MIT 18.01SC Unit 2

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UNIT 2. Applications Of Differentiation

2.A. Approximation

2.A.14

2A-14* Suppose that a piece of bubble gum has volume 4 cubic centimeters.

a) Use a linear approximation to calculate the thickness of a bubble of inner radius 10 centimeters.
(Hint: Start with the relation between the volume V of a sphere and the radius r, and derive the approximate relation between ΔV and Δr.)

b) Find the exact answer.

c) To how many significant figures is the linear approximation accurate? In other words, find the order of magnitude of the difference between the approximation and the exact answer. (Be sure you use enough digits of π to reflect correctly this accuracy!)

d) Use a quadratic approximation to the exact formula for the thickness that you found in part (b) to get an even more accurate estimate.

e) Why is the quadratic term comparable to the error in the accuracy of the linear approximation?

$$\text{Volume of the Gum} = 4 \text{ cm}^3$$

a.
We know that,

$$V(r) = \frac{4}{3}\pi r^3 \quad \Rightarrow \quad \delta V(r) = 4\pi r^2 \, \delta r.$$

This gives us the Linear Approximation result.
Note that in the Linear Approximation, the term $4\pi r^2$ represents the Surface Area of the bubble. The $\delta V(r)$ represents the change in Volume for the change in the radius by δr. So the volume of gum in the bubble $= \delta V(r)$ and the thickness of the gum layer $= \delta r$.

According to the question, $\delta V(r) \approx 4$. Therefore,

$$4\pi r^2\, \delta r \approx 4 \quad \Rightarrow \quad \delta r = \frac{4}{4\pi r^2} = \frac{1}{100\pi}.$$

Thus, Thickness of the bubble

$$\delta r = \frac{1}{100\pi} \approx \mathbf{0.003183} \text{ cm}.$$

b.
Let Δr be the exact thickness. Then the Volume of the gum is

$$\frac{4}{3}\pi\big[(r+\Delta r)^3 - r^3\big] = 4.$$

Substituting $r = 10$, we get

$$\frac{4}{3}\pi\big[(10+\Delta r)^3 - 1000\big] = 4.$$

Dividing both sides by $\frac{4}{3}\pi$,

$$(10+\Delta r)^3 = 1000 + \frac{3}{\pi}.$$

Expanding,

$$1000 + 300\Delta r + 30(\Delta r)^2 + (\Delta r)^3 = 1000 + \frac{3}{\pi}.$$

Thus the exact cubic equation is

$$(\Delta r)^3 + 30(\Delta r)^2 + 300(\Delta r) = \frac{3}{\pi}.$$

Solving, we obtain

$$\Delta r = \mathbf{0.00318} \text{ cm}.$$

c.
We can see that the Linear Approximation is accurate till the 5th decimal place, i.e. accurate to about **5** significant figures.

d.
We have $V''(r) = 8\pi r$. Therefore the Quadratic Approximation Formula is

$$\delta V(r) \approx V'(r)\delta r + \frac{1}{2}V''(r)(\delta r)^2 = 4\pi r^2 \delta r + 4\pi r(\delta r)^2 \approx 4.$$

Substituting $r = 10$,

$$400\pi\,\delta r + 40\pi\,(\delta r)^2 = 4.$$

Dividing by 40π,

$$(\delta r)^2 + 10\,\delta r - \frac{1}{10\pi} = 0.$$

Solving this quadratic, we get

$$\delta r \approx \mathbf{0.00318}\text{ cm.}$$

e.

The quadratic term is comparable due to the fact that the thickness is extremely small with respect to the inner radius. Hence higher-order corrections contribute negligibly.

2.D. Maxima/Minima

2.D.3

2D-3 Coffee in a cup at a temperature $y(t_0)$ at time t_0 in a room at temperature a cools according to the formula (derived in 3F-4); assume $a = 20°\text{C}$ and $c = 1/10$:

$$y(t) = (y(t_0) - a)e^{-c(t-t_0)} + a, \quad t \geq t_0$$

You are going to add milk so that the cup has 10% milk and 90% coffee. If the coffee has temperature T_1 and the milk T_2, the temperature of the mixture

will be $\frac{9}{10}T_1 + \frac{1}{10}T_2$.

The coffee temperature is 100°C at time $t = 0$, and you will drink the mixture at time $t = 10$. The milk is refrigerated at 5°C. What is the best moment to add the milk so that the coffee will be hottest when you drink it?

Let us consider that we add the milk at time $t = t_1$.
Temperature of the coffee before adding milk at t_1 is

$$y(t_1) = (y(t_0) - a)e^{-c(t_1-t_0)} + a$$

Let this be T_1.

According to the question, the milk is at a temperature 5°C.
Temperature at t_1 after adding the milk is

$$0.9T_1 + 0.1 \times 5 = 0.9\big((y(t_0) - a)e^{-c(t_1-t_0)} + a\big) + 0.5$$

According to the question, $y(t_0) = 100$°C as $t_0 = 0$.
So the temperature after adding milk is

$$0.9\big((100 - a)e^{-ct_1} + a\big) + 0.5.$$

If we consider that we drink the coffee at $t = 10$, then we are drinking it after $(10 - t_1)$ time of adding the milk.
So considering our new initial as

$$T_{\text{mix}} = 0.9(100 - a)e^{-ct_1} + 0.9a + 0.5,$$

the final temperature after $10 - t_1$ is

$$T = \big(T_{\text{mix}} - a\big)e^{-c(10-t_1)} + a.$$

Substituting T_{mix},

$$T = \Big(0.9(100 - a)e^{-ct_1} + (0.5 - 0.1a)\Big)e^{-c(10-t_1)} + a.$$

This simplifies to

$$T = a + 0.9(100 - a)e^{-10c} + (0.5 - 0.1a)e^{-c(10-t_1)}.$$

Now substitute the given constants $a = 20$ and $c = \dfrac{1}{10}$. From the cooling law and mixing rule we can simplify the final temperature at $t = 10$ as a function of the milk–adding time t_1 to

$$T(t_1) = 20 + 72e^{-1} - 1.5e^{-1}e^{t_1/10} = 20 + e^{-1}\big(72 - 1.5e^{t_1/10}\big).$$

Now we have got the temperature of the coffee we will drink as a function of the time at which we will add milk.

Differentiating with respect to t_1,

$$\frac{dT}{dt_1} = e^{-1}\left(-1.5 \cdot \frac{1}{10}e^{t_1/10}\right) = -\frac{3}{20}\,e^{-1}e^{t_1/10}.$$

Since

$$e^{-1}e^{t_1/10} > 0 \quad \text{for all } t_1,$$

we have

$$\frac{dT}{dt_1} < 0 \quad \text{for all } 0 \leq t_1 \leq 10.$$

Therefore $T(t_1)$ is strictly decreasing as a function of t_1, and the coffee will be hottest at $t = 10$ if the milk is added as early as possible, i.e. at

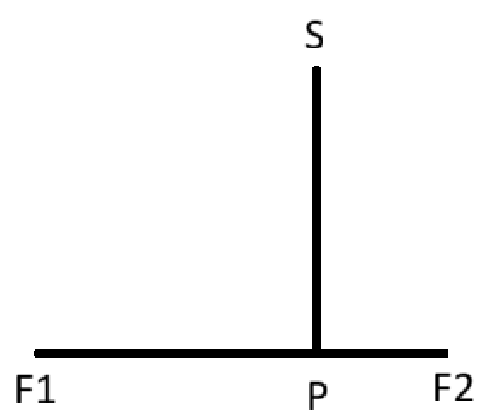
$$t_1 = 0.$$

2.E. Related Rates

2.E.8

2E-8 One ship is sailing east at 60km an hour and another is sailing south at 50km an hour. The slower ship crosses the path of the faster ship at noon when the faster ship was there one hour earlier. Find the time at which the two ships were closest to each other.

We will consider our notations as follows: t represents the amount of time we see backwards from the moment the slower ship crossed the path of the faster ship. So the path crossing took place at $t = 0$.
Refer the image given,



Here the point P denotes the position of the crossing. S denotes the position of the slower ship t time before the crossing took place.

There is a probability of the faster ship to either be at a position like F_1 or at a position like F_2. It depends on the value of t. But there will be no problem for that since we will only use the magnitude of the lengths SP and F_1P or F_2P. Say the latter be called FP.

So $SP = 50t$ and $FP = 60t - 60$

By Pythagoras Theorem, we get the distance between the two ships to be

$$D = \sqrt{(50t)^2 + (60t - 60)^2}$$
$$= 10\sqrt{61t^2 - 72t + 36}$$

So we got the distance between the ships as a function of t.

$$\frac{dD}{dt} = \frac{5(122t - 72)}{\sqrt{61t^2 - 72t + 36}}$$

Solving $\frac{dD}{dt} = 0$, we get,

$$t = \frac{72}{122} = \frac{36}{61} \approx 35\text{min } 24\text{s}$$

So the Ships were closest to each other at the time **11:24:36 am**

2.F. Newton's Method

2.F.5

2F-5 a) Find an initial value x_1 for the zero of $x - x^3 = 0$ for which Newton's method gives an undefined quantity for x_2.

b) Find an initial value x_1 for the zero of $x - x^3 = 0$ such that Newton's method bounces back and forth between two values forever. Hint: use symmetry.

c) Find the largest interval around each of the roots of $x - x^3 = 0$ such that Newton's method converges to that root for every initial value x_1 in the interval. Hint: Parts (a) and (b) should help.

a.
For the initial value to get undefined,

$$f'(x_o) = 0$$

$$x = \pm\frac{1}{\sqrt{3}}$$

b.
We are given

$$f(x) = x - x^3.$$

Then

$$f'(x) = 1 - 3x^2.$$

Newton's Method defines the iteration

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

If we consider this as a function $N(x)$ (the Newton map), we have

$$N(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x - x^3}{1 - 3x^2}$$

$$N(x) = \frac{x(1-3x^2) - (x - x^3)}{1 - 3x^2} = \frac{x - 3x^3 - x + x^3}{1 - 3x^2} = \frac{-2x^3}{1 - 3x^2},$$

for $x \neq \pm\frac{1}{\sqrt{3}}$ (where $f'(x) = 0$).

Applying Newton's method again, the second step is

$$x_2 = N(x_1) = N(N(x)).$$

$$N(N(x)) = N(x) - \frac{f(N(x))}{f'(N(x))}.$$

$$N(x) = \frac{-2x^3}{1 - 3x^2}.$$

$$N(N(x)) = \frac{-2\big(N(x)\big)^3}{1 - 3\big(N(x)\big)^2}.$$

$$\big(N(x)\big)^2 = \frac{4x^6}{(1-3x^2)^2}, \qquad \big(N(x)\big)^3 = \frac{-8x^9}{(1-3x^2)^3}.$$

Therefore

$$N(N(x)) = \frac{-2\left(\dfrac{-8x^9}{(1-3x^2)^3}\right)}{1 - 3\left(\dfrac{4x^6}{(1-3x^2)^2}\right)} = \frac{16x^9}{(1-3x^2)^3} \cdot \frac{1}{1 - \dfrac{12x^6}{(1-3x^2)^2}}.$$

Hence

$$N(N(x)) = \frac{16x^9}{(1-3x^2)^3} \cdot \frac{(1-3x^2)^2}{(1-3x^2)^2 - 12x^6} = \frac{16x^9}{(1-3x^2)\big[(1-3x^2)^2 - 12x^6\big]}.$$

Now we solve for a 2-cycle by solving

$$N(N(x)) = x.$$

So

$$\frac{16x^9}{(1-3x^2)\big[(1-3x^2)^2 - 12x^6\big]} = x.$$

We first assume $x \neq 0$ (the case $x = 0$ will be handled separately, since it is an obvious fixed point of N). Multiplying both sides by the denominator,

$$16x^9 = x(1-3x^2)\big[(1-3x^2)^2 - 12x^6\big].$$

Dividing by x (since $x \neq 0$),

$$16x^8 = (1-3x^2)\big[(1-3x^2)^2 - 12x^6\big].$$

We expand

$$(1-3x^2)^2 = 1 - 6x^2 + 9x^4,$$

so

$$(1-3x^2)^2 - 12x^6 = 1 - 6x^2 + 9x^4 - 12x^6.$$

Therefore

$$(1-3x^2)\big[(1-3x^2)^2 - 12x^6\big] = (1-3x^2)\big(1 - 6x^2 + 9x^4 - 12x^6\big).$$

Let $a = x^2$. Then

$$(1-3a)(1-6a+9a^2-12a^3) = 1 - 9a + 27a^2 - 39a^3 + 36a^4.$$

So our equation becomes

$$16x^8 = 1 - 9x^2 + 27x^4 - 39x^6 + 36x^8.$$

Bringing all terms to one side,

$$0 = 1 - 9x^2 + 27x^4 - 39x^6 + 36x^8 - 16x^8 = 1 - 9x^2 + 27x^4 - 39x^6 + 20x^8.$$

Let $u = x^2$. Then $u \geq 0$ and the equation becomes

$$20u^4 - 39u^3 + 27u^2 - 9u + 1 = 0.$$

We can factor this quartic. First observe that $u = 1$ is a root:

$$20 - 39 + 27 - 9 + 1 = 0.$$

So $(u-1)$ is a factor, and polynomial division gives

$$20u^4 - 39u^3 + 27u^2 - 9u + 1 = (u-1)\big(20u^3 - 19u^2 + 8u - 1\big).$$

Next we notice that $u = \dfrac{1}{5}$ is also a root of the cubic:

$$20\left(\frac{1}{5}\right)^3 - 19\left(\frac{1}{5}\right)^2 + 8\left(\frac{1}{5}\right) - 1 = \frac{20}{125} - \frac{19}{25} + \frac{8}{5} - 1 = 0.$$

So $(u - \frac{1}{5})$ is a factor of the cubic, and we obtain

$$20u^3 - 19u^2 + 8u - 1 = (u - \tfrac{1}{5})\big(20u^2 - 15u + 5\big) = 5(u - \tfrac{1}{5})(4u^2 - 3u + 1).$$

Thus the full factorization is

$$20u^4 - 39u^3 + 27u^2 - 9u + 1 = 5(u-1)\left(u - \frac{1}{5}\right)(4u^2 - 3u + 1).$$

The quadratic $4u^2 - 3u + 1$ has discriminant

$$\Delta = (-3)^2 - 4 \cdot 4 \cdot 1 = 9 - 16 = -7 < 0,$$

so it has no real roots. Therefore the real solutions for u are

$$u = 1 \quad \text{or} \quad u = \frac{1}{5}.$$

Recalling that $u = x^2$, we obtain

$$x^2 = 1 \quad \text{or} \quad x^2 = \frac{1}{5},$$

so

$$x = \pm 1, \qquad x = \pm \frac{1}{\sqrt{5}}.$$

We also remember that we excluded $x = 0$ earlier by dividing by x, but $x = 0$ is clearly a fixed point of the Newton map $N(x)$, since $f(0) = 0$. Altogether, the real solutions of $N(N(x)) = x$ are

$$x = 0, \ \pm 1, \ \pm \frac{1}{\sqrt{5}}.$$

Here $x = 0, \ \pm 1$ are fixed points (period 1). The remaining two points

$$x = \pm \frac{1}{\sqrt{5}}$$

are genuine period-2 points: starting from $x_1 = \frac{1}{\sqrt{5}}$, the Newton iteration alternates forever between $-\frac{1}{\sqrt{5}}$ and $\frac{1}{\sqrt{5}}$.

c.

For $f(x) = x - x^3$, the Newton map is

$$N(x) = \frac{-2x^3}{1 - 3x^2}.$$

The roots $-1, 0, 1$ are fixed points of N.

Convergence to 0 requires

$$|N(x)| < |x| \iff 5x^2 < 1,$$

so Newton's method converges to 0 for all

$$x_1 \in \left(-\tfrac{1}{\sqrt{5}}, \ \tfrac{1}{\sqrt{5}}\right).$$

The derivative vanishes at $\pm\frac{1}{\sqrt{3}}$, which split the basins for ± 1. Thus,

$$x_1 > \frac{1}{\sqrt{3}} \implies x_n \to 1, \qquad x_1 < -\frac{1}{\sqrt{3}} \implies x_n \to -1.$$

So the largest intervals are:

$$(-\infty, -\tfrac{1}{\sqrt{3}}),\ (-\tfrac{1}{\sqrt{5}}, \tfrac{1}{\sqrt{5}}),\ (\tfrac{1}{\sqrt{3}}, \infty)$$

2.G. Mean Value Theorem

2.G.4

2G-4 A polynomial $p(x)$ of degree n has at most n distinct real roots, but it may have fewer — for instance, $x^2 + 1$ has no real roots at all. However, show that if $p(x)$ does have n distinct real roots, then $p'(x)$ has $n - 1$ distinct real roots.

Let x_1 and x_2 be two consecutive roots of p then we get:

1. $p(x)$ is continuous in $[x_1, x_2]$ because it is a Polynomial.
2. $p(x)$ is differentiable in (x_1, x_2) because it is a Polynomial.
3. $p(x_1) = p(x_2) = 0$

By the above three facts, we can say $p(x)$ satisfies the conditions of Roles Theorem in $[x_1, x_2]$.
By Roles Theorem, we can say that $p'(x)$ has a root in (x_1, x_2).

Expanding this fact over n roots of $p(x)$, taking the consecutive roots at a time, we can see that,
$p'(x)$ **has** $n - 1$ **roots.**