Analysing LTI: A Study on Linear Time Invarient Systems

Nilakna D. Warushavithana

June 2, 2024

1 Introduction

In the analysis of signals and systems, linear time invarient systems (LTI) allows understanding of many systems in the world and extends beyond by approximating complex systems. This article looks into different methods of analysing LTI and mathematical tools we use to simplify the process.

2 LTI Systems - Linear Time Invarient Systems

2.1 Properties

A system in general is simply a *input - process - output* model with different propoerties. Let $x(t)$ be the input to our system and $y(t)$ be the system output. [fig. 1] In signals and systems context, we refer to the input as 'signal' and the output as 'response'. Usually we take time 't' as the variable but it can be of any choice.

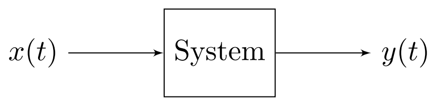

Figure 1: A System

Now, let's understand what LTI is. LTI system has two properties, namely linearity and time invarience. A system is linear if the following property holds true for any input x(t).

Let $x_1(t)$ and $x_2(t)$ be inputs to the system and $y_1(t)$ and $y_2(t)$ be outputs or system reponses to the inputs.

$$x_1(t) \longrightarrow y_1(t)$$
$$x_2(t) \longrightarrow y_2(t)$$

If the weighted sum of individual inputs result in the weighted sum of individual responses, we call the system linear. i.e. if,

$$x_1(t) + x_2(t) \longrightarrow y_1(t) + y_2(t) \Longrightarrow Linear$$

More generally,

$$\sum a_i x_i(t) \longrightarrow \sum a_i y_i(t) \Longrightarrow Linear$$

Essentially, if the input is zero, output is also zero, for linear systems; hence called zero in - zero out propoerty as well. For example $y(t) = x(t) + 3$ is non-linear and it can be clearly identified by giving a zero input and checking the output.

If the system response does not change with time, it it called a time invarient system. In other words, time invarience property tells that a time shift in the input results in an identical time shift in the output [1]: for a time shift t_0, the time input $x(t-t_0)$ gives the same output as $y(t-t_0)$. This is satisfied if the response depends on t inderectly only through x(t).

$$y(t) = f(x(t), t) = f(x(t))$$

For example, $y(t) = sin(x(t))$ is a time invarient system because $y(t-t_0) = sin(x(t-t_0))$. Conversely, $y(t) = tsin(x(t))$ is time varying since $y(t-t_0) \neq tsin(x(t-t_0))$.

Sytems with both propoerties linarity and time invarience are called Linear Time-Invarient systems, in short, LTI systems, and we will continue out study on analysing them.

2.2 Characterizing

To analysie LTI we first need to characterize the system. By characterizing it means to find a system function/ transfer function which represents what a system does to any input. The characterising function should be able to capture a large class of signals and simple enough to do calculations. Examples for such fundamental functions are impulse function, unit step function, and complex exponential function.

2.2.1 Impulse Response and Convolution

We can represent any signal as a convolution with the impulse function as follows for continuous time (CT) and discrete time (DT).

$$x(t) * \delta(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau = \int_{-\infty}^{+\infty} x(t)\delta(t-\tau)d\tau = x(t)\int_{-\infty}^{+\infty} \delta(t-\tau)d\tau = x(t)\cdot 1 = x(t)$$

$$x[n] * \delta[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k] = \sum_{k=-\infty}^{+\infty} x[n]\delta[n-k] = x[n]\sum_{k=-\infty}^{+\infty} \delta[n-k] = x[n]\cdot 1 = x[n]$$

Let the system response to an impulse be h[n].

$$\delta[n] \xrightarrow{LTI} h[n]$$

For each k fixed, x[k] is also fixed and acts as a scaling factor to $x[k]\delta_n[k]$. Because of time invarience, we can write $\delta_n[k] = \delta[n-k]$.

$$\forall k \in \mathbb{Z}, \quad x[k]\delta[n-k] \xrightarrow{LTI} x[k]h[n-k]$$

Due to linearity, when we take the sum of weighted impulses over all k, the output is the weighted sum of impulse response over all k.

$$...+x[-1]\delta[n+1]+x[0]\delta[n]+x[1]\delta[n-1]+... \xrightarrow{LTI} ...+x[-1]h[n+1]+x[0]h[n]+x[1]h[n-1]+...$$

$$\sum_{k=-\infty}^{+\infty} x[k]\delta[n-k] \xrightarrow{LTI} \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

Similarly, for continuous time,

$$\int_{-\infty}^{+\infty} x(t)\delta(t-\tau)d\tau \xrightarrow{LTI} \int_{-\infty}^{+\infty} x(t)h(t-\tau)d\tau$$

We can summerise the LTI system reponses for CT and DT using convolution with impulse response as follows. So, if we know the impulse response of a system we can find system reponse (output) to any input. Hence, the impulse response characterizes the system.

$$CT : x(t) * \delta(t) \xrightarrow{LTI} x(t) * h(t) \tag{1}$$

$$DT : x[n] * \delta[n] \xrightarrow{LTI} x[n] * h[n] \tag{2}$$

2.2.2 Unit Step Response

By substituting $x(t)$ with unit step function $u(t)$ (or $u[n]$ in DT), we find the unit step response $s(t)$ (or $s[n]$ in DT) to the system with impulse response $h(t)$.

$$u[n] = u[n] * \delta[n] \longrightarrow u[n] * h[n] = \sum_{k=-\infty}^{+\infty} u[k]h[n-k] = \sum_{k=0}^{+\infty} h[n-k] = \sum_{k=-\infty}^{n} h[k] = s[n]$$

If unit step response s[n] is known, we can derive the impulse response as $h[n] = s[n] - s[n-1]$, essentially characterizing the LTI system.

Similarly, for continous time,

$$u(t) = u(t) * \delta(t) \longrightarrow u(t) * h(t) = \int_{-\infty}^{+\infty} u(\tau)h(t-\tau)d\tau = \int_{0}^{+\infty} h(t-\tau)d\tau = \int_{-\infty}^{t} h(\tau)d\tau = s(t)$$

Again, if unit step response $s(t)$ is known, impulse response can be derived as $h(t) = \frac{ds(t)}{dt} = s'(t)$ characterizing the LTI system. Unit impulse response is the first derivative of the unit step response.

Unit step response is particularly useful in analsysing RC circuits. When we turn a switch on or off we make voltage a step function (system input) across the capacitor and observe the current (system response).

2.2.3 Complex Exponentials and Eigenfunctions

Let us now find the response to a complex exponential signal. Let $s \in \mathbb{C}$ and $x(t) = e^{st}$. Then the system response $y(t)$ is,

$$y(t) = e^{st} * h(t) = \int_{-\infty}^{+\infty} h(\tau)e^{s(t-\tau)}d\tau = e^{st} \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau = e^{st}H(s)$$

Hence, e^{st} is an eigen function to LTI system where H(s) is the eigenvalue and complex exponential is the eigen-function / eigen-vector.

$$e^{st} \xrightarrow{LTI} H(s)e^{st} \tag{3}$$

where

$$H(s) = \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau \tag{4}$$

Since we can retrieve h(t) the system impulse response by the inverse transformation, H(s) also characterizes the LTI system.

2.3 Analysis

For some s_0, $X(s_0)$ is fixed, and the eqn 3 gives,

$$X(s_0)e^{s_0 t} \xrightarrow{LTI} H(s_0)X(s_0)e^{s_0 t}$$

If we can represent a signal as a sum or an integral of complex exponentials we can utilize this eigen properties of the sytem. Using property in eqn 3,

$$x(t) = const. \int X(s)e^{st}ds \xrightarrow{LTI} const. \int H(s)X(s)e^{st}ds = const. \int Y(s)e^{st}ds = y(t)$$

We can summerize the above as,

$$x(t) \longleftrightarrow X(s)$$
$$h(t) \longleftrightarrow H(s)$$
$$y(t) = x(t) * h(t) \longleftrightarrow X(s)H(s) = Y(s)$$

We now see that finding system response in time domain involves convolution but in frequency domain it is only a simple multiplication. To use this property for LTI analysis, we need to develop ways to convert signals from time domain to frequency domain and vice versa. This opens up a broad field of mathematics called fourier analysis. We develop our study for continous time and discrete time domains and classes of signals of periodic, aperiodic, and unstable. To do the transforms and inverse transforms, from time domain to frequency domain and vice versa, we use the following mathematical tools.

	Continous Time (CT)	Discrete Time (DT)
Periodic	Fourier Series	DT Fourier Series
+Aperiodic	Fourier Transform	DT Fourier Series
+Unstable	Laplace Transform	Z Transform

3 Transforms and Inverse Transforms

3.1 Equations

	Continous Time (CT)	Discrete Time (DT)
Periodic	Fourier Series $$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$ $$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$	DT Fourier Series $$a_k = \frac{1}{N}\sum_{k=<N>} x[n]e^{-jk(2\pi/N)n}$$ $$x(t) = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}$$
+Aperiodic	Fourier Transform $$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$ $$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$$	DT Fourier Series $$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$ $$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$
+Unstable	Laplace Transform $$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \; ; s = \sigma + j\omega$$ $$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds$$ Unilateral Laplace Transform $$X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt \; ; s = \sigma + j\omega$$	Z Transform $$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \; ; z = re^{j\omega}$$ $$x[n] = \frac{1}{2\pi j}\int X(z)z^{n-1}dz$$ Unilateral Z Transform $$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

References

[1] A. V. Oppenheim, A. S. Willsky, and S. H. Nawab, *Signals & Systems.* Pretince Hall, 1983.