MIT 18.02SC Problem Set 2

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Part II

Problem 2

A manufacturing process mixes three raw materials M_1, M_2, and M_3 to produce three products P_1, P_2, and P_3. The ratios of the amounts of the raw materials (in the order M_1, M_2, M_3) which are used to make up each of the three products are as follows:

For P_1 the ratio is 1:2:3; for P_2 the ratio is 1:3:5; and for P_3 the ratio is 3:5:8. In a certain production run, 137 units of M_1, 279 units of M_2, and 448 units of M_3 were used. The problem is to determine how many units of each of the products P_1, P_2, and P_3 were produced in that run.

a) Set this problem up in matrix form. Use the letter A for the matrix, and write down the (one-line) formula for the solution in matrix form.

b) Compute the inverse matrix of A and use it to solve for the production vector P.

c) Find a choice for the ratios for the third product (in lowest form), different from the other two ratios, and for which the resulting system has non-unique solutions.

Solutions:-

a.
For the sake of simplicity, we will assume that the products and materials are produced and used in whole number ratios. In other words we will frame the problem as :

1 unit of P_1 need 1 unit of M_1 + 2 unit of M_2 + 3 unit of M_3. and so on...

So our Mathematical equations becomes,

$P_1 = 1M_1 + 2M_2 + 3M_3$
$P_2 = 1M_1 + 3M_2 + 5M_3$
$P_3 = 3M_1 + 5M_2 + 8M_3$

Now let us consider x_1 units of P_1, x_2 units of P_2 and x_3 units of P_3 are produced.

So we get,

Amount of M_1 consumed $= 1x_1 + 1x_2 + 3x_3$

Amount of M_2 consumed $= 2x_1 + 3x_2 + 5x_3$

Amount of M_3 consumed $= 3x_1 + 5x_2 + 8x_3$

According to question,

$1x_1 + 1x_2 + 3x_3 = 137$

$2x_1 + 3x_2 + 5x_3 = 279$

$3x_1 + 5x_2 + 8x_3 = 448$

In matrix notation it becomes,

$$\begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 5 \\ 3 & 5 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 137 \\ 279 \\ 448 \end{bmatrix}$$

The question states us to write it as $AX = \begin{bmatrix} 137 \\ 279 \\ 448 \end{bmatrix}$ or, $X = A^{-1} \begin{bmatrix} 137 \\ 279 \\ 448 \end{bmatrix}$

b.

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 5 \\ 3 & 5 & 8 \end{bmatrix}$$

Note:- a rough work for finding Adjoint of the A,

```
3       5       1       3
    o       o       o
5       8       3       5
    o       o       o
2       3       1       2
    o       o       o
3       5       1       3
```

Elements of $adj.(A)$ is given by the o in the above rough work. Each o is the value of determinant formed by the 4 elements surrounding it.

So, $adj.(A) = \begin{bmatrix} -1 & 7 & -4 \\ -1 & -1 & 1 \\ 1 & -2 & 1 \end{bmatrix}$

Now $|A| = 1(3 \times 8 - 5 \times 5) - 1(2 \times 8 - 3 \times 5) + 3(2 \times 5 - 3 \times 3) = 1$

$A^{-1} = \frac{adj.(A)}{|A|} = \begin{bmatrix} -1 & 7 & -4 \\ -1 & -1 & 1 \\ 1 & -2 & 1 \end{bmatrix}$

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 & 7 & -4 \\ -1 & -1 & 1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 137 \\ 279 \\ 448 \end{bmatrix} = \begin{bmatrix} 24 \\ 32 \\ 27 \end{bmatrix}
$$

So $x_1 = 24$, $x_2 = 32$, and $x_3 = 27$

This means 24 units of P_1, 32 units of P_2 and 27 units of P_3 were produced.

c.

Let the ratio for the third product be $a : b : c$. Then our coefficient matrix becomes

$$
A = \begin{bmatrix} 1 & 1 & a \\ 2 & 3 & b \\ 3 & 5 & c \end{bmatrix}.
$$

The determinant of A is

$$
|A| = a + c - 2b.
$$

For the system to have non-unique solutions, we need $|A| = 0$. Therefore

$$
a + c = 2b.
$$

For consistency of the system $AX = b$, the third row of A must be the same linear combination of the first two rows as the third entry 448 is of the first two RHS entries 137 and 279.

Let those constants be p and q. Then we need

$$
448 = 137p + 279q.
$$

Taking $q = 1$ gives

$$
137p = 448 - 279 = 169, \qquad p = \frac{169}{137}.
$$

Now the third row is the combination

$$
(a, b, c) = p(1, 2, 3) + q(1, 3, 5) = \left(\tfrac{306}{137}, \tfrac{749}{137}, \tfrac{1192}{137} \right).
$$

Multiplying by 137 to get an integer ratio, we obtain

$$
a : b : c = 306 : 749 : 1192.
$$

This ratio is different from the given ratios $1 : 2 : 3$ and $1 : 3 : 5$, and it satisfies $a + c = 2b$, so the matrix is singular. The system is consistent, and therefore the solutions are non-unique.

Problem 3

For any plane P which is not parallel to the x-y plane, define the steepest direction on P to be the direction of any vector which lies in P and which makes the largest (acute) angle with the x-y plane.

(a) Let P be the plane through the origin with normal vector n. Derive a formula, in terms of n, for a vector w which points in the steepest direction on P.

(b) Now let P be the plane through the origin which contains two non-parallel vectors u and v, where u and v do not both lie in the x-y plane. Derive a formula, in terms of u and v, for a vector w which points in the steepest direction on P.

Solutions:-

Let us consider the vector, such that it lies on a plane P. Now think about this: the vector which makes the largest angle with x-y plane, is the one which is parallel the orthogonal projection of the z-axis on the plane.

Consider the following diagram:
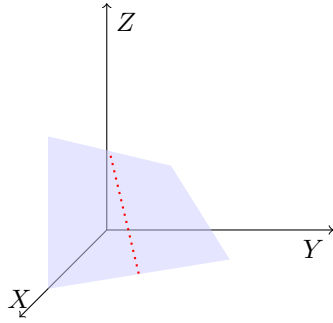
   The dotted line becomes the steepest slope if it is the Orthogonal projection of $\hat{k}$ on the plane.

a.

So, w is orthogonal projection of $\hat{k}$ on the plane P, which passes through the origin.

Therefore, $\overrightarrow{w} = \hat{k} - \frac{\hat{k} \cdot \overrightarrow{n}}{||\overrightarrow{n}||^2} \overrightarrow{n}$

b.

If the plane contains two non-parallel vectors u and v, the vector normal to the plane is $u \times v$ this is kind of equivalent to n.

So, our answer becomes, $\overrightarrow{w} = \hat{k} - \frac{\hat{k} \cdot \overrightarrow{u} \times \overrightarrow{v}}{||\overrightarrow{u} \times \overrightarrow{v}||^2} (\overrightarrow{u} \times \overrightarrow{v})$