MIT 18.02SC Problem Set 11

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Part 2

Problem 4

Problem 4 (7 2,2,3)
Let $f(x, y, z) = 1/\rho$.

a) Compute $\mathbf{F} = \boldsymbol{\nabla} f$ and show div$\mathbf{F} = 0$.

b) Find the outward flux of $\mathbf{F}$ through the sphere of radius a centered at the origin. Why does this not contradict the divergence theorem?

c) Imititating what we did with Green's theorem, use the extended divergence theorem to show that the flux of $\mathbf{F}$ through any closed surface surrounding the origin is -4π.

Solution :-

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

a.

$$F = \nabla(f) = \nabla\left(\frac{1}{\sqrt{x^2 + y^2 + z^2}}\right)$$

Hence we get,

$$F = -\frac{< x, y, z >}{\rho^3}$$

$$\nabla \cdot (F) = -\nabla \cdot \left(\frac{< x, y, z >}{\rho^3}\right) = -\Big[\frac{1}{\rho^3}\nabla \cdot (< x, y, z >) + < x, y, z > \cdot \nabla\Big(\frac{1}{\rho^3}\Big)\Big]$$

$$= -\Big[\frac{3}{\rho^3} + < x, y, z > \cdot -\frac{3 < x, y, z >}{\rho^5}\Big] = -\Big[\frac{3}{\rho^3} - \frac{3(x^2 + y^2 + z^2)}{\rho^5}\Big]$$

$$= -\Big[\frac{3}{\rho^3} - \frac{3\rho^2}{\rho^5}\Big] = 0$$

b.

Let $\hat{n}$ denotes the Unit Vector Perpendicular to the Surface of the Sphere i.e. a Vector pointing Radially Outward. This gives us

$$\hat{n} = \frac{< x, y, z >}{\rho}$$

On the Surface of the Sphere, every Point is at a Distance a from the Origin i.e. $\rho = a$

Therefore on the Surface of the Sphere, we get

$$F = -\frac{< x, y, z >}{a^3} \text{ and } \hat{n} = \frac{< x, y, z >}{a}$$

$$\implies F \cdot \hat{n} = -\frac{< x, y, z >}{a^3} \cdot \frac{< x, y, z >}{a} = -\frac{1}{a^2}$$

Therefore Flux of F through the Sphere is

$$\oiint_S F \cdot \hat{n} dS \text{ here } dS \text{ is Surface Element}$$

$$= \oiint_S -\frac{1}{a^2} dS = -\frac{1}{a^2} \oiint_S dS = -\frac{1}{a^2} 4\pi a^2$$

Hence,

Flux of F through the Sphere is -4π

Divergence Theorem is not violated here because the Theorem itself is not applicable here. For Divergence Theorem to be applicable, F must be smooth everywhere in the Volume bounded by S. But F has a Singularity at $(0, 0, 0)$

c.

Although $\nabla \cdot F = 0$ for $r \neq 0$, the flux through any closed surface enclosing the origin is nonzero.

Let S_C be a Sphere of Radius ϵ surrounding the Origin and S be any Closed Surface Surrounding the Origin.
$\epsilon \to 0$
Let V be the Volume Enclosed between S and S_C.

Therefore, the Modified Divergence Theorem is Given by

$$\oiint_S F \cdot \hat{n} dS = \iiint_V \nabla \cdot (F) dV + \oiint_{S_C} F \cdot \hat{n} dS_C$$

$$= \iiint_V 0 dV - \frac{1}{\epsilon^2} \oiint_{S_C} dS_C = 0 - \frac{1}{\epsilon^2} 4\pi\epsilon^2$$

Hence, flux of F through any closed surface surrounding the origin is

$$-4\pi$$