MIT 18.02SC Problem Set 10

Nilangshu Sarkar

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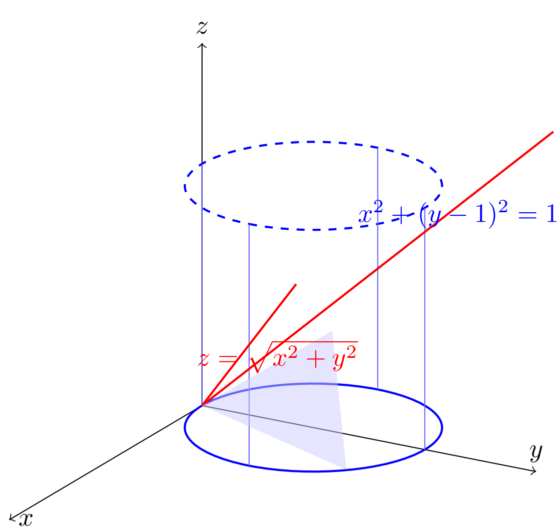
Part 2

Problem 1

Problem 1 (6: 3,3)
Let $\mathcal{G}$ be the solid region in 3-space which lies inside the surface $x^2 + (y-1)^2 = 1$, above $z = 0$, and below the surface $z = \sqrt{x^2 + y^2}$.
a) Find the volume of $\mathcal{G}$.
b) Find the z-coordinate of the centroid of $\mathcal{G}$.

Solution :-

The surfaces are plotted as follows:-



a.

The Volume is given by the Triple Integral

$$\int_y \int_x \int_z dzdxdy$$

Note that:

$$0 \leq z \leq \sqrt{x^2 + y^2}$$

So our Triple Integral becomes

$$\int_y \int_x \int_0^{\sqrt{x^2+y^2}} dzdxdy = \int_y \int_x \left(\sqrt{x^2 + y^2}\right) dxdy$$

Converting to Cylindrical Polar Coordinate, we get

$$x = r\cos(\theta) \text{ and } y = r\sin(\theta);$$

$$x^2 + (y-1)^2 \leq 1 \implies (r\cos(\theta))^2 + (r\sin(\theta) - 1)^2 \leq 1 \implies r^2 - 2r\sin(\theta) \leq 0$$

$$\implies 0 \leq r \leq 2\sin(\theta);$$

$$0 \leq \theta \leq \pi$$

Therefore our Integral becomes

$$\int_{\theta=0}^{\pi} \int_{r=0}^{2\sin(\theta)} \left(\sqrt{(r\cos(\theta))^2 + (r\sin(\theta))^2}\right) rdrd\theta = \int_{\theta=0}^{\pi} \int_{r=0}^{2\sin(\theta)} r^2 drd\theta = \frac{32}{9}$$

Hence

$$\textbf{Volume } V = \frac{32}{9}$$

b.

$$\bar{z} = \frac{1}{V} \int_{\theta=0}^{\pi} \int_{r=0}^{2\sin(\theta)} \int_{z=0}^{\sqrt{(r\cos(\theta))^2 + (r\sin(\theta))^2}} zdzrdrd\theta$$

$$= \frac{1}{2V} \int_{\theta=0}^{\pi} \int_{r=0}^{2\sin(\theta)} r^3 drd\theta = \frac{27\pi}{128}$$

Therefore

$$\textbf{Z-Coordinate of the Centroid is } \frac{27\pi}{128}$$