Colliding Blocks and Pi

Nilakna D. Warushavithana

July 27, 2023

1 Introduction

The mathematical constant pi (π) has captivated human minds for centuries. It represents the ratio of the circumference to the diameter of a circle and is interestingly important in various fields, including mathematics, physics, and engineering. Throughout history, numerous methods have been developed to calculate π with ever-increasing precision, with these techniques evolving from ancient times to present day.

2 Methods of calculating pi

Geometric Methods: The most straightforward way to approximate π is through geometry. Archimedes, one of the earliest mathematicians, used inscribed and circumscribed polygons to bound the value of π, leading to a more accurate approximation. Trigonometric Methods: Trigonometric functions are also utilized to calculate π. For instance, the arctangent function can be employed to find π by using its Taylor series expansion. Series Formulas: There are several infinite series that converge to π. One of the most famous series is the Leibniz formula: $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + 1/9 - ...$ Calculus: Integrals and derivatives have been employed to calculate π in various ways. For instance, the integral of the square root of $(1 - x^2)$ can be used to evaluate π. Monte Carlo Method: This probabilistic method involves random sampling. By generating random points within a square and counting the number that falls within a quarter of a circle inscribed in that square, we can approximate π. Continued Fractions: π can be expressed as an infinite continued fraction, providing a method to approximate its value.

3 A more interesting method

Among these various methods of calculating pi, rather a simple but interesting experimental method was presented by mathematician Galperin in his paper Playing Pool with pi in 2003. It is completely deterministic and can compute pi to any arbitrary precision.

Let us consider the following system of two masses and a vertical wall. We push the block with the higher mass towards the other block, and let them collide with each other and the wall. Then, we count the number of collisions between the blocks plus the hits with the wall, considering each collision as absolutely elastic.

An intriguing result emerges when the mass ratio of the colliding blocks is a power of 100, denoted as N. In this scenario, the collision count precisely corresponds to the first (N+1) digits of pi.

4 Mathematics behind the result

When the blocks collide, they change their velocities following the laws of conservation of momentum and energy, as the blocks experience no external force and the collisions are perfectly elastic.

Let m_1, m_2 be higher and lower mass and v_1, v_2 be their velocities. Then, by the conservation of momentum,

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = c_1 \tag{1}$$

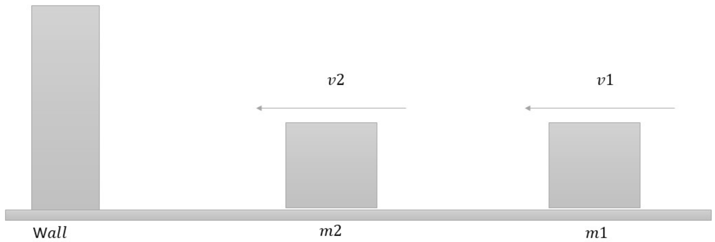

Figure 1: Colliding blocks system

where c is a constant and is the total energy of the system, which is the same as the total kinetic energy. By the conservation of momentum,

$$m_1 v_1 + m_2 v_2 = c_2 \tag{2}$$

By solving equations (1) and (2), we can determine the velocities after each collision. Plotting these two equations with v_1 against v_2 as coordinate axes gives an ellipse and a straight line, respectively, and the points of intersection correspond to the velocity value pairs before and after the collision.

But if we choose the coordinate axes to be $\sqrt{m_1}v_1$ and $\sqrt{m_2}v_2$, equation (1) transforms into an equation of a circle,

$$(\sqrt{m_1}v_1)^2 + (\sqrt{m_2}v_2)^2 = c_1'$$
$$x^2 + y^2 = c_1'$$

and equation (2) as below.

$$\sqrt{m_1}.(\sqrt{m_1}v_1) + \sqrt{m_2}.(\sqrt{m_2}v_2) = c_2'$$
$$\sqrt{m_1}x + \sqrt{m_2}y = c_2'$$

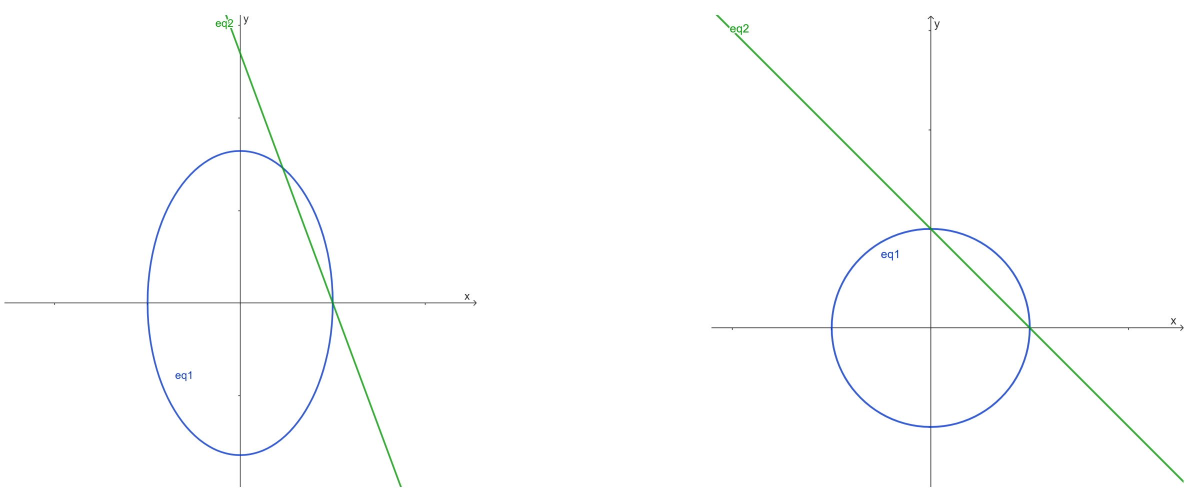

Figure 2: (a) original configuration space (v_1, v_2) (b) transformed configuration space $(\sqrt{m_1}v_1, \sqrt{m_2}v_2)$

The x-intercept point corresponds to the initial velocity values while the other intersection corresponds to the velocities after the first collision, which is between the blocks (x_1, y_1). At the second collision, which is between the small mass and the wall, its velocity changes in the opposite direction while the other block moves with the velocity unchanged. This corresponds to a point on the circle such that $(x_1, -y_1)$. Now we can draw a straight line, through this new point parallel to the first line, for the new momentum equation, and get the velocities after the third collision. By continuing this process, we can find all the velocity value pairs corresponding to each collision at each intersection point between the two curves.

When the velocity value pair has the property $-v_1 > -v_2$ the blocks will not collide again. This corresponds to the region of $(180, 180+45)$ angle of $v_1 - v_2$ plane, enclosed by v_1 and $v_1 = v_2$ line. In the transformed plane, $v_1 = v_2$ is transformed to $x/\sqrt{m_1} = y/\sqrt{m_2}$, and the angle region to $(180, 180+\alpha)$ such that

$$\alpha = \arctan(-\sqrt{m_1}/\sqrt{m_2})$$

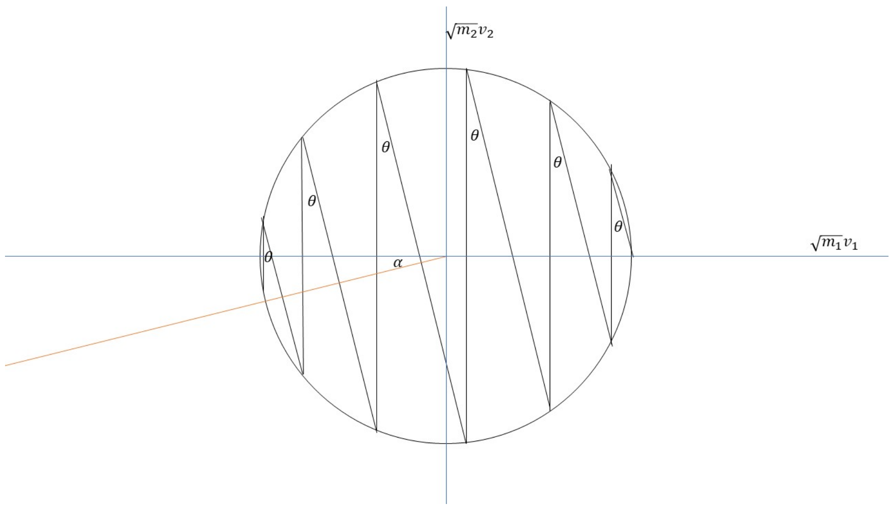

Figure 3: .

As soon as the next configuration point comes to this region the collisions discontinue. Note that the arcs between adjacent configuration points are equal in length as each makes an angle of θ on the circumference and 2θ on the center.

$$\theta = \arctan \frac{\sqrt{m_2}}{\sqrt{m_1}} \tag{3}$$

So, the maximum number of arcs that can be on the circumference without overlapping will give the maximum number of collisions. Let r be the radius and $N \in Z^+$ be the total number of collisions. The above condition satisfies,

$$\begin{aligned} r2\theta N &< 2\pi r \\ N &< \frac{\pi}{\theta} \end{aligned} \tag{4}$$

Let $\frac{m_2}{m_1}$ be a power of 100 denoted d. Then,

$$\theta = \arctan \sqrt{\frac{m_2}{m_1}} = \arctan 10^{-d} \approx 10^{-d}$$

$$N < \pi 10^d$$

Since N is the maximum positive integer less than the given value it takes the integer part of $\pi 10^d$ which represents $d+1$ digits of pi.

5 The simulation

The simulation allows the user to perform collisions varying the mass ratio of the blocks and count the total number of collisions in each scenario. Different options for the mass ratio are given to the user to choose from, including powers of 100 up to the fourth power (10,000) calculating up to the first three digits of pi. The simulation is built with Unity2D.

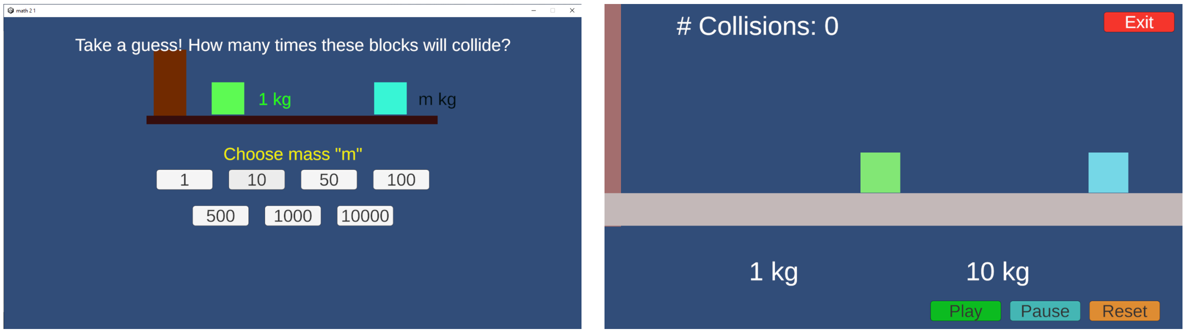

Figure 4: (a) User Interface of the home page (b) Sample simulation page

[1] [2]

References

[1] G. Galperin. Playing pool with π (the number π from a billiard point of view). *Regular and Chaotic Dynamics*, 8, 01 2003.

[2] G. Sanderson. *Why do colliding blocks compute pi.* https://www.3blue1brown.com/lessons/clacks-solutiontitle, 2019.