MIT 18.02SC Problem Set 12

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Part 2

Problem 1

Problem 1 (5: 1,2,1,1)
Let $\mathbf{F}(x,y,z) = \left(\frac{-z}{x^2+z^2}\right)\mathbf{i} + y\,\mathbf{j} + \left(\frac{x}{x^2+z^2}\right)\mathbf{k}$ defined for all points (x,y,z) in 3-space not on the y-axis (that is, all points for which $x^2+z^2>0$).
a) By direct computation, show that $\nabla \times \mathbf{F} = \mathbf{0}$ for all points not on the y-axis.
b) By direct computation, show that $\oint_{C_1} \mathbf{F}\cdot d\mathbf{r} = 0$ where C_1 is the closed curve defined by $x^2+y^2=1,\ z=1$.
c) Can you use Stokes' Theorem and the fact (from part (a)) that $\nabla \times \mathbf{F} = \mathbf{0}$ to conclude that $\oint_{C_2} \mathbf{F}\cdot d\mathbf{r} = 0$ when C_2 is the closed curve defined by $x^2+z^2=1,\ y=0$? Why/ why not ?
d) Compute out $\oint_{C_2} \mathbf{F}\cdot d\mathbf{r}$ and see what happens.

Solution :-

a.

$$\nabla \times (F) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \left(\frac{-z}{x^2+z^2}\right) & y & \left(\frac{x}{x^2+z^2}\right) \end{vmatrix} = 0$$

b.

Parameterizing the Curve C_1, we get

$$x = \cos(\theta)\ ;\ y = \sin(\theta)\ ;\ z = 1$$

$$dx = -\sin(\theta)d\theta\ ;\ dy = \cos(\theta)d\theta\ ;\ dz = 0$$

So our Integral Computes as

$$\oint_{C_1} F \cdot dr = \int_{C_1} \Big(\frac{-z}{x^2+z^2}\Big)dx + ydy + \Big(\frac{x}{x^2+z^2}\Big)dz$$

$$= \int_0^{2\pi} \left(\Big(\frac{-1}{(\cos(\theta))^2+1^2}\Big)(-\sin(\theta)d\theta) + (\sin(\theta))(\cos(\theta)d\theta) + \Big(\frac{\cos(\theta)}{(\cos(\theta))^2+1^2}\Big)0\right)$$

$$= -\int_0^{2\pi} \Big(\frac{1}{\cos^2(\theta)+1} + \cos(\theta)\Big)d(\cos\theta) = 0$$

c.

We can't use Stokes' Theorem for the Curve C_2 since for Stokes Theorem to be applicable F needs to be smooth at every point on the Surface we choose Bounded by the Curve. But F has a Singularity at every point on y axis and there can be no Surface with C_2 as Boundary which does not have any common Point with y axis.

d.

Parameterizing the Curve C_1, we get

$$x = \cos(\theta)\ ;\ y = 0\ ;\ z = \sin(\theta)$$

$$dx = -\sin(\theta)d\theta\ ;\ dy = 0\ ;\ dz = \cos(\theta)d\theta$$

So our Integral Computes as

$$\oint_{C_2} F.dr = \int_{C_2} \Big(\frac{-z}{x^2+z^2}\Big)dx + ydy + \Big(\frac{x}{x^2+z^2}\Big)dz$$

$$= \int_0^{2\pi} \left(\Big(\frac{-\sin(\theta)}{(\cos(\theta))^2+(\sin(\theta))^2}\Big)(-\sin(\theta)d\theta) + (0)(0) + \Big(\frac{\cos(\theta)}{(\cos(\theta))^2+(\sin(\theta))^2}\Big)(\cos(\theta)d\theta)\right)$$

$$= \int_0^{2\pi} (d\theta) = 2\pi$$

Problem 2

Problem 2 (5: 2,2,1)
Suppose the field $\mathbf{F}$ in the previous problem is replaced by the field
$\mathbf{G}(x,y,z) = \left(\dfrac{x}{x^2+y^2+z^2}\right)\mathbf{i} + \left(\dfrac{y}{x^2+y^2+z^2}\right)\mathbf{j} + \left(\dfrac{z}{x^2+y^2+z^2}\right)\mathbf{k}$
defined for all points $(x,y,z) \neq (0,0,0)$.

a) Show that $\nabla \times \mathbf{G} = \mathbf{0}$ for all points $(x,y,z) \neq (0,0,0)$

b) Can you use Stokes' Theorem in this case and the fact that $\nabla \times \mathbf{G} = \mathbf{0}$ for all points $(x,y,z) \neq (0,0,0)$ to conclude that $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ for all simple closed curves C which do not pass through the origin?

c) Explain the difference between these two cases in term of the connectedness type of the domains of definition of the two fields.

Solution :-

a.

$$\nabla \times (G) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{\sqrt{x^2+y^2+z^2}} & \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{z}{\sqrt{x^2+y^2+z^2}} \end{vmatrix} = \frac{0}{\sqrt{x^2+y^2+z^2}} = 0$$

b.

Yes. It is applicable to use Stokes Theorem here. We can get Surfaces having those Curves, not passing through Origin as Boundaries such that G is smooth at all points on those Curves.

c.

The difference between the two cases lies in the connectedness of the domains on which the vector fields are defined.

In Problem 1, the vector field is not defined along an entire axis. As a result, the domain is not simply connected: there exist closed curves which do not pass through the singularity, but which cannot be spanned by any surface that avoids the singular set. Hence, Stokes' Theorem cannot be applied to such curves, and it is possible for the circulation to be nonzero even though the curl vanishes everywhere in the domain.

In contrast, for the field G in this problem, the domain is $\mathbb{R}^3 \setminus \{(0,0,0)\}$, which is simply connected. Any closed curve that does not pass through the origin can be spanned by a surface that also avoids the origin. Since $\nabla \times \mathbf{G} = 0$ everywhere on such a surface, Stokes' Theorem applies and implies that the circulation around any such closed curve is zero.

Thus, the key difference is that the domain of G is simply connected, while the domain of the field in Problem 1 is not.