

A. Motivation

The motivation for thermal emission lies in a classical phenomenon 'Superradiance' [15–18] in which a rotating (or charged) black hole can extract energy from the rotation (or charge) of the black hole. Experimentally, we radiate energy in certain frequency modes on this rotating or charged black hole and it scatters off with a higher amplitude. Till now, we have only talked about things that make sense mathematically but may or may not have any observable effects. Luckily, superradiance has been observed experimentally using analogous sonic radiation in Bose-Einstein condensates [19–22] which obey a similar physics to that of gravity, so we are not shining light into a black hole and hoping for it to return.

Observations of superradiance suggest that a black hole will steadily emit spontaneous radiation in these superradiant modes which carry along with them the mass (since they are particles with positive energy) and angular momentum and/or charge of the black hole. With all this talk of particle creation, one must keep in mind that for any particle to be created, we need to have a mixing of positive and negative frequency modes. This does not seem to happen in a static solution to the Einstein equations.

Hence, to understand the creation of particles, we need to consider not only the static Schwarzschild solution but also the time-dependent formation stages of a black hole (which generally involve gravitational collapse). By the no-hair theorems, we expect that the final solution for the thermodynamics of a black hole should depend only on the mass, charge, and angular momentum of the black hole, and not on the details of the collapse.

Consider the simplest spherically symmetric, non-rotating, uncharged black hole, the Schwarzschild solution. The metric is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \quad (6)$$

We, being physicists, look at pretty pictures. Consider the following Penrose diagrams.

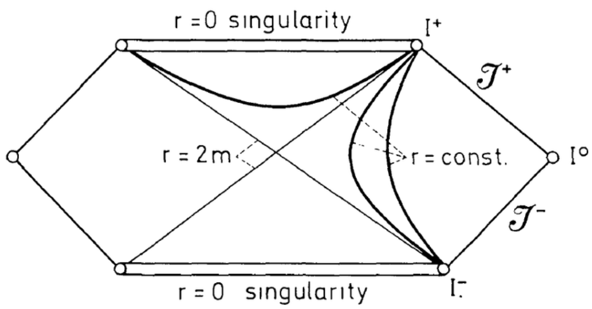

FIG. 1. Penrose diagram of an analytically extended Schwarzschild solution. $\mathcal{J}^+$ is the future null infinity, $\mathcal{J}^-$ is the past null infinity, $\mathcal{I}^+$ is the future timelike infinity and $\mathcal{I}^-$ is the past timelike infinity. $r = 0$ is the singularity, and constant r surfaces are plotted.

We are, however, not considering only static solutions.

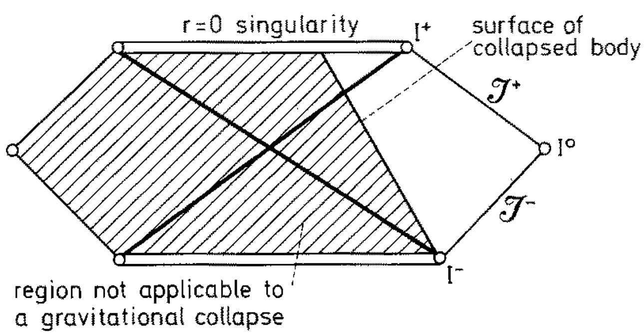

FIG. 2. Only the region outside the collapsing body is relevant for a black hole formed by gravitational collapse. Inside the body, the solution is completely different.

The surface of the collapsing body is represented by a timelike geodesic starting at past timelike infinity. Inside the body, the metric is completely different. The past event horizon, the past $r = 0$ singularity, and the other asymptotically flat region do not exist and are replaced by a timelike curve representing the origin of polar coordinates.

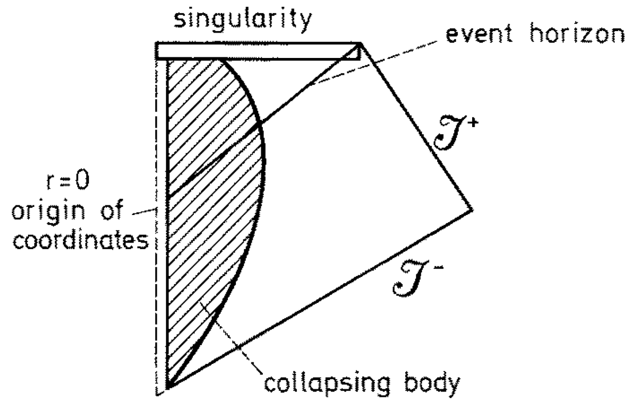

FIG. 3. $r = 0$ is the origin of coordinates and not a singularity. We have used a conformal transformation to map the origin of coordinates to a vertical line.

We can obtain the conformally invariant[23] massless Hermitian scalar field wave equation for pure-photon radiation

$$\Box\phi + \frac{1}{6}R\phi = 0 \tag{7}$$

and in our discussion, $R = 0$ for a Schwarzschild solution.

B. Quantum Effects

Since we are discussing the radiation of massless scalar particles, we need to consider null infinities in our analysis. Solutions $\{f_i\}$ form a complete and orthonormal basis for the space of solutions of the wave equation, following 5 where S is $\mathcal{J}^-$. By this condition, we can find positive frequency modes with respect to the canonical affine time coordinate on $\mathcal{J}^-$.

Choosing the null hypersurface S to be $\mathcal{J}^-$, the operators $\boldsymbol{a_i}$ and $\boldsymbol{a_i^\dagger}$ have a well-defined meaning. They are annihilation and creation operators for ingoing particles. The field $\boldsymbol{\phi}$ is completely determined by its data on $\mathcal{J}^-$.

We now observe that near the event horizon, we can again define the same field by its data on $\mathcal{J}^+$ and on the event horizon. Decomposing the field into purely outgoing (zero Cauchy data on the event horizon, $\{p_i\}$ is the complete basis) and no outgoing modes (zero Cauchy data on $\mathcal{J}^+$, $\{q_i\}$ is the complete basis),

$$\boldsymbol{\phi} = \sum_i p_i\boldsymbol{b_i} + \overline{p_i}\boldsymbol{b_i^\dagger} + q_i\boldsymbol{c_i} + \overline{q_i}\boldsymbol{c_i^\dagger} \tag{8}$$

As in the case of $\{f_i\}$, $\{p_i\}$ contains only positive frequency modes with respect to the canonical affine time coordinate on $\mathcal{J}^+$. This, similar to before, defines $\boldsymbol{b_i}$ and $\boldsymbol{b_i^\dagger}$ as annihilation and creation operators for outgoing particles.

We have so far decomposed the field into two types of solutions. Both of these must be the same, so $\{p_i\}$ and

$\{q_i\}$ must be related to $\{f_i\}$ and $\{\overline{f_i}\}$ by linear combinations, which will furnish some linear relation between $\boldsymbol{a_i}$, $\boldsymbol{a_i^\dagger}$, $\boldsymbol{b_i}$, and $\boldsymbol{c_i}$.

$$p_i = \sum_j \alpha_{ij} f_j + \beta_{ij} \overline{f}_j \tag{9}$$

$$q_i = \sum_j \gamma_{ij} f_j + \delta_{ij} \overline{f}_j \tag{10}$$

$$\boldsymbol{b_i} = \sum_j \overline{\alpha}_{ij} \boldsymbol{a_j} - \overline{\beta}_{ij} \boldsymbol{a_j^\dagger} \tag{11}$$

$$\boldsymbol{c_i} = \sum_j \overline{\gamma}_{ij} \boldsymbol{a_j} - \overline{\delta}_{ij} \boldsymbol{a_j^\dagger} \tag{12}$$

The question to ask now is, are the vacuum states of these two representations of the field the same? The answer is no. The vacuum state of the $\{f_i\}$ representation is the Minkowski vacuum state, which obeys the condition 3. This is the vacuum seen by an observer at rest at $\mathcal{J}^-$; this means that there are no incoming particles in this vacuum state.

However, the vacuum state for an observer at rest at $\mathcal{J}^+$ is different. This observer writes the equations in terms of $\{p_i\}$ and $\{q_i\}$. Let's say she wants to find the number of particles coming in her direction in the vacuum state. This means that she should find the expectation of the number operator which corresponds to no Cauchy data at the event horizon. $N_i^{\text{outgoing}} = \boldsymbol{b_i^\dagger b_i}$ is the number operator for outgoing particles in the ith mode.

$$\langle 0|\boldsymbol{b_i^\dagger b_i}|0\rangle = \sum_j |\beta_{ij}|^2 \tag{13}$$

Recall the discussion in Section I B and at the beginning of this section. We expect these coefficients β_{ij} to depend on only the mass of the black hole (for a Schwarzschild solution), at least asymptotically. Moreover, we can expect some terms related to the surface gravity of the black hole to appear in the expression for the number of particles because κ measures in some way the gradient of the Killing vector. We expect that, asymptotically, there should be a steady outflow of particles from the black hole. Note that the collapse phase will result in a transient phase of particle emission, and the steady state will be reached only after the collapse is complete. During this transient state, there will be emission of a finite number of particles depending on the details of the collapse.

Analysis of the value of the coefficients β_{ij} is made significantly easier by employing the retarded and advanced time parameters in the Eddington-Finkelstein coordinates. These are defined by

$$v = t + r^* \tag{14}$$

$$u = t - r^* \tag{15}$$

where r^* is the tortoise coordinate defined by $dr^* = dr/(1-2M/r)$. Then, using reasonable boundary conditions, $r^* = r + 2M \log\left|\frac{r}{2M} - 1\right|$

Lines of constant v and u are null geodesics, relating to incoming and outgoing null rays respectively.

Using the continuum normalization of the modes (hoping to extract the relevant physical finite normalization solutions by superposing the found solutions), we can now break the field into its Fourier components. In this case, spherical symmetry allows us to write the field as an integral over the spherical harmonics.

$$\textbf{Incoming}: \quad f_{\omega' lm} = \frac{F_{\omega'}(r) Y_{lm}(\theta,\phi)}{\sqrt{2\pi\omega' r^2}} e^{i\omega' v} \tag{16}$$

$$\textbf{Outgoing}: \quad p_{\omega lm} = \frac{P_{\omega}(r) Y_{lm}(\theta,\phi)}{\sqrt{2\pi\omega r^2}} e^{i\omega u} \tag{17}$$

in the region outside the collapsing spherical body. The functions $F_{\omega'}(r)$ and $P_\omega(r)$ are the radial parts of the solutions. Each solution $p_{\omega lm}$ is equivalent to an integral over ω' of the incoming solutions, with the same l and m.

$$p_\omega = \int_0^\infty d\omega' \{\alpha_{\omega\omega'} f_{\omega'} + \beta_{\omega\omega'} \overline{f}_{\omega'}\} \tag{18}$$

Our calculation now has transformed to the computation of $d\omega \int_0^\infty d\omega' |\beta_{\omega\omega'}|^2$ which is the number of particles emitted in the frequency range ω to $\omega + d\omega$. You may ask, why is this easier? The answer is that we can try to figure out independently the forms of the functions p_ω and f_ω and apply an appropriate inverse Fourier transform to obtain the coefficients $\beta_{\omega\omega'}$.

C. Geometric Optics Approximation

What kind of waves belong to p_ω? Let's reverse our arrow of time, and say that a part of p_ω starts at $\mathcal{J}^+$ and is reflected by the static external Schwarzschild field to $\mathcal{J}^-$. These scattered waves must clearly have the same frequency as the incoming radiation from $\mathcal{J}^+$ because they do not interact with the collapsing matter and hence do not contribute to the production of particles. According to this logic, the other part of p_ω enters the collapsing body and is partly scattered, partly reflected through the non-singular center back to $\mathcal{J}^-$. This part of the wave interacts with the collapsing matter and contributes to the production of particles.

The first part can be effectively ignored for particle production. It is important to note that $\alpha_{\omega\omega'}$ corresponding to these will have a $\delta(\omega-\omega')$ term multiplied somewhere.

To solve for the coefficients $\beta_{\omega\omega'}$ corresponding to the second part, we first need to make sensible approximations for the asymptotic steady emission. A reasonable assumption follows from the observation that the density of constant phase outgoing null geodesics increases exponentially with the affine parameter along incoming geodesics as one approaches the event horizon. This suggests that we can make a geometric optics (high frequency or WKB in other terms) approximation for the observed frequency near the event horizon. Following these geodesics back in advanced time, one sees that there exists some finite advanced time v_0 after which all the waves that enter the collapsing matter are completely swallowed by the event horizon and cannot escape.

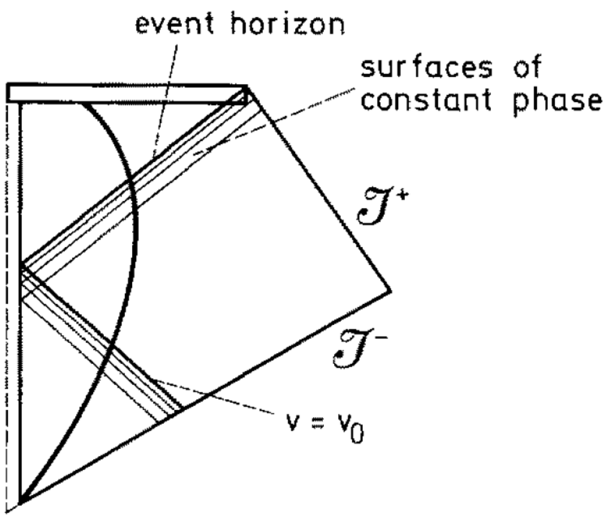

FIG. 4. Solution p_ω has an infinite number of cycles near the event horizon and near the surface $v = v_0$.

Observe that lines connecting the event horizon and surfaces of constant u are past-directed radially pointing null vectors if they are perpendicular to the tangent at the event horizon. Let the tangent along the event horizon be l^a and the future-directed radially pointing normal to the event horizon be n^a. For simplicity, consider the normalization $l^a n_a = -1$.

Consider a small positive number ϵ. Then $-\epsilon n^a$ connects surfaces of constant u to the event horizon. We now check the dependence of the phase of the wave $p_\omega^{(2)}$ on ϵ. Think more graphically. Extending this diagram analytically, we can get that there exists some affine parameter λ which parameterizes the past event horizon such that $\lambda = 0$ and $\frac{dx^a}{d\lambda} = n^a$ at the point of intersection of past and future event horizons respectively.

We have the freedom of choosing the affine parameter in such a way that the Lagrangian when written in terms of u is independent of the affine parameter. This constraint leads to the equation

$$\lambda = -C \cdot \exp(-\kappa u) \tag{19}$$

where κ is the surface gravity of the black hole defined with respect to the time translation Killing vector.

We also have

$$\omega_{\text{obs}} = \omega \cdot u \tag{20}$$

where ω_{obs} is the observed frequency of the wave. Using equation 19, we can write

$$\omega_{\text{obs}} = -\frac{\omega}{\kappa}(\log\epsilon - \log C) \tag{21}$$

Observe that $n^a = D \cdot K^a$ where K^a is the time translation Killing vector and D is some constant determined with respect ϵ since they are parallel. Now, comparing dimensions, we can write $\epsilon = (v_0 - v)/D$. The phase can now be written as

$$-\frac{\omega}{\kappa}\left(\log(v_0 - v) - \log D - \log C\right) \tag{22}$$

for $v < v_0$ otherwise zero.

$$p_\omega^{(2)} \sim \frac{P_\omega(2M)}{\sqrt{2\pi\omega}\cdot 2M}\exp(-i\frac{\omega}{\kappa}\log(\frac{v_0 - v}{CD})) \tag{23}$$

This expression is valid for $v < v_0$ small only. At earlier advanced times, the amplitude will be different. This is only the asymptotic form that we are concerned with.

We can now write the Fourier components of this field.

$$\alpha_{\omega\omega'}^{(2)} \approx \frac{P_\omega(2M)(CD)^{\frac{i\omega}{\kappa}}}{2\pi\sqrt{\omega}}\exp(i(\omega-\omega')v_0)\Gamma\left(1 - \frac{i\omega}{\kappa}\right)(\omega')^{1/2}(-i\omega')^{-1+\frac{i\omega}{\kappa}} \tag{24}$$

$$\beta_{\omega\omega'}^{(2)} \approx -i\alpha_{\omega(-\omega')}^{(2)} \tag{25}$$

$\alpha_{\omega\omega'}$ is analytic in the upper half plane of ω'. However, it has a logarithmic singularity at $\omega' = 0$ stemming from the last term. We want to extend this to the lower half plane to calculate $\beta_{\omega\omega'}$. Since it is a logarithmic singularity, we rotate anticlockwise by π to get the lower half-plane. This gives us a relation between the magnitudes of $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$.

$$|\beta_{\omega\omega'}| = |\alpha_{\omega\omega'}| \cdot \exp\Big(-\frac{\pi\omega}{\kappa}\Big) \tag{26}$$

Recall that the number of particles emitted in some frequency range is determined by the square of the magnitude of the coefficients $\beta_{\omega\omega'}$. One can similarly extend our analysis and say that the number of particles absorbed in the same frequency range is determined by the square of the magnitude of the coefficients $\alpha_{\omega\omega'}$. Note that we will consider the net number of particles going out, which is different from just integrating over frequency in $\beta_{\omega\omega'}$. We expect that if the ratio of this number of particles absorbed to the number of particles emitted is of the form $(\exp(-\beta H) - 1)^{-1}$, where H is the Hamiltonian of the system and β is the inverse temperature, then the system is in thermal equilibrium and behaves like a black body (for bosons). Our analysis produces exactly this in an asymptotic limit.

The factor is $(\exp(2\pi\omega\kappa^{-1}) - 1)^{-1}$ in our case. Taking $H = \hbar\omega = \omega$ and $\beta = 1/k_{\mathrm{B}}T = 1/T$ where T is the temperature of the black hole, we can identify $T = \kappa/2\pi$. This is the temperature of the black hole.

D. Limitation and Extensions

It should be noted that you cannot apply a $M \to 0$ approximation for Minkowski spacetime since this would imply that we are measuring T at the origin. In this limiting case, the singularity and horizon coincide, which is not physical for a no-mass-at-all situation. It is possible to change our sense of κ by fixing the distance at which we measure it and taking the limit of $M \to 0$. This will give a zero temperature, which is expected. In Minkowski spacetime, there is no surface gravity since there is no curvature (not even apparent). We can directly take $\kappa \to 0$. This could give in Equation 26 that no particles are created, even when any arbitrary number of particles are absorbed. However, in the case of a Minkowski spacetime, there are no particle absorbed.

I now without proof summarise the following results:

1. Similar results hold for electromagnetic and linearised gravitational fields. This means that all massless boson fields in a Schwarzschild solution will behave like black body radiation with the same temperature-dependent only on the mass of the black hole. Our solution for a scalar field can be extended to photons (spin 1) and gravitons (spin 2) with the same temperature.

2. For massless fermions fields, like neutrinos, negative frequency components give a positive contribution to the probability flux of the collapsing body. The $|\beta|^2$ is now negative. This will lead to a plus sign instead of a minus sign in the denominator of the Planck distribution. This is expected. The black hole can emit massless fermions at the same temperature as the bosons.

3. Massive fields will only be emitted if the temperature is higher than the rest mass of the particle.

4. This solution can be extended to non-spherical collapse as well. A similar phase difference is obtained as it only depends on geometric optics approximation.

5. Analysis can be extended to rotating and charged black holes to show that radiation emission carries with the mass, charge, and angular momentum of the black hole. ω is replaced by $\omega - m\Omega - q\Phi$ where Ω and Φ are the angular velocity and electric potential of the black hole.

6. Emission of radiation leads to a back reaction on the metric, which manifests as black hole evaporation.

7. Fermionic fields do not exhibit superradiance due to Pauli exclusion.

III. PATH INTEGRAL APPROACH TO HAWKING RADIATION

So far, our derivation of the Hawking radiation has been based on the analysis of time-reversal symmetry of the field equations and the difference in nature of paths followed by null geodesics through the center of a collapsing body. We can think of a different method to think of the same problem. In black body radiation, absorption and emission probabilities are compared. It is also a common method to associate weights to trajectories in a phase space and sum over all possible paths. Both these ideas combined may lead to a path integral formulation of the Hawking radiation.

In classical general relativity, our main goal while reaching the Einstein equations is to find a covariant action for the system. Observe that the propagator in Schrodinger's equation is manifestly covariant [2] due to it being a functor of the action. Here, I present directly the path integral formulation of the Hawking radiation instead of motivating it. One can refer to [2] for a detailed derivation.

For simplicity, let's consider the case of a massive scalar field in curved spacetime. Let x^α describe the spacetime coordinates with respect to an affine coordinate w. Let the path be $(x = x', w = 0)$ to $(x = x, w = W)$. The action functional is then,

$$S[x(w)] = \frac{1}{4}\int_0^W dw\; g(\dot{x}, \dot{x}) \tag{27}$$

for a metric g. The equations of motion for this action give the geodesic equation with respect to w as an affine parameter. One can take w to be the proper time for timelike paths and proper distance for spacelike paths. We have taken this action because it is correct and can be analytically continued from timelike to spacelike paths.

For such a path, the amplitude to go from x' to x in parameter time W is given by $F(x, x', W)$ defined as

$$F(x, x', W) = \int \mathcal{D}x(w) \exp\left[\frac{1}{4} i \int_0^W g(\dot{x}, \dot{x})\ dw\right] \quad (28)$$

where the integral is over all unobserved paths.

The probability amplitude to go from x' to x is in general given by $K(x, x')$ and it does not refer to the parameter time W. It is constructed in two steps: first finding the amplitude to go from x' to x in time W and then integrating over all W. We have calculated (roughly) the first part as $F(x, x', W)$. We give a weight [24] of $i \exp(-im^2 W)$ to each path taking parameter time W. Note that this weight is non-existent for massless fields.

$$K(x, x') = i \int_0^\infty dW\ F(x, x', W) \exp(-im^2 W) \quad (29)$$

here, $W > 0$ is always positive so that the particle propagates only forward in parameter time. Observe that $K(x, x')$ is symmetric in its arguments.

In the present form, the integral is not well-defined. If we analytically continue the parameter time to negative imaginary values (so that the exponential factor is converging) and coordinates to complex values such that the metric has signature +4, then the integral is well-defined. This is the Wick rotation. Let's say the wick-rotated expressions for w, W, and g are ω, Ω, and γ respectively. Then, the Euclidean path integral is given by

$$F(x, x', \Omega) = \int \mathcal{D}x(\omega) \exp\left[-\frac{1}{4} \int_0^\Omega \gamma(\dot{x}, \dot{x})\ d\omega\right] \quad (30)$$

We can also derive the parabolic differential equation satisfied by $F(x, x', \Omega)$

$$\frac{\partial F}{\partial \Omega} = (\tilde{\Box} - \frac{1}{3} R) F \quad (31)$$

where $\tilde{\Box} = \gamma^{ab} \tilde{\nabla}_a \tilde{\nabla}_b$. This is derived in the appendix of [2].

A. Schwarzschild Solution

The Euclidean metric for a Schwarzschild solution in Wick-rotated Kruskal coordinates is given by

$$d\sigma^2 = (32M^3 e^{-r/2M}/r)(d\zeta^2 + dy^2) + r^2 d\Omega^2 \quad (32)$$

where $z = i\zeta$ and r is defined by

$$\zeta^2 + y^2 = (r/2M - 1)e^{r/2M} \quad (33)$$

This has a topology of $R^2 \times S^2$ which is regular when covered by the union of two overlapping charts (one covering $\theta = 0$ and the other $\theta = \pi$). The integral defined by Equation 30 is now well-defined. We assume that once we analytically continue back to the Lorentzian signature (real values of coordinates and parameter time), we get the correct propagator.

This Euclidean metric is positive definite for real values of arguments. Analytically continuing to allow the parameter and coordinates to assume complex values, we get a metric in which singularities are poles. The condition $r > 2M$ obtained by Equation 33 is now relaxed and r is allowed to assume all values which are solutions to equation 33 with complex ζ and y. The metric is now meromorphic with poles at $r = 0$. Path integrals involving paths that extend into $r < 0$ depend on the prescription of contour integration around the poles. Observe that all paths that cross at the same pole coordinates will be equivalent in the sense that they can be continuously deformed into each other except at poles. We choose contours of integration which are analytic continuations of those in the positive-definite section.

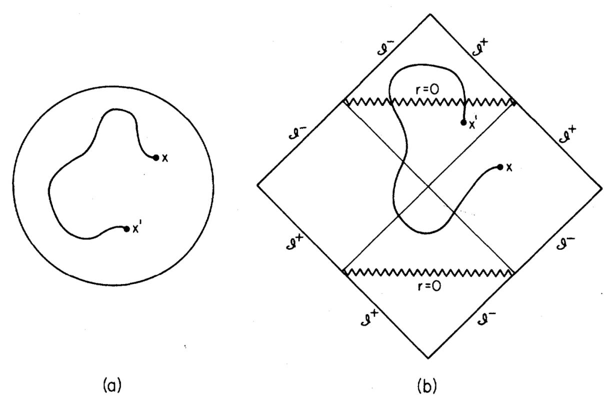

FIG. 5. (a) A general path between x' and x in the complactified Euclidean Schwarzschild metric. There are no singularaities. (b) Typical path in analytically extended Schwarzschild metric. The path crosses the pole at $r = 0$.

Equation (31) in this metric is now reduced to the diffusion equation

$$\frac{\partial F}{\partial \Omega} = \Box F \quad (34)$$

We can continue this solution back to the Lorentzian signature and obtain a Schrodinger equation

$$i \frac{\partial F}{\partial W} = -\Box F \quad (35)$$

with $\Box$ being the d'Alembertian operator. Our next task is to define appropriate boundary conditions for this equation.

B. Boundary Conditions on $F(x, x', W)$

In order to compute $F(x, x', W)$ as a solution of Equation 35, we need to specify some conditions in the small W, and large W limits, and beware of any divergences. A key idea in this section is to regularise the propagator by some method.

For $\Omega = 0$, the propagator is the delta function $\delta^{(4)}(x - x') \cdot \gamma^{-1/2}$. This is essentially imposing that the field is localised at 0 Euclidean time. Another condition is for $\Omega \to \infty$, $F \to 0$. This follows from the exponential damping of the integral in Euclidean signature.

In the small W limit, we can safely say that the paths that contribute the most to the integral are those that are close to the classical path, i.e. the geodesics. Hence, we do not need to figure out the trajectory. It is possible to extract more information from this limit by employing the WKB approximation and constructing an action

$$S(x, x', W) = \frac{1}{4} s(x, x')/W \tag{36}$$

such that we know the trajectory from the numerator and the denominator controls the weight of the path. Here, $s(x, x')$ is the square of the geodesic distance between x and x'. In general, there can be numerous geodesics connecting x and x'. For the next step consider just one geodesic connects the two points.

$$F(x, x', W) = \exp\left[is(x, x')/4W\right] N(x, x', W) \tag{37}$$

where $N(x, x', W)$ is a normalization. By usual WKB methods and noting that $\nabla_\mu s \cdot \nabla^\mu s = 4s$ we can solve equation (35) for small W behaviour of N [25] to get

$$N = D(x, x')W^{-2} + \dots \tag{38}$$

The general form is

$$F(x, x', W) = \sum_c e^{is_c(x,x')/(4W)} \cdot W^{-2}[D_c(x, x') + O(W^{-1})] \tag{39}$$

where the sum is over all classes of geodesics connecting x and x'. It must be noted that this small W behavior breaks down at caustics, at which $s(x, x')$ should have a branch point. Observe that $F(x, x', W)$ does not have branch points.

By computing Equation 34 in any reasonable software, one can determine that the solution for large Ω decreases at least as fast as Ω^{-2} and we can say for sure that the integral converges in the large W limit.

To regularise it in the small W limit (where it is currently diverging), one introduces a convergence factor in the weight such that

$$K(x, x') = i \int_0^\infty dW \ F(x, x', W) \exp(-im^2 W - \epsilon/W) \tag{40}$$

where ϵ is a small positive number. After computing the integral, we can take $\epsilon \to 0$ limit to obtain the correct propagator. This technique has been popularly applied in quantum field theory of scalar field in flat spacetime.

C. Analyticity Properties of the Propagator

The propagator $K(x, x')$ defined by Equation40 is a well-defined object. In the process of deriving this expression, we came across a parameter time Schrodinger equation, and boundary conditions on F. These conditions now imply that $K(x, x')$ is a solution of the inhomogeneous wave equation in Schwarzschild background.

$$(\Box - m^2)K(x, x') = -\delta(x, x') \tag{41}$$

where $\delta(x, x')$ is the delta function mentioned just before Equation 36.

Let's look at this problem from a different approach. Let $K(x, x')$ be a solution of the wave equation with suitable boundary conditions. We can now derive the appropriate boundary conditions on $K(x, x')$ by considering the analytic properties of the propagator.

Detour: *Example of Flat Spacetime*
A massless scalar field in flat spacetime has

$$F(x, x', W) = i(4\pi W)^{-2} \exp\left[is(x, x')/4W\right] \tag{42}$$

with $s(x, x')$ being the square of the interval between x and x'. The propagator is then

$$K(x, x') = -\frac{i}{4\pi^2} \frac{1}{s(x, x') + i\epsilon} \tag{43}$$

As is known, this is the same as the propagator calculated by Feynman in Minkowski spacetime. Analytically continuing the Minkowski spacetime, $s(x, x') = -i\epsilon$ is the pole and this corresponds to null geodesics in the $\epsilon \to 0$ limit. Observe that it is the location of these poles in the complex plane and regularity at infinity that characterizes $K(x, x')$ as a solution of the wave equation. Here, it becomes important to notice the positivity of ϵ.

The solution to poles in usual Minkowski coordinates are

$$t - t' = \pm(|x - x'| - i\epsilon) \tag{44}$$

which shows that future-directed null geodesics lie below the real t-axis and past-directed null geodesics lie above the real t-axis. A similar kind of idea will transpire in the black hole solution but with respect to Kruskal coordinates.

In null Kruskal coordinates U and V,

$$ds^2 = -(32M^3 e^{-r/2M}/r)dUdV + r^2 d\Omega^2 \tag{45}$$

with r defined by

$$UV = (1 - r/2M)e^{r/2M} \tag{46}$$

Here the future event horizon is at $U = 0$ and the past event horizon is at $V = 0$. Analytically continuing V on the future event horizon we get the future complexified horizon, and similarly for the past event horizon. Since the metric is analytic in the Kruskal coordinates on the complexified horizons, the propagator $F(x, x', W)$ is also analytic in these coordinates. Thus, any singularities in $K(x, x')$ arise due to integration over W. We know that the integral converges for large W, so the pole is at the $W = 0$ endpoint.

In the Schwarzschild metric, the conserved quantities for a complexified null geodesic (as obtained from the Lagrangian) correspond to the energy of a particle (e) and its angular momentum (l), both of which are now allowed to take complex values. We represent their ratio as b which is equivalent to the impact parameter in the real case. Scale freedom in the affine parameter indicates that the ratio b uniquely determines any null geodesic.

In fact, when we write the equations of motion for a null geodesic and solve for advanced Eddington-Finkelstein time coordinate on the future complexified horizon, the only free parameter is b. These equations are derived from the standard Lagrangian method. We solve for ν and ϕ with respect to r in the Eddington-Finkelstein coordinate system. If we want the geodesic to cross the future complexified horizon on the real section, then we get the constraint that $0 < b < 3\sqrt{3}M$ with all allowed contours of integration between $r = r' > 2M$ and $r = 2M$.

A more stringent constraint is obtained when we solve the equations of motion for ϕ and observe that ϕ/b is multivalued in the above prescription of contours. To absolve this, it is necessary to restrict b^2 to a given Riemann sheet surface of ϕ/b (called the physical sheet). This sheet must include the real axis in $0 < b^2 < 27M^2$ which correspond to physical real null geodesics. It is simple to observe that any such contour can be continuously deformed into a contour which lies along the real axis.

It is important to note that the condition that ϕ is real on the intersection of null geodesics (starting from some real point) with the future complexified horizon implies that ν, the advanced Eddington-Finkelstein time coordinate, is real on the future complexified horizon on that intersection point. Thus, complex null geodesics starting from x' intersect the complexified future horizon at a real point on for real V. A similar result must hold for U. We immediately obtain that for values of V and U displaced from their real values, the intersection points corresponding to $s(x, x') = -i\epsilon$ are poles of the propagator. But are there singularities in both directions? In which half of the complex plane of V (and U) is the propagator analytic?

This can be answered if we try to find solutions of $s(x, x') = -i\epsilon$. Consider

$$s(x, x') = \left(\frac{\partial s}{\partial V}\right)_{x_0} (V - V_0) + \dots \tag{47}$$

which is the behaviour of s near some point x_0 on the future horizon (V_0 is the value of V at x_0). We know that geodesics in the neighborhood of null geodesics which go into the real future horizon are timelike since they represent massive particles being absorbed by the black hole. So, the sign of the derivative in Equation 47 is negative. This means that the poles of the propagator are in the upper half plane of V. So, $K(x, x')$ is analytic in the lower half plane of V. A similar argument can be made for U, which would imply that the propagator is analytic in the upper half plane of U.

We now investigate analyticity in Schwarzschild coordinate t. Let $t = \tau + i\sigma$ where τ and σ are real. By usual change of coordinates, we write

$$U = |U| \exp(-i\kappa\sigma) \tag{48}$$

$$V = |V| \exp(i\kappa\sigma) \tag{49}$$

which indicates that to keep the propagator well-defined and analytic, we must have $-\pi/\kappa < \sigma < 0$. This ensures that U is in the upper half plane and V is in the lower half-plane. Note that since the metric is independent of t (and hence σ), the wave equation is also independent of σ. This implies that the Cauchy data is regular given our prescription of analyticity.

We have so far derived that the propagator $K(x, x')$ is analytic in a strip of width π/κ below the real t axis. On either side of this strip, there exist poles that correspond to real null geodesics. What are these geodesics and what points do they connect?

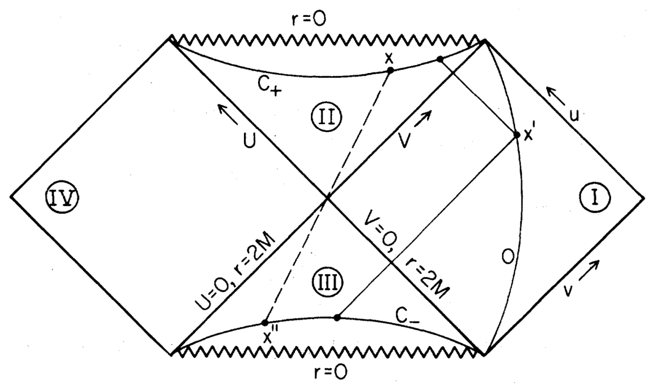

FIG. 6.

Consider the following cases:

1. $x, x' \in$ **Region I:**
   σ is periodic with period $2\pi/\kappa$. There exist real values of t in both the future and past of x' which connect it to the constant r surface of x by real null geodesics. For $t > 0$ real null geodesics correspond to poles slightly displaced above the real t axis. For $t < 0$, the poles are displaced slightly below the real t axis. There is no pole at $\sigma = \pm\pi/\kappa$ because now x lies in Region IV, which is spacelike separated from Region I.

2. $x \in$ **Region II,** $x' \in$ **Region I:**
   σ is again periodic with period $2\pi/\kappa$. Only future-directed real null geodesics can connect x' to x. The poles are displaced slightly above the real t axis.

3. $x \in$ **Region III,** $x' \in$ **Region I:**
   σ is again periodic with period $2\pi/\kappa$. Only past-directed real null geodesics can connect x' to x. The poles are displaced slightly below the real t axis.

The key idea when deriving black hole radiance comes when relating cases 2 and 3. If we make the transformation $t \to t - i\pi/\kappa$ in $x \in$ Region II, then we get some $x'' \in$ Region III. This relates incoming and outgoing radiation.

All these periodic analyticity properties of the propagator $K(x, x')$ are beautifully summarised in Figures 7 and 8.

These properties of the propagator can be extended to more physical and geometrical properties of quantum systems, such as the Berry phase for adiabatic time evolution. I review this effect in Section VI B.

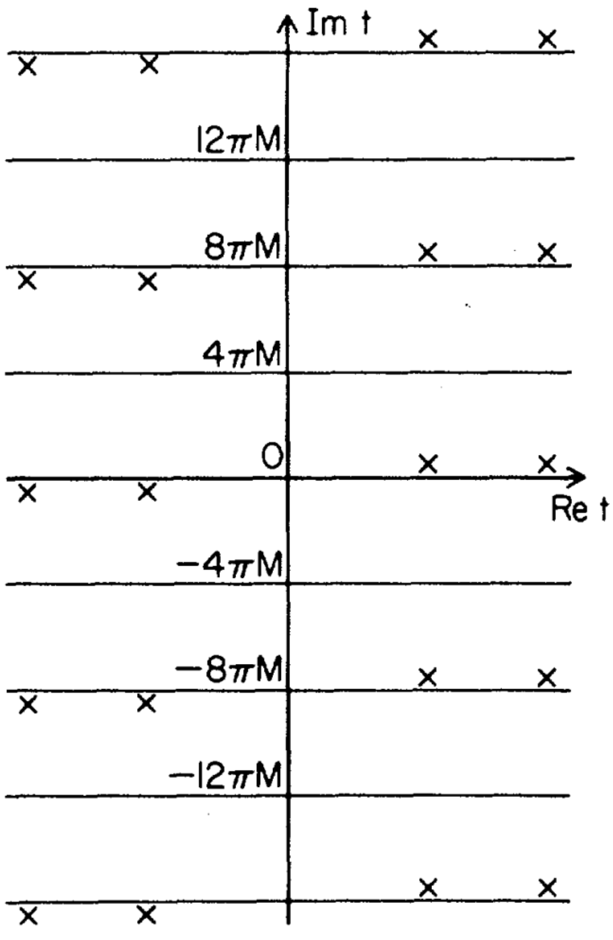

FIG. 7. For Case 1

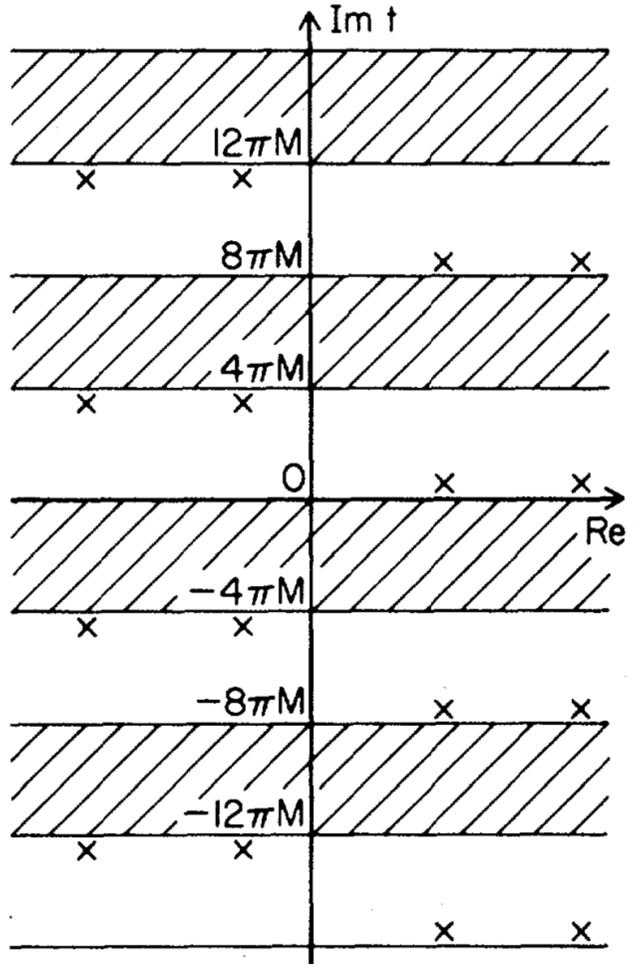

FIG. 8. For Case 2 and 3

Unruh [5] had proposed a similar propagator for the Schwarzschild solution. His work demonstrates that observers moving on lines of constant spatial position in the Schwarzschild metric will see a thermal spectrum of radiation. This can be extended to uniformly accelerating observers in Minkowski spacetime, which observe a thermal bath. Interestingly, we can also see that an observer moving along the horizon will not see any particles. We will see this kind of thing in the next section when we discuss Rindler coordinates to cover a wedge of the Minkowski spacetime.

D. Temperature of the Black Hole

Instead of going into complete mathematical rigor, we apply the methods developed so far in a rough back-of-the-envelope calculation. We want to relate the amplitude of particle emission and absorption. From Section 2, we know that these are related by time-reversal symmetry. We also derived at the end of section 3.C that a periodicity in imaginary time can relate incoming and outgoing radiation.

Let's analytically continue to time coordinate in the Schwarzschild metric. We write the amplitude for a particle with energy E to go from $\vec{R}'$ to $\vec{R}$ as

$$\mathcal{S}_E(\vec{R},\vec{R}') = \int_{-\infty}^{\infty} dt \exp(-iEt) K(0,\vec{R}';t,\vec{R}) \tag{50}$$

This expression is symmetric under the exchange of x and x' which can be attributed to Equation 28.

$$\mathcal{S}_E(\vec{R},\vec{R}') = \int_{-\infty}^{\infty} dt \exp(-iEt) K(t,\vec{R};0,\vec{R}') \tag{51}$$

We know that we can apply the transformation $t \rightarrow t - i\pi/\kappa$ and distort the contour of integration to a strip below the one containing the real t axis. This transformation works because of the analyticity properties of the propagator developed in Section III C.

$$\mathcal{S}_E(\vec{R},\vec{R}') = e^{-\pi E/\kappa} \int_{-\infty}^{\infty} dt e^{-iEt} K(t - i\pi/\kappa, \vec{R}; 0, \vec{R}') \tag{52}$$

We now exchange the arguments in the propagator again. This is equivalent to outgoing radiation leaving from inside the past horizon to some point outside the event horizon. We must square both sides of this equation to relate to probabilities.

$$\mathbb{P}(\text{Schwarzschild BH emits particle with energy E}) = e^{-2\pi E/\kappa} \times \mathbb{P}(\text{Schwarzschild BH absorbs particle with energy E}) \tag{53}$$

We identify the temperature of the black hole as $T_{BH} = \kappa/2\pi$ following Boltzmann, and the subscript BH is kept to honor Beckenstein and Hawking.

Our extended derivation of T_{BH} has now come to an end. It should be kept in mind that we have yet not considered the back reaction of the emission of particles to the metric. It is expected that this is a quasistationary process and the black hole will lose mass and eventually evaporate. When the mass of the black hole reaches the order of Planck mass, then the semiclassical approximation breaks down. We expect the black hole to radiate away all its mass and disappear. This may be used to explain the abundance of photons in the universe in comparison to baryonic matter.

We can, again, extend our analysis to Kerr-Newman black holes. A major assumption is that the propagator is the solution of the inhomogeneous wave equation, following boundary conditions specified by its analytic properties, and that we can generalize the notion of this propagator to higher-order spins and charges. The assumption lies in the fact that exactly the same imaginary time periodicity is observed in the propagator for all fields.

In the case of Kerr black holes, both the imaginary part of time and imaginary part of ϕ are periodic in their analyticity and the periodicity in ϕ depends on the period of $\mathrm{Im}(t)$ (this can be shown by introducing a new azimuthal Killing vector but we skip that here). A similar case is made for Reisner-Nordstrom black holes. These solutions are additive in the sense described at the end of Section 2.

IV. RINDLER COORDINATES

As hinted before, the application of quantum field theory to relativity depends greatly on what coordinate system is used. Here we consider the simple example of two different coordinate systems in the Minkowski spacetime, the Rindler coordinates and the Minkowski coordinates. We will observe that vacuum states in both are different.

Minkowski metric is maximal on the flat spacetime. It covers the entire spacetime in one coordinate chart and is globally hyperbolic. Its vacuum state for massless scalar fields is one with no particles. There is no particle creation in this vacuum state.

A. Davies' Method

The Rindler wedge covers one-fourth of the flat spacetime. Consider a 3+1 D Minkowski spacetime and the transformations:

$$\rho = (x^2 - t^2)^{1/2} \quad 0 < \rho < \infty \tag{54}$$

$$\eta = \tanh^{-1}(t/x) \quad -\infty < \eta < \infty \tag{55}$$

This covers a wedge from $x = t$ to $x = -t$ with $x > 0$. Lines of constant η are straight lines passing through the origin. They are lines of simultaneity for an observer moving with a constant velocity (x/t). Lines of constant ρ are hyperbolas. They represent the worldlines of uniformly accelerating observers moving with an acceleration of ρ^{-1}. $(\rho = 0, \eta = \infty)$ is the future event horizon and $(\rho = 0, \eta = -\infty)$ is the past event horizon.

We can do a very quick analysis following Section 3.

The transformed metric is:

$$ds^2 = d\rho^2 - \rho^2 d\eta^2 + dy^2 + dz^2 \tag{56}$$

$\rho = 0$ is a coordinate singularity that can be removed by using two coordinate charts as in usual polar coordinates. Note that this geometry is flat since it is embedded in the Minkowski flat spacetime. It should remain flat when transformed into an Euclidean theory.

Applying a Wick rotation to η, we get

$$ds^2 = d\rho^2 + \rho^2 d\theta^2 + dy^2 + dz^2 \tag{57}$$

where $\theta = i\eta$. The metric is that of a 4D Euclidean space when we identify θ with period 2π (for it to be regular at the origin with no conical defect or excess). This is again flat.

From our experience with the Schwarzschild solution by Hawking, we know that periodicity in the imaginary part of some coordinate is related in some way to temperature. This will lead to a false result that $T = 1/2\pi$. There is no term of ρ in this temperature. We now formulate reasonable physical grounds to put this mathematical motivation to test.

Davies' [4] method is similar to the quantization scheme proposed by Hawking. He considers a reflecting wall at some arbitrary $x = a$. This wall reflects incoming null geodesics to outgoing null geodesics, which is very similar to the collapsing body in the black hole solution. The similarity does not end here, since the Rindler coordinates cover the patch of the Minkowski spacetime which is causally similar to the black hole solution in its Kruskal representation. We can think of Rindler coordinates as covering the exterior of a black hole.

Solving the massless Hermitian scalar field equation in this metric gives us:

$$\phi(\rho, \eta, y, z) = A_\omega \exp[i\omega(\eta \pm \ln \rho)] \tag{58}$$

with A_ω prescribing the normalization.

For a continuum normalization, we can write the incoming field at large negative η as

$$\phi(\rho, \eta) = \sum_\omega \omega^{-1/2} \left(a_\omega f_\omega + a_\omega^\dagger \overline{f}_\omega\right) \tag{59}$$

where a_ω and $a_\omega^\dagger$ are annihilation and creation operators and $f_\omega \approx \exp[i\omega(\eta + \ln \rho)]$.

We can also break this field, by time-reversal symmetry, into reflected and absorbed modes. Our main task at hand, as in Section 2, is to find the ratio of the amplitudes of these two modes. I do not carry out these calculations here, but the result is that the ratio is $\exp(-2\pi\omega)$. This again points to a temperature of $T = 1/2\pi$. For an observer moving with acceleration ρ^{-1}, wavepackets of particles are different from the observer at rest. Frequency scales as $\omega \times \rho^{-1}$ and the temperature is $T = \rho^{-1}/2\pi$. This can be visualized as a spherically collapsing body, with its exterior wall moving inwards with acceleration κ exuding a temperature of $T = \kappa/2\pi$ with respect to an external observer at rest.

There are, of course, subtle differences between the Black Hole solutions presented above and the present treatment. The most important is that in Hawking radiation, the temperature is an asymptotic constant for null geodesics passing to smaller values of κ but in the Rindler coordinates, the temperature falls off as ρ^{-1} for larger ρ. This is traced to the fact that Rindler coordinates represent an inverse-first power gravitational field, but the black hole solution represents an inverse-square gravitational field. An additional redshift factor reduces the temperature in Rindler for higher ρ.

B. Hartman's Density Matrix Formalism

Hartman's 2015 Physics 7661 course covers this problem in much detail by applying a completely general path integral method. He applies his method to the simple case of a 2D Minkowski spacetime.

For a Rindler coordinate system in this spacetime, we can apply a 2D QFT on a line. Breaking left and right wedges of the Minkowski geometry into Region A and B, following $x > 0$ and $x < 0$ respectively. We can break the whole density matrix $\rho = |0\rangle\langle 0|$ (where these are the Minkowski vacuum states) into ρ_A and ρ_B.

$$\langle \phi_2 | \rho_A | \phi_1 \rangle = \langle \phi_2 | \mathrm{Tr}_B \rho | \phi_1 \rangle \tag{60}$$

gives a diagrammatic representation as

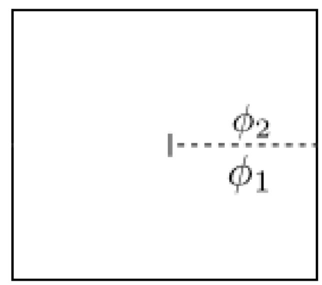

FIG. 9. Transition amplitude in Region A. Summing over all possible paths that do not cross the line $x > 0$, with states ϕ_1 and ϕ_2 specified as initial and final conditions.

Another way to sum over these paths is to Wick rotate η to a Euclidean theory by $\eta = i\theta$, and employing a generator of translations in θ to sum over all paths. This is a generalization of the path integral method. Let $H_{Rindler}$ be the generator of θ translations, and by that logic, it is the operator which governs θ-evolution of any arbitrary operator. Then, the density matrix is reduced to

$$\rho_A = e^{-2\pi H_{Rindler}} \tag{61}$$

which looks like a thermal state with temperature $T = 1/2\pi$ in the Euclidean theory. Since η is the timelike coordinate in our Lorentzian metric, we can analytically continue back to the Lorentzian signature to say that this Hamiltonian is the boost generator corresponding to the causal development in the Rindler wedge. By comparing proper times and frequencies, and keeping track of ρ throughout, one can say that any uniformly accelerating observer in the Rindler wedge will see a thermal bath of particles with temperature $T = \kappa/2\pi$ for acceleration κ. This is the Unruh effect.

We have now completed a review of elementary results in quantum field theory in curved spacetime. We have seen that the path integral method is a powerful tool. Mathematical tools like analytic continuation and Wick rotation have been used to derive results that are physically intuitive. Possible generalizations up to the No-Hair theorems and the Unruh effect have been discussed. An extension to generic accelerations will prove to be important in our understanding of these effects.

V. RESPONSE FUNCTIONS

In 1988, Baurle and Koning[3] produced a breakthrough paper covering the following ideas.

A DeWitt detector can be idealized as an atom in a massless scalar background. The coupling of the atom to the field is given by the interaction Lagrangian

$$\mathcal{L}_{int} = j(x)\phi(x) \tag{62}$$

where $j(x)$ is the current density of the atom given by $j(x) = M(\tau)\delta^3(x - \boldsymbol{\zeta}(\tau))$ for a detector moving along a worldline $x = \zeta(\tau) = (\zeta^0(\tau), \boldsymbol{\zeta}(\tau))$, and $M(\tau)$ describes the internal structure and mechanism of the detector. This toy model treats the atom as a point particle. τ is the proper time of the atom.

Let the atom have some internal Hamiltonian H_0 with a discrete energy spectrum $\{E_n\}$

$$H_0|E_n\rangle = E_n|E_n\rangle \tag{63}$$

Let the atom and the field be a coupled system. Their initial state is

$$|i\rangle = |0\rangle \otimes |E_0\rangle \tag{64}$$

where $|0\rangle$ is the vacuum state of the field. The final state is

$$|f\rangle = |\psi\rangle \otimes |E_n\rangle \tag{65}$$

where $|\psi\rangle$ is an element of an orthonormal basis of the state vector space of the scalar field. We construct the probability to go from $|i\rangle$ to $|f\rangle$ as the square of the transition amplitude. By first-order perturbation theory in ϕ

$$A_{i\to f} = \langle f| - i\int_{-\infty}^{\tau_0} d\tau H_{int}(\tau)|i\rangle \tag{66}$$

$$A_{i\to f} = i\langle E_n|M(0)|E_0\rangle \int_{-\infty}^{\tau_0} d\tau e^{i(E_n-E_0)\tau}\langle\psi|\phi(\zeta(\tau))|0\rangle \tag{67}$$

From which we

$$\mathbb{P}(\tau_0, E_n) := \sum_{\psi} |A_{i\to f}|^2 \tag{68}$$

$$\mathbb{P}(\tau_0, E_n) = |\langle E_n|M(0)|E_0\rangle|^2 \int_{-\infty}^{\tau_0} d\tau \int_{-\infty}^{\tau_0} d\tau' e^{i(E_n-E_0)(\tau-\tau')}\langle 0|\phi(\zeta(\tau))\phi(\zeta(\tau'))|0\rangle \tag{69}$$

where the completeness of the basis of states $|\psi\rangle$ has been used.

From this definition of the probability, we define the excitation rate

$$R(\tau_0, E) := \frac{d\mathbb{P}(\tau_0, E)}{d\tau_0} \tag{70}$$

and the response function

$$S(\tau_0, E) := \frac{R(\tau_0, E)}{|\langle E|M(0)|E_0\rangle|^2} \tag{71}$$

We can write the response function in two forms

$$S(\tau_0, E) = \int_{-\infty}^{\tau_0} d\tau e^{-iE(\tau_0-\tau)}\langle 0|\phi(\zeta(\tau_0))\phi(\zeta(\tau))|0\rangle + \int_{-\infty}^{\tau_0} d\tau e^{-iE(\tau-\tau_0)}\langle 0|\phi(\zeta(\tau))\phi(\zeta(\tau_0))|0\rangle \tag{72}$$

$$S(\tau_0, E) = -i\left(\int_{-\infty}^{\tau_0} d\tau e^{-iE(\tau_0-\tau)}K(\zeta(\tau_0), \zeta(\tau)) + \int_{-\infty}^{\tau_0} d\tau e^{-iE(\tau-\tau_0)}K(\zeta(\tau), \zeta(\tau_0))\right) \tag{73}$$

for the propagator defined by Equation 43, modified such that the denominator is of the form $|t-t'-i\epsilon|^2 - |x-x'|^2$ equation of motion of null geodesics remains the same. We have made this trivial change to make the computation of the response function easier. It makes no physical difference since the only extra terms are second order in ϵ or go to zero in the limit $\epsilon \to 0$. One problem this leads to, however, is that we cannot symmetrize the response function in τ and τ_0, like done earlier. We will see this is not too much of a problem for accelerating observers in a Minkowski background spacetime, since we can use the modified propagator of Equation 43.

A. Intertial Observers

For such cases, the denominator reduces to $|\tau-\tau'-i\epsilon|^2$.

$$S(\tau_0, E) = -\frac{1}{4\pi^2}\int_{-\infty}^{\infty} d\tau \cdot e^{-iE\tau}\frac{1}{(\tau - i\epsilon)^2} \tag{74}$$

By closing the contour in the lower half of the complex τ-plane (basically going around the point $i\epsilon$) one finds

$$S(\tau_0, E) = 0 \tag{75}$$

As expected, there is no excitation for inertial observers.

B. Uniformly Accelerating Observers

We parameterize the world-line of a uniformly accelerating observer ($\rho = \text{const.}$) for which we can use hyperbolic coordinates

$$t = \rho\sinh(\tau/\rho) \quad x = \rho\cosh(\tau/\rho) \tag{76}$$

where τ is the proper time of the observer.

$$S(\tau_0, E) = \frac{-\rho^{-2}}{16\pi^2}\int_{-\infty}^{\infty} d\tau \cdot e^{-iE\tau}\frac{1}{\sinh^2(\frac{1\tau}{2\rho} - i\epsilon)} \tag{77}$$

By the theorem of residues,

$$S(\tau_0, E) = \frac{E}{2\pi}\frac{1}{\exp(2\pi E\rho) - 1} \tag{78}$$

We know the acceleration is $\rho^{-1} = a$, so the temperature is $T = a/2\pi$.

C. Just Starting to Accelerate Forever

Let's think of an observer who is at rest at some $x = \rho$ up to $\tau = 0$ and then starts accelerating indefinitely. We can write his world-line as:

$$\zeta(\tau) = \begin{cases} (t, \rho, 0, 0) & \tau < 0 \\ (\rho\sinh(\tau/\rho), \rho\cosh(\tau/\rho), 0, 0) & \tau > 0 \end{cases} \tag{79}$$

The world-line is differentiable and the four-velocity is continuous. The response function is

$$S(\tau_0, E) = 0 \quad \tau_0 < 0 \tag{80}$$

And for $\tau_0 > 0$

$$\begin{aligned} S(\tau_0, E) = \frac{-1}{4\pi^2\rho^2}\int_0^{\tau_0} d\tau \cdot e^{-iE(\tau_0-\tau)}&\frac{1}{(\sinh(\tau_0/\rho) - \sinh(\tau/\rho) - i\epsilon)^2 - (\cosh(\tau_0/\rho) - \cosh(\tau/\rho))^2} \\ + \frac{-1}{4\pi^2\rho^2}\int_{-\infty}^{0} d\tau \cdot e^{-iE(\tau_0-\tau)}&\frac{1}{(\sinh(\tau_0/\rho) - \tau/\rho - i\epsilon)^2 - (\cosh(\tau_0/\rho) - 1)^2} \\ &+ \text{c.c.} \end{aligned} \tag{81}$$

In dimensionless variables $x := \tau/\rho$, and $\varepsilon := E\rho$, we can write the the dimensionless response function $s = 2\pi S\rho$ as

$$s = \frac{\varepsilon}{e^{2\pi\varepsilon} - 1} + f(x_0, \varepsilon) \tag{82}$$

where $f(x_0, \varepsilon)$ is a function of $x_0 = \tau_0/\rho$ and ε, given by equation (83)

$$f(x_0, \varepsilon) = \frac{1}{\pi}\int_{x_0}^{\infty} dx\cos(\varepsilon x)\left[\frac{1}{2(\cosh x - 1)} - \frac{1}{(x - x_0)^2 + 2(x - x_0)\sinh x_0 + 2(\cosh x_0 - 1)}\right] \tag{83}$$

This function approaches the Planckian distribution (equation 78) for large x_0. Physically, this means that the longer the observer has been accelerating, the more he behaves like an observer who has been accelerating forever.

D. No Fuel

Let's consider an observer who has been accelerating forever, but suddenly stops accelerating at $\tau = 0$. Instead of a rigorous analysis, we try to physically reason what should happen. Numerical integrations of equation (82) show that the response function approaches the Planckian distribution for large τ_0 but not monotonically; it attenuates to that value. In the reversed case we can expect a similar behaviour. This time, the response function will initially behave very close to a Planckian distribution, but will swiftly attenuate to zero.

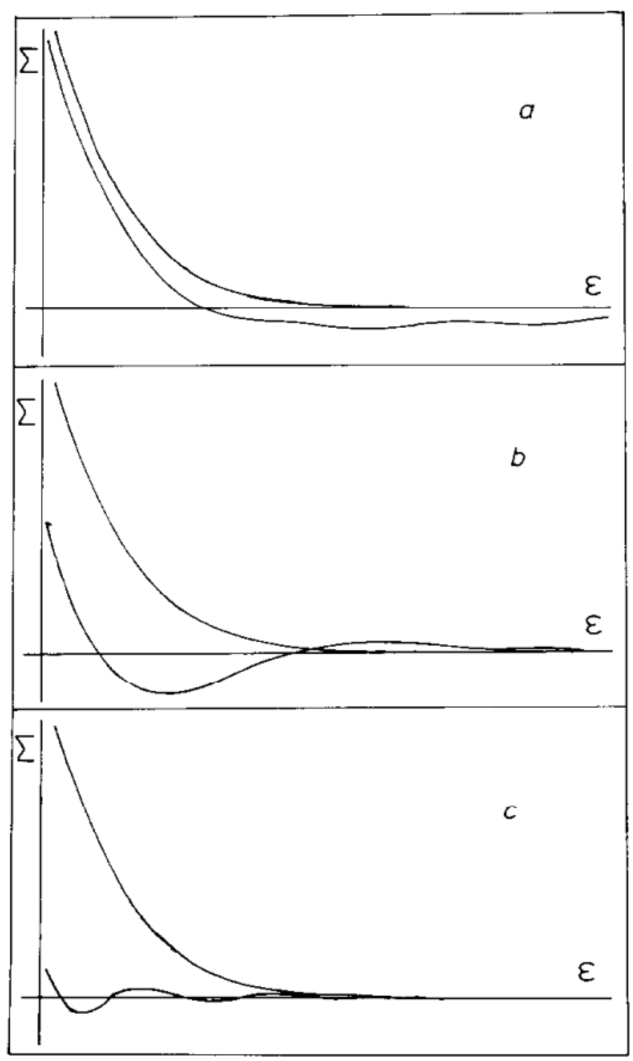

FIG. 10. Σ is the response function for subsection D. a, b, c correspond to $x_0 = 0.2, 2, 10$ respectively. The repeated monotonically decreasing function is the Planckian distribution.

VI. PHENOMENOLOGY

Chronologically, Fulling [13] in 1972 explained why elementary particles are observer dependent. He was followed by Davies [4] in 1975 who derived the Unruh effect. Unruh [5] added substantially to these ideas by proving theoretically that the Minkowski vacuum was thermal with respect to uniformly accelerating observers. In [26], Matsas and Vanzella argued about the existence of the FDU effect by studying the decay of an accelerating proton. They show that there exist two ways to compute the lifetime of uniformly accelerated protons in a standard QFT. One way is to compute it with respect to an inertial observer and the other with respect to a Rindler observer. Though these ideas correspond to different decay processes, the lifetimes turn out to be the same up to an order of 10^{-16}.

Inertial frames observe decay as

$$p^+ \to n^0 + e^+_M + \nu_M \tag{84}$$

which is the usual decay.

Rindler frames observe decay as

$$p^+ + e^-_R \to n^0 + \nu_R \tag{85}$$

$$p^+ + \overline{\nu}_R \to n^0 + e^+_R \tag{86}$$

$$p^+ + \overline{\nu}_R + e^-_R \to n^0 \tag{87}$$

which correspond to the absorption of Rindler electrons and/or antineutrinos to stimulate decay.

A. In the LHC

From our experience of response functions, we need to devise a system that couples small energy scales to very high accelerations. Such a system has been executed in LHC scales [27, 28] where highly energetic positrons were channeled into single crystal silicone sensitive to the channeling oscillation, recoil radiation, and other processes in the comoving frame.

The basic idea behind this experiment is to investigate radiation reaction (recoil) in accelerated systems. The first-order corrected Unruh-DeWitt detector response function (obtained by including quantum corrections to the Larmor formula for accelerated charges) is given by

$$S = \frac{2}{3}\alpha a^2 \left[1 - \frac{12}{m}T\right] \tag{88}$$

where α is the fine structure constant, a is the acceleration, m is the mass of the particle, and T is the temperature of the radiation bath. This corresponds to a correction in the fine structure constant for non-inertial observers. The correction is seen exactly in the Lorentz-Abraham-Dirac equation for the motion of a charged particle.

$$m\frac{du^\mu}{ds} = qF^{\mu\nu}u_\nu + \frac{2}{3}\alpha\left[1 - \frac{24}{m}T\right][J^\mu + a^2u^\mu] \quad (89)$$

To test this, we fit data of $dS/d\omega$ (where ω is the frequency of the radiation) vs ω to the theoretical prediction. This fitting gives values of T and α which are consistent with our model up to a very small chi-squared value. The presence of recoil is enough to confirm the Unruh temperature. One can extend this prediction to the generalized Beckenstein-Hawking entropy law by recalling that Rindler coordinates cover the exterior of a black hole.

B. Using Berry's Phase

Martin-Martinez et al. [29] noticed a very interesting fact. Recall how the propagator is periodic in imaginary time, and how the Rindler coordinates are Euclidean regular and thus periodic in Wick-rotated time. We remarked there (see III C) that such periodicity can be related to Berry's phase.

Let's write the Hamiltonian of an Unruh-DeWitt detector in a Minkowski background, coupled with some massless scalar field and peaked at some frequency mode k.

$$H_T = \Omega_a a^\dagger a + \Omega_b b^\dagger b + \lambda(b^\dagger + b) \times (a^\dagger f_{\Omega_a} + a\overline{f}_{\Omega_a}) \quad (90)$$

where a and b are the annihilation operators of the field and the detector respectively, Ω_a and Ω_b are the frequencies of the field and the detector, and λ is the coupling constant. f_{Ω_a} is the field mode at frequency Ω_a.

$\varphi = kx - \Omega$ is the phase which is periodic for an inertial observer. Details on the adiabaticity of time evolution by choosing the mentioned phase can be found in [29] Defining the Berry phase γ as

$$i\gamma = \oint_R \boldsymbol{A} \cdot d\boldsymbol{R} \quad (91)$$

where $A_i = \langle\psi|\partial_{R_i}|\psi\rangle$ and R is a closed trajectory in the parameter space of H_T.

For the Rindler observer, the state is not pure but mixed. This is a key difference between the inertial and the Rindler observer.

$\delta = \gamma_{Inertial} - \gamma_{Rindler}$ is the difference in the Berry phase for the inertial and the Rindler observer. This depends on the acceleration of the Rindler observer. Computational details can be found in existing literature, but the main idea is that we can observe δ to depend on the acceleration of the Rindler observer. A structured experiment to verify this has not been devised yet.

VII. CONCLUSION

A problem that began with the question of non-uniqueness of quantization in generic metrics [13] led to many monumental discoveries in the field of quantum field theory in curved spacetime. The first application of Fulling's idea was successfully carried out by Davies, and in the same year applied to black hole radiance by Hawking. Inspired by some remarks in Hawking's research, Unruh carried out a detailed analysis of black hole evaporation.

Once the Unruh effect was established theoretically, transient phenomena in accelerating systems in a Minkowski background were studied. Quantum measurement theory was applied to these systems to derive response functions and later devise experiments to detect the predicted phenomena. Meanwhile, rigorous mathematical methods were developed to understand and apply density matrix formalism to path integrals and Euclidean field theory.

Entire branches of physics have been developed from these ideas. A recent example is quantum information paradoxes in black holes and detailed studies of black hole thermodynamics. Radiative effects from black holes may have serious implications in theories of background radiations, dark matter, primordial black holes, and the early universe.

In this paper, I covered detailed derivations of some of these phenomena and their applications. I also discussed some recent experiments and their implications. Four methods (see II, III, IV A and IV B) were used to derive black hole radiance and the Unruh effect. Response functions for generic accelerations were constructed (see V) and their implications were discussed. Finally, I discussed some recent experiments and their implications (see VI).

Proposed ideas to devise experiments to test the Unruh effect (see VI B) are still in their infancy. Claims made in Section VI A are disputed by some researchers. Effort should be made in analogous BEC models or in LHC energy scales to verify the Unruh effect.

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[1] S. W. Hawking, Particle Creation by Black Holes, Commun. Math. Phys. **43**, 199 (1975), [Erratum: Commun.Math.Phys. 46, 206 (1976)].

[2] J. B. Hartle and S. W. Hawking, Path-integral derivation of black-hole radiance, Phys. Rev. D **13**, 2188 (1976).

[3] G. Bäuerle and A. Koning, Transient phenomena in the unruh effect, Physica A: Statistical Mechanics and its Applications **152**, 189 (1988).

[4] P. C. W. Davies, Scalar production in Schwarzschild and Rindler metrics, Journal of Physics A: Mathematical and General **8**, 609 (1975).

[5] W. G. Unruh, Notes on black-hole evaporation, Phys. Rev. D **14**, 870 (1976).

[6] S. W. Hawking, The event horizon, Black Holes. New York, London, Paris: Gordon and Breach (1973).

[7] J. M. Bardeen, B. Carter, and S. W. Hawking, The Four laws of black hole mechanics, Commun. Math. Phys. **31**, 161 (1973).

[8] J. D. Bekenstein, Generalized second law of thermodynamics in black-hole physics, Phys. Rev. D **9**, 3292 (1974).

[9] R. Hagedorn, Thermodynamics of Strong Interactions at High Energy and its Consequences for Astrophysics, **5**, 184 (1970).

[10] J. D. Bjorken and S. D. Drell, *Relativistic quantum mechanics*, International series in pure and applied physics (McGraw-Hill, New York, NY, 1964).

[11] J. A. Wheeler, Geons, Phys. Rev. **97**, 511 (1955).

[12] D. R. Brill and J. B. Hartle, Method of the self-consistent field in general relativity and its application to the gravitational geon, Phys. Rev. **135**, B271 (1964).

[13] S. A. Fulling, *Scalar Quantum Field Theory in a Closed Universe of Constant Curvature*, Doctoral dissertation, princeton university (1972).

[14] S. A. Fulling, Nonuniqueness of canonical field quantization in riemannian space-time, Phys. Rev. D **7**, 2850 (1973).

[15] C. Misner, Stability of Kerr black holes against scalar perturbations, in *Bulletin of the American Physical Society*, Vol. 17 (1972) p. 472.

[16] W. H. Press and S. A. Teukolsky, Floating Orbits, Superradiant Scattering and the Black-hole Bomb, Nature **238**, 211 (1972).

[17] A. A. Starobinskii, Amplification of waves during reflection from a rotating ublack hole, Zh. Eksp. Teor. Fiz. **64**, 48 (1972).

[18] A. A. Starobinskii and S. M. Churilov, Amplification of electromagnetic and gravitational waves scattered by a rotating black hole, Zh. Eksp. Teor. Fiz. **65**, 3 (1973).

[19] C. Barceló, S. Liberati, and M. Visser, Towards the observation of hawking radiation in bose–einstein condensates, International Journal of Modern Physics A **18**, 3735 (2003).

[20] J. Steinhauer, Observation of quantum hawking radiation and its entanglement in an analogue black hole, Nature Physics **12**, 959 (2016).

[21] J. R. M. de Nova, K. Golubkov, V. I. Kolobov, and J. Steinhauer, Observation of thermal hawking radiation and its temperature in an analogue black hole, Nature **569**, 688 (2019).

[22] V. I. Kolobov, K. Golubkov, J. R. M. de Nova, and J. Steinhauer, Observation of stationary spontaneous hawking radiation and the time evolution of an analogue black hole, Nature Physics **17**, 362 (2021).

[23] P. A. M. Dirac, Wave equations in conformal space, Annals Math. **37**, 429 (1936).

[24] R. P. Feynman, Space-time approach to quantum electrodynamics, Phys. Rev. **76**, 769 (1949).

[25] C. Morette, On the definition and approximation of feynman's path integrals, Phys. Rev. **81**, 848 (1951).

[26] G. E. A. MATSAS and D. A. T. VANZELLA, The fulling–davies–unruh effect is mandatory: The proton's testimony, International Journal of Modern Physics D **11**, 1573 (2002), https://doi.org/10.1142/S0218271802002918.

[27] T. N. Wistisen, A. Di Piazza, H. V. Knudsen, and U. I. Uggerhøj, Experimental evidence of quantum radiation reaction in aligned crystals, Nature Communications **9**, 795 (2018), arXiv:1704.01080 [hep-ex].

[28] T. N. Wistisen, A. Di Piazza, C. F. Nielsen, A. H. Sørensen, and U. I. Uggerhøj (CERN NA63), Quantum radiation reaction in aligned crystals beyond the local constant field approximation, Phys. Rev. Res. **1**, 033014 (2019).

[29] E. Martín-Martínez, I. Fuentes, and R. B. Mann, Using berry's phase to detect the unruh effect at lower accelerations, Phys. Rev. Lett. **107**, 131301 (2011).