MIT 18.02SC Problem Set 7

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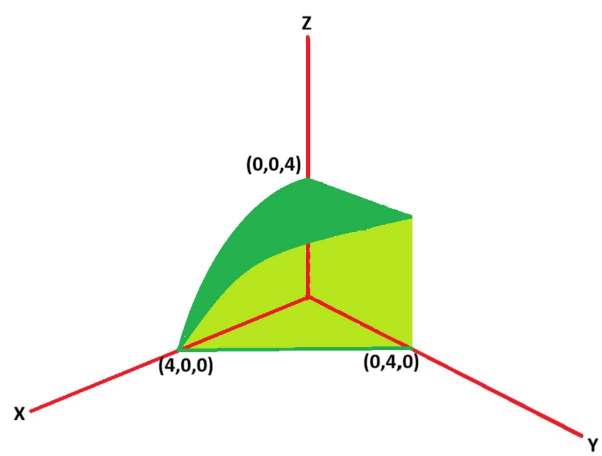
Part 2

Problem 1

Problem 1 (3: 1,2)
a) Sketch the solid in the first octant bounded by the xy, yz and xz coordinate planes, the plane $x + y = 4$ and the surface $z = \sqrt{4 - x}$.
b) Find the volume of the solid of part (a).

Solution:-

a.



b.

Here,

$$0 \leq x \leq 4$$

$$0 \leq y \leq 4 - x$$

$$0 \leq z \leq \sqrt{4 - x}$$

So, our Volume becomes :

$$\int_{x=0}^{4} \int_{y=0}^{4-x} \sqrt{4-x} dydx$$

$$= \int_{x=0}^{4} (4-x)^{\frac{3}{2}} dx$$

$$= [-\frac{2}{5}(4-x)^{\frac{5}{2}}]_{x=0}^{4}$$

$$= \frac{64}{5}$$

Problem 4 & 5

Background for problems 4 and 5:
In fluid mechanics the *fluid flow map* φ is defined as follows: if (x, y, z) is the position of a point mass in the flow at time $t = 0$, then $(X, Y, Z) = \varphi(x, y, z, t)$ is the downstream position of that same point mass after an elapsed time t.
The standard assumptions on φ are that it is smooth and one-to-one.
We will call a flow *volume incompressible* if for any bounded space region $\mathcal{R}$ in the flow, the volume of $\varphi(\mathcal{R}, t)$ is the same as the volume of $\mathcal{R}$ for all t. In other words, if $\mathcal{R}_t = \varphi(\mathcal{R}, t) = \{\varphi(x, y, z, t) \mid (x, y, z) \text{ in } \mathcal{R}\}$ is the the region formed by the points from $\mathcal{R}$ which have been carried downstream by the flow, then $\mathcal{R}_t$ can have a different shape but must have the same volume at all times, if the flow is 'v-i'.
In problems 4 & 5, we'll take the simpler case of a 2D flow (which could be e.g. a 2D section of a flow in 3D). Let $(X(x, y, t), Y(x, y, t)) = \varphi(x, y, t)$. Note that then by definition, the velocity vectors $\mathbf{v}(x, y, t)$ of the flow are given by $\mathbf{v}(x, y, t) = \langle \frac{\partial X}{\partial t}, \frac{\partial Y}{\partial t} \rangle$.
A v-i flow in this case is one that preserves area, since area is the 2D version of volume.

For a fixed value of t, let $J(x, y, t) = \dfrac{\partial(X, Y)}{\partial(x, y)}$ be the Jacobian of the transformation $(x, y) \mapsto (X(x, y, t), Y(x, y, t))$. The general change-of-variables formula says that if a region $\mathcal{R}$ goes to a region $\mathcal{R}'$ by a transformation $(x, y) \mapsto (X, Y)$ with Jacobian $\frac{\partial(X,Y)}{\partial(x,y)}$, then the areas of $\mathcal{R}$ and $\mathcal{R}'$ are related by $A(\mathcal{R}') = \iint_{\mathcal{R}} \mid J(x, y) \mid dA$. Here this gives that $A(\mathcal{R}_t) = \iint_{\mathcal{R}} \mid J(x, y, t) \mid dA$, and therefore that a 2D flow is v-i if and only if $\mid J(x, y, t) \mid = 1$ for all (x, y, t).
In problems 5 & 6 we will look at three examples of 2D flows, v-i and non-vi, in order to illustrate this idea.
Example A: $\varphi(x, y, t) = ((1 + t)x, (1 + t)y)$;
$\mathcal{R}$ = the triangle with vertices at $(0, 0)$, $(1, 1)$ and $(1, -1)$.
Example B: $\varphi(x, y, t) = (x \cos t - y \sin t, x \sin t + y \cos t)$;
$\mathcal{R}$ = the triangle with vertices at $(0, 0)$, $(2, 0)$ and $(2, 1)$.
Example C: $\varphi(x, y, t) = \left((1 + t)x, (\frac{1}{1+t})y\right)$;
$\mathcal{R}$ = the rectangle with vertices at $(1, 1)$, $(1, 4)$, $(2, 1)$ and $(2, 4)$.

Problem 4 (4: 2,2)
In each of the cases A, B and C:
i) compute the Jacobian $J(x, y, t)$
ii) compute the area $A(\mathcal{R}_t)$

Problem 5 (9: 3,3,3)
In each of the cases A, B and C:
i) Sketch the pattern of the flow paths over time, including some starting from points in $\mathcal{R}$.
ii) Compute the velocity vectors of the flow and sketch in a few on the flow lines.
iii) Sketch the regions $\mathcal{R}$ and $\mathcal{R}_t$ and check this against the areas calculated in problem 5 to see if it looks correct in each case. Use the following values for t:
A: $t = 2$, B: $t = \frac{\pi}{2}$, and C: $t = 3$.
Suggestion for sketching $\mathcal{R}_t$: see where the corners of $\mathcal{R}$ end up on $\mathcal{R}_t$.
For the flow lines in case C, also note that3$X(x, y, t)Y(x, y, t) = xy$ for all values of t.
iv) Identify which flows are v-i, using the computed results (and the sketches).
v) In the cases A and B, describe the what the flow is doing.

Solution 4:-

i.

Case A

$$J(x,y,t) = \begin{vmatrix} \frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} \\ \frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y} \end{vmatrix} = \begin{vmatrix} (1+t) & 0 \\ 0 & (1+t) \end{vmatrix} = (1+t)^2$$

Case B

$$J(x,y,t) = \begin{vmatrix} \frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} \\ \frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y} \end{vmatrix} = \begin{vmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{vmatrix} = 1$$

Case C

$$J(x,y,t) = \begin{vmatrix} \frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} \\ \frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y} \end{vmatrix} = \begin{vmatrix} (1+t) & 0 \\ 0 & \frac{1}{(1+t)} \end{vmatrix} = 1$$

ii.

Case A

$$A(\mathcal{R}_t) = \int\int_{\mathcal{R}} (1+t)^2 dxdy = (1+t)^2 \int\int_{\mathcal{R}} dxdy = (1+t)^2 A(\mathcal{R})$$

Case B

$$A(\mathcal{R}_t) = \int\int_{\mathcal{R}} 1dxdy = \int\int_{\mathcal{R}} dxdy = A(\mathcal{R})$$

Case C

$$A(\mathcal{R}_t) = \int\int_{\mathcal{R}} 1dxdy = \int\int_{\mathcal{R}} dxdy = A(\mathcal{R})$$

Solution 5:-

ii.

Case A

$$v =< \frac{\partial X}{\partial t}, \frac{\partial Y}{\partial t} >=< x, y >$$

Case B

$$v =< \frac{\partial X}{\partial t}, \frac{\partial Y}{\partial t} >=< -x\sin(t) - y\cos(t), x\cos(t) - y\sin(t) >$$

Case C

$$v =< \frac{\partial X}{\partial t}, \frac{\partial Y}{\partial t} >=< x, \frac{-y}{(1+t)^2} >$$

iv.

For a flow to be 'v-i', the Area should remain constant.
Therefore the Condition for a flow to be 'v-i' comes out to

$$A(\mathcal{R}_t) = A(\mathcal{R}) \text{ for any } t$$

Case A

$$A(\mathcal{R}_t) = (1+t)^2 A(\mathcal{R})$$

Therefore it is not 'v-i'

Case B

$$A(\mathcal{R}_t) = A(\mathcal{R})$$

Therefore it is 'v-i'

Case C

$$A(\mathcal{R}_t) = A(\mathcal{R})$$

Therefore it is 'v-i'