# MIT 18.02SC Problem Set 8

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## Part 2

### Problem 1

**Problem 1** (3 :1,2)
a) Write down in $xy$-coordinates the vector field $\mathbf{F}$ whose vector at $P = (x, y)$ runs in the vertical direction from $P$ to the line $L : x + y = 1$.

b) Show *without calculation* that for this field, $\int_C \mathbf{F} \cdot d\mathbf{r} \neq 0$, if $C$ is any positively oriented right triangle with legs parallel to the axes and hypotenuse on $L$.

### Solution :-

a.

Since the Vector Field runs Vertically, it does not have any $x$ component. Let us consider the Field to be

$$F =< 0, \bar{y} >$$

Now $\bar{y}$ is the Vertical Distance of the point to the line $x+y = 1 \implies y = 1-x$
At a point $(x, y)$ this distance comes out to be the distance between $(x, y)$ and $(x, 1 - x)$

Now the Direction of pointing is towards the line. This gives us

$$\bar{y} = (1 - x) - (y) = 1 - x - y$$

Hence,

$$F =< 0, 1 - x - y >$$

b.

Let $C = C_1 + C_2 + C_3$

- $C_1 \implies (a, b)$(here $a + b < 1$) to $(1 - b, b)$

- $C_2 \implies (1-b, b)$ to $(a, 1-a)$
- $C_3 \implies (a, 1-a)$ to $(a, b)$

This gives us

$$\int_C F\cdot dr = \int_{C_1} F\cdot dr + \int_{C_2} F\cdot dr + \int_{C_3} F\cdot dr$$

Calculating for $C_1$

$$\int_{C_1} F\cdot dr = \int_{C_1} < 0, 1-x-y > \cdot dr = \int_{C_1} (1-x-y)dy$$

$$= \int_{C_1} (1-x-b)0 \text{ [Since } y = b; dy = 0$$

$$= 0$$

Calculating for $C_2$

$$\int_{C_2} F\cdot dr = \int_{C_2} < 0, 1-x-y > \cdot dr = \int_{C_2} (1-x-y)dy$$

$$= \int_{C_2} (0)dy \text{ [Since in } C_2 : x + y = 1$$

$$= 0$$

Calculating for $C_3$

$$\int_{C_3} F\cdot dr = \int_{C_3} < 0, 1-x-y > \cdot dr = \int_{C_3} (1-x-y)dy$$

$$= \int_{C_3} (1-a-y)dy \text{ [Since } x = a$$

$$= \int_{1-a}^{b} (1-a-y)dy \text{ [Since } y \text{ varies from } 1-a \text{ to } b$$

$$= (1-a)(a+b-1) - \frac{b^2-(1-a)^2}{2} = -\frac{(a+b-1)^2}{2}$$

This gives us

$$\int_C F\cdot dr = \int_{C_1} F\cdot dr + \int_{C_2} F\cdot dr + \int_{C_3} F\cdot dr$$

$$= 0 + 0 + (-\frac{(a+b-1)^2}{2})$$

$$\implies \int_C F\cdot dr = -\frac{(a+b-1)^2}{2}$$

Therefore,

$$\int_C F\cdot dr \neq 0$$

## Problem 3

**Problem 3** (4 :2,2)
a) Let $\mathbf{F} = -\boldsymbol{\nabla}(\ln r)$. Show that $\mathbf{F}$ is the field in problem 2, when $c = 1$.

b) Show that for any path $C$ joining two points $P_1$ and $P_2$ the values of $\int_C \mathbf{F} \cdot d\mathbf{r}$ depends only on the ratio $r_2/r_1$, where $r_i$ is the distance of $P_i$ to the origin.

## Solution :-

a.

$$r = \sqrt{x^2 + y^2}$$

$$\begin{aligned} F &= -\nabla(\ln(r)) = -\nabla(\ln(\sqrt{x^2+y^2})) \\ &= -\Big(\frac{x}{x^2+y^2}\hat{i} + \frac{y}{x^2+y^2}\hat{j}\Big) \end{aligned}$$

$$\implies F = -\frac{x\hat{i} + y\hat{j}}{x^2+y^2}$$

b.

$$\int_C F \cdot dr = -\int_{C:P_1 \,\mathrm{to}\, P_2} \nabla(\ln(r)) \cdot dr = -(\ln(r_2) - \ln(r_1))$$

where $r_i = |P_i|$ is the distance of $P_i$ from the origin.

**This gives us**

$$\int_C F \cdot dr = -\ln\left(\frac{r_2}{r_1}\right)$$

**Hence,** $\int_C F \cdot dr$ **depends only on** $\frac{r_2}{r_1}$

## Problem 4

**Problem 4** (6 :1,2,1,2)
Let $\mathbf{F} = \boldsymbol{\nabla}(x^2y + 2xy^2)$. Let $C$ be the quarter ellipse $9x^2 + 4y^2 = 1$ running from the positive $x$-axis to the positive $y$-axis.

a) Write the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ in the form $\int_C M\,dx + N\,dy$.

b) By parametrizing the curve $C$ write the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ as an explicit definitie integral in $t$. (*Do not evaluate.*)

c) Use the Fundamental Theorem to (easily) compute $\int_C \mathbf{F} \cdot d\mathbf{r}$.

d) Use path independence to choose a different path and (again easily) compute $\int_C \mathbf{F} \cdot d\mathbf{r}$.

## Solution :-

a.

$$F = \nabla(x^2y + 2xy^2) =< (2xy + 2y^2), (x^2 + 4xy) >$$
$$\implies F \cdot dr =< (2xy + 2y^2), (x^2 + 4xy) > \cdot < dx, dy >$$
$$= (2xy + 2y^2)dx + (x^2 + 4xy)dy$$

Therefore,

$$\int_C F \cdot dr = \int_C [(2xy + 2y^2)dx + (x^2 + 4xy)dy]$$

b.

Parametrizing $C$ in terms of $\theta$, where $\theta$ is the Angle made by the Line Segment joining Origin and the Point with the $x$ axis, we get,

$$x = \frac{\cos\theta}{3} \text{ and } y = \frac{\sin\theta}{2} \text{ where } 0 \leq \theta \leq \frac{\pi}{2}$$

$$\implies dx = -\frac{\sin\theta}{3}d\theta \text{ and } dy = \frac{\cos\theta}{2}d\theta$$

So our Integral becomes

$$\int_C F \cdot dr = \int_0^{\frac{\pi}{2}} [(2\frac{\cos\theta}{3}\frac{\sin\theta}{2} + 2\frac{\sin^2\theta}{2^2})(-\frac{\sin\theta}{3}) + (\frac{\cos^2\theta}{3^2} + 4\frac{\cos\theta}{3}\frac{\sin\theta}{2})(\frac{\cos\theta}{2})]d\theta$$

c.

Here, $P_{initial} = (\frac{1}{3}, 0)$ and $P_{final} = (0, \frac{1}{2})$

$$\int_C F \cdot dr = \int_C \nabla(x^2y + 2xy^2).dr$$

$$= [x^2y + 2xy^2]_{x=0,y=\frac{1}{2}} - [x^2y + 2xy^2]_{x=\frac{1}{3},y=0}$$

$$\implies \int_C F \cdot dr = 0$$

d.

We will select the easy path in the format $C = C_1 + C_2$

- $C_1$: $(\frac{1}{3}, 0)$ to $(\frac{1}{3}, \frac{1}{2})$
- $C_2$: $(\frac{1}{3}, \frac{1}{2})$ to $(0, \frac{1}{2})$

Calculating for $C_1$

$$x = \frac{1}{3}; dx = 0$$

$$\int_{C_1} F \cdot dr = \int_{C_1} [(2xy + 2y^2)dx + (x^2 + 4xy)dy]$$

$$= \int_{y=0}^{\frac{1}{2}} [\frac{1}{9} + \frac{4y}{3}]dy = \frac{2}{9}$$

Calculating for $C_2$

$$y = \frac{1}{2}; dy = 0$$

$$\int_{C_2} F \cdot dr = \int_{C_2} [(2xy + 2y^2)dx + (x^2 + 4xy)dy]$$

$$= \int_{x=\frac{1}{3}}^{0} [x + \frac{1}{2}]dx = -\frac{2}{9}$$

Therefore,

$$\int_{C} F \cdot dr = \int_{C_1} F \cdot dr + \int_{C_2} F \cdot dr = \frac{2}{9} - \frac{2}{9} = 0$$