MIT 18.02SC Problem Set 9

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Part 2

Problem 1

Problem 1 (4: 2,2)

$\mathbf{F}(x, y) = (y^3 - 6y)\,\mathbf{i} + (6x - x^3)\,\mathbf{j}$.

a) Using Green's Theorem, find the simple closed curve C for which the integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ (with positive orientation) will have the largest positive value.

b) Compute this largest positive value.

Solution :-

a.

$$F = (y^3 - 6y)\hat{i} + (6x - x^3)\hat{j}$$

$$\nabla \times F = \frac{\partial}{\partial x}(6x - x^3) - \frac{\partial}{\partial y}(y^3 - 6y) = -3(x^2 + y^2 - 4)$$

$$\oint_C F \cdot dr = \iint_R (\nabla \times F) dA = \iint_R -3(x^2 + y^2 - 4)dxdy$$

Now in order to search for a Curve C for the Integral $\oint_C F \cdot dr$ to be maximum, we need to find the Region R where the Double Integral $\iint_R -3(x^2+y^2-4)dxdy$ is maximum.

Note that the value of $-3(x^2 + y^2 - 4)$ Decreases with Increase of both x and y.

So, we can set our region to be the one where $-3(x^2 + y^2 - 4)$ is positive. This gives us

$$-3(x^2 + y^2 - 4) \geq 0 \implies x^2 + y^2 \leq 2^2 \text{ ...this gives our required } R$$

The C bounding this R is the

Circle $x^2 + y^2 = 2^2$

b.

Converting to Polar Coordinate, we get

$$\iint_{x^2+y^2\leq 2^2} -3(x^2+y^2-4)dxdy = \int_{\theta=0}^{2\pi}\int_{r=0}^{2} -3\Big((r\cos(\theta))^2+(r\sin(\theta))^2-4\Big)rdrd\theta$$

Hence our Integral computes out to be

$$\int_{\theta=0}^{2\pi}\int_{r=0}^{2} -3\Big((r\cos(\theta))^2+(r\sin(\theta))^2-4\Big)rdrd\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{2} -3\Big(r^3-4r\Big)drd\theta = 24\pi$$

Therefore, the **Largest Positive Value is** 24π

Problem 2

Problem 2 (6: 2,2,2)

In the reading V4.2 (pp.1-3) it is shown that in the context of 2D fluid flows, Green's theorem in normal form combined with the principle of conservation of mass imply that $\mathrm{div}(\mathbf{F})$, the divergence of the flow field $\mathbf{F}(x, y)$, represents the (signed) rate of mass per unit time per unit area which originates at the point (x, y), or the source or sink rate for short. This extends to non-steady flows $\mathbf{F}(x, y, t)$, and leads directly to the *Equation of Continuity* for fluid flows, which is the statement of conservation of mass and hence one of the basic physical principles of fluid dynamics. We'll continue to use ρ for the density (instead of δ used in the Notes).

The divergence of a vector field $\mathbf{F}(x, y, t)$ in this context is defined with respect to the space variables only, that is, if $\mathbf{F}(x, y, t) = \langle M(x, y, t), N(x, y, t)\rangle$ is a smooth vector field, then $\mathrm{div}(\mathbf{F}) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$.

Then for the case of a flow field $\mathbf{F}(x, y, t) = \rho(x, y, t)\,\mathbf{v}(x, y, t)$ with density $\rho(x, y, t)$ and velocity $\mathbf{v}(x, y, t)$, the equation of continuity reads

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\mathbf{F}) \;=\; 0.$$

Note that for *steady* flows, which by definition means $\rho = \rho(x, y)$ and $\mathbf{v} = \mathbf{v}(x, y)$, the equation of continuity holds if and only if $\mathrm{div}(\mathbf{F}) = 0$. Thus conservation of mass for steady flows is equivalent to the absence of any sources or sinks, which makes sense.

a) For non-steady flows, assuming that the physical interpretation of $\mathrm{div}(\mathbf{F})$ is the same as in the case of steady flows (at each time t), explain why the equation of continuity is in fact the statement of conservation of mass.

Hint: take an arbitrary bounded region $\mathcal{R}$ and integrate both terms of the continuity equation over $\mathcal{R}$. Then use Green's theorem in normal form.

b) Let $g(x, y, t)$ be a smooth scalar function, and again define the gradient of $g(x, y, t)$ in this case to be with respect to just the space variables: $\nabla g = \langle g_x,\, g_y\rangle$. Then if and $\mathbf{G}(x, y, t) = \langle M(x, y, t), N(x, y, t)\rangle$ is a smooth vector field, use the product rule to show that

$$\mathrm{div}\,(\mathrm{g}\;\mathbf{G}) \;=\; \mathrm{g}\;\mathrm{div}(\mathbf{G}) \;+\; \mathbf{G}\cdot\nabla \mathrm{g}$$

c) Refer to the definition of the convective derivative $\dfrac{Df}{Dt}$ given in p-set 5 #2, and the definition of incompressibility for flows $\dfrac{D\rho}{Dt} = 0$, as given in p-set 5 #3.

Combining: the equation of continuity; the result of part(b) above; and the result of p-set 5 #2, show that the flow $\mathbf{F}(x, y, t) = \rho(x, y, t)\,\mathbf{v}(x, y, t)$ is incompressible if and only if

$$\mathrm{div}(\mathbf{v}) \;=\; 0.$$

This is thus an equivalent condition for the incompressibility of a flow.

Solution :-

a.

Equation of Continuity :-

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (F) = 0$$

$$\implies \iint_R \frac{\partial \rho}{\partial t} dA + \iint_R \nabla \cdot (F) dA = \iint_R 0 dA = 0 \text{ [Taking Double Integral on both Sides]}$$

$$\implies \frac{\partial}{\partial t} \iint_R (\rho) dA + \oint_C F \cdot \hat{n} ds = 0 \text{ [By Green's Theorem (Normal Form), } \oint_C F \cdot \hat{n} ds = \iint_R \nabla \cdot (F) dA]$$

$$\implies \frac{\partial M}{\partial t} + \oint_C F \cdot \hat{n} ds = 0 \text{ [Since } \iint_R (\rho) dA = M]$$

Note:-

- $$\frac{\partial M}{\partial t}$$
  denotes the Mass Increment with respect to Time at each Position.
- $$\oint_C F \cdot \hat{n} ds$$
  denotes the Flux of Flow at each position i.e. positive outward.
- Sum of these two factors equals Zero implies that Mass is Conserved.

Hence proved that the **Equation of Continuity implies Law of Conservation of Mass**.

b.

$$\nabla \cdot (gG) = \nabla \cdot (gM + gN) = \frac{\partial}{\partial x}(gM) + \frac{\partial}{\partial y}(gN)$$

$$= g_x M + g M_x + g_y N + g N_y = g(M_x + N_y) + (g_x M + g_y N)$$

$$= g(M_x + N_y) + \nabla(g) \cdot < M, N >= g\nabla \cdot (G) + \nabla(g) \cdot (G)$$

Hence proved,

$$\nabla \cdot (gG) = g\nabla \cdot (G) + \nabla(g) \cdot (G)$$

c.

The Expression obtained in p-set 5

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + v \cdot \nabla f$$

$$\implies \frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + v \cdot \nabla \rho = -\nabla \cdot (F) + v \cdot \nabla(\rho) \text{ [By Equation of Continuity]}$$

$$= -\nabla \cdot (\rho\, v) + v \cdot \nabla(\rho) \text{ [Given } F = \rho\, v\text{]}$$

$$= -\rho \nabla \cdot (v) - \nabla(\rho) \cdot (v) + v \cdot \nabla(\rho) \text{ [From result of part b.]}$$

$$= -\rho \nabla \cdot (v)$$

Therefore,

$$\frac{D\rho}{Dt} = 0 \implies -\rho \nabla \cdot (v) = 0$$

Hence proved that for a Fluid to be incompressible

$$\nabla \cdot (v) = 0$$