MIT 18.01SC Unit 4

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December 2025

Unit 4. Application of Integration

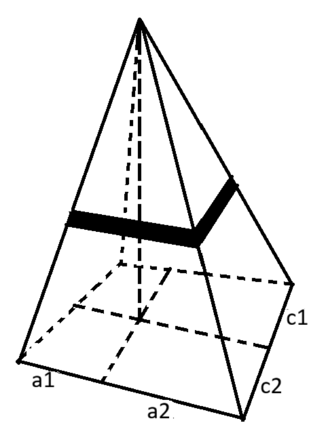
4.B. Volume by Slicing

4.B.3

Show that the volume of a pyramid with a rectangular base is $bh/3$, where b is the area of the base and h is the height. (Show in the process that the proportions of the rectangle do not matter.)

Solution:-

Let the base has a Dimension of $(a1 + a2) \times (c1 + c2)$
Consider the figures below :-



The black Strip in second figure is considered to be an Elementary Volume horizontal to the ground at a length x from the tip.

Let that Elementary Volume has cross-section with sides $(s1_1 + s1_2)$ and $(s2_1 + s2_2)$. Such that,

$$(s1_1 + s1_2) \parallel (a1 + a2) \text{ and } (s2_1 + s2_2) \parallel (c1 + c2)$$

By similarity Property of Triangle we get,

$$\frac{x}{s1_1} = \frac{h}{a1} \; ; \; \frac{x}{s1_2} = \frac{h}{a2} \; ; \; \frac{x}{s2_1} = \frac{h}{c1} \; ; \; \frac{x}{s2_2} = \frac{h}{c2}$$

This gives us

$$s1_1 = \frac{a1}{h}x \; ; \; s1_2 = \frac{a2}{h}x \; ; \; s2_1 = \frac{c1}{h}x \; ; \; s2_2 = \frac{c2}{h}x$$

Volume of the Elementary Volume is given by

$$dV = (s1_1 + s1_2)(s2_1 + s2_2)dx$$

$$= (a1 + a2)(c1 + c2)\frac{x^2}{h^2}dx$$

This Gives us Volume of the Pyramid

$$V = \int_0^h (a1 + a2)(c1 + c2)\frac{x^2}{h^2}dx$$

$$= (a1 + a2)(c1 + c2)\frac{h}{3}$$

Note that

$$(a1 + a2)(c1 + c2) = b = \text{Area of the Base}$$

Therefore Volume becomes

$$V = \frac{bh}{3}$$

This shows that the Volume is independent of $(a1 + a2)$ and $(c1 + c2)$.
Hence the Proportion of Base Does not Matter.

4.F. Arc Length

4.F.6

a) The cycloid given parametrically by $x = t - \sin(t)$, $y = 1 - \cos(t)$ describes the path of a point on a rolling wheel. If t represents time, then the wheel is rotating at a constant speed. How fast is the point moving at each time t? When is the forward motion (dx/dt) largest and when is it smallest?
b) Find the length of the cycloid for one turn of the wheel. (Use a half angle formula.)

Solution:-

a.

Let s be the length along the Path. So $\frac{ds}{dt}$ represents the Speed of the Point We know the Formula for differential length along the Curve is

$$ds = \sqrt{(dx)^2 + (dy)^2}$$

$$= \sqrt{(d(t - \sin(t)))^2 + (d(1 - \cos(t)))^2}$$

$$= \sqrt{(1 - \cos(t))^2 + (\sin(t))^2}dt$$

$$ds = \sqrt{2 - 2\cos(t)}dt$$

$$\frac{ds}{dt} = \sqrt{2 - 2\cos(t)}$$

Therefore Speed of the moving Point is given by

$$\sqrt{2 - 2\cos(t)}$$

Let v_x be the forward motion.

$$v_x = \frac{dx}{dt} = 1 - \cos(t)$$

From $-1 \leq \cos(t) \leq 1$, we get

$$(v_x)_{<max>} = 2 \textbf{ at } t = (2n + 1)\pi$$

$$(v_x)_{<min>} = 0 \textbf{ at } t = 2m\pi$$

$m, n \in Z$

b.

Parametric equation of a general Cycloid in terms of angle of rotation (θ) is given by

$$< (b\theta - a\sin(\theta)), (b - a\cos(\theta)) >$$

Here b is the radius of the Disc that is rolling, a is the Distance of the point from the Center of the Disc and θ is the Angular Rotation of the Disc.
Now θ can be given by

$$\theta = \omega t$$

Here ω is the constant Angular velocity and t is the Rotation
So our equation can also be expressed as

$$< (b\omega t - a\sin(\omega t)), (b - a\cos(\omega t)) >$$

Now comparing the general equation $< (b\omega t - a\sin(\omega t)), (b - a\cos(\omega t)) >$ with the equation given in the question $< (t - \sin(t)), (1 - \cos(t)) >$, we get

$$\omega = 1; a = 1; b = 1$$

So

$$\theta = t$$

Hence, we can write the equation of motion of the given cycloid as

$$< (\theta - \sin(\theta)), (1 - \cos(\theta)) >$$

For one Complete turn of a wheel, we can take t varies from 0 to 2π.

Therefore, Length covered by a Cycloid is given as by

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{(1-\cos\theta)^2 + \sin^2\theta}\, d\theta.$$

Simplify the integrand:

$$\sqrt{(1-\cos\theta)^2 + \sin^2\theta} = \sqrt{2 - 2\cos\theta} = 2\sin\frac{\theta}{2} \qquad (0 \le \theta \le 2\pi).$$

Therefore

$$L = \int_0^{2\pi} 2\sin\frac{\theta}{2}\, d\theta = 4\big[-\cos\frac{\theta}{2}\big]_0^{2\pi} = 4\big(-\cos\pi + \cos 0\big) = 4(1+1) = 8.$$

Hence the point travels a distance $L = 8$ in one complete turn of the wheel.