MIT 18.03 IF

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Part 2

2.

Problem 2: [Solutions to linear equations] Almost all the radon in the world today was created within the past week or so by a chain of radioactive decays beginning mainly from uranium, which has been part of the earth since it was formed. This cascade of decaying elements is quite common, and in this problem we study a "toy model" in which the numbers work out decently. This is about Tatooine, a small world endowed with unusual elements.

A certain isotope of Startium, symbol St, decays with a half-life t_S. Strangely enough, it decays with equal probability into a certain isotope of either Midium, Mi, or into the little known stable element Endium. Midium is also radioactive, and decays with half-life t_M into Endium. All the St was in the star-stuff that condensed into Tatooine, and all the Mi and En arise from the decay route described. Also, $t_M \neq t_S$.

Use the notation $x(t)$, $y(t)$, and $z(t)$, for the amount of St, Mi, and En on Tatooine, in units so that $x(0) = 1$. Also, assume $y(0) = 0$ and $z(0) = 0$.

(a) Make rough sketches of graphs of x, y, z, as functions of t. What are the limiting values as $t \to \infty$?

(b) Write down the differential equations controlling x, y, and z. Be sure to express the constants that occur in these equations correctly in terms of the relevant decay constants. Use the notation σ (Greek letter sigma) for the decay constant for St and μ (Greek letter mu) for the decay constant for Mi. Your first step is to relate σ to t_S and μ to t_M. A check on your answers: the sum $x + y + z$ is constant, and so we should have $\dot{x} + \dot{y} + \dot{z} = 0$.

(c) Solve these equations, successively, for x, y, and z.

(d) At what time does the quantity of Midium peak? (This will depend upon σ and μ.)

(e) Suppose that instead of $x(0) = 1$, we had $x(0) = 2$. What change will this make to $x(t)$, $y(t)$, and $z(t)$?

(f) Unrelated question: Suppose that $x(t) = e^t$ is a solution to the differential equation $t\dot{x} + 2x = q(t)$. What is $q(t)$? What is the general solution?

b.

$$\dot{x} = -\sigma x$$
$$\dot{y} = -\mu y + \sigma \frac{x}{2}$$
$$\dot{z} = +\sigma \frac{x}{2} + \mu y$$

We can see that,

$$\dot{x} + \dot{y} + \dot{z} = 0$$

Relating, σ to t_S

$$\frac{x(0)}{2} = x(0)e^{-\sigma t_S}$$

$$\implies t_S = \frac{\ln 2}{\sigma}$$

Similarly,

$$t_M = \frac{\ln 2}{\mu}$$

c.

$$\dot{x} = -\sigma x \implies x = x(0)e^{-\sigma t}$$

$$\implies x = e^{-\sigma t}$$

$$\dot{y} = -\mu y + \sigma \frac{x}{2} = -\mu y + \sigma \frac{(e^{-\sigma t})}{2} \implies \dot{y} + \mu y = \frac{\sigma}{2} e^{-\sigma t}$$
$$\implies y e^{\mu t} = \int \frac{\sigma}{2} e^{-\sigma t} e^{\mu t} dt$$
$$\implies y = \frac{\sigma e^{-\sigma t}}{2(\mu - \sigma)} + c e^{-\mu t}$$

Using $y(0) = 0$

$$y = \frac{\sigma(e^{-\sigma t} - e^{-\mu t})}{2(\mu - \sigma)}$$

$$\dot{z} = +\sigma \frac{x}{2} + \mu y \implies \dot{z} = +\sigma \frac{(e^{-\sigma t})}{2} + \mu(\frac{\sigma(e^{-\sigma t} - e^{-\mu t})}{2(\mu - \sigma)})$$
$$\dot{z} = \frac{2\mu\sigma - \sigma^2}{2(\mu - \sigma)} e^{-\sigma t} - \frac{\mu\sigma}{2(\mu - \sigma)} e^{-\mu t}$$
$$z = \frac{2\mu - \sigma}{2(\sigma - \mu)} e^{-\sigma t} - \frac{\sigma}{2(\sigma - \mu)} e^{-\mu t} + c$$

Using $z(0) = 0$

$$z = \frac{2\mu - \sigma}{2(\sigma - \mu)} e^{-\sigma t} - \frac{\sigma}{2(\sigma - \mu)} e^{-\mu t} + 1$$

d.

$$y = \frac{\sigma(e^{-\sigma t} - e^{-\mu t})}{2(\mu - \sigma)}$$

$$\implies \dot{y} = \frac{\sigma}{2(\mu - \sigma)} \left(-\sigma e^{-\sigma t} + \mu e^{-\mu t}\right)$$

For maximum amount of medium,

$$\dot{y} = 0 \implies \mu e^{-\mu t} = \sigma e^{-\sigma t}$$

$$\implies e^{(\sigma - \mu)t} = \frac{\sigma}{\mu}$$

$$t = \frac{1}{\sigma - \mu} \ln\left(\frac{\sigma}{\mu}\right)$$