MIT 18.03 Numerical Methods

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Part 2

1.

Problem 1: [Euler's method] **(a)** Write y for the solution to $y' = 2x$ with $y(0) = 0$. What is $y(1)$? What is the Euler approximation for $y(1)$, using 2 equal steps? 3 equal steps? What about n steps, where n can now be any natural number? (It will be useful to know that $1 + 2 + \cdots + (n-1) = n(n-1)/2$.) As $n \to \infty$, these approximations should converge to $y(1)$. Do they?

(b) In the text and in class it was claimed that for small h, Euler's method for stepsize h has an error which is at most proportional to h. The n-step approximation for $y(1)$ has $h = 1/n$. What is the exact value of the difference between $y(1)$ and the n-step Euler approximation? Does this conform to the prediction?

a.

$$y' = 2x \implies \frac{dy}{dx} = 2x \implies y = x^2 + c$$

Since $y(0) = 0$, we have

$$c = 0$$

Therefore,

$$y = x^2 \implies y(1) = 1$$

The Euler Approximation Formula gives us

$$x_{n+1} = x_n + h \implies x_n = x_{n-1} + h$$

$$y_{n+1} = y_n + hy'_n = y_n + 2hx_n \implies y_n = y_{n-1} + 2hx_{n-1}$$

2-Step Euler Approximation

$$h = \frac{1-0}{2} = 0.5$$

n	$x_n = x_{n-1} + h$	$y_n = y_{n-1} + hy'_{n-1}$	$y'_n = 2x_n$	hy'_n
0	$x_0 = 0$	$y_0 = 0$	0	0
1	0.5	0	1	0.5
2	1	0.5		

So this gives us

$$y(1) \approx 0.5$$

3-Step Euler Approximation

$$h = \frac{1-0}{3} = \frac{1}{3}$$

n	$x_n = x_{n-1} + h$	$y_n = y_{n-1} + hy'_{n-1}$	$y'_n = 2x_n$	hy'_n
0	$x_0 = 0$	$y_0 = 0$	0	0
1	1/3	0	2/3	2/9
2	2/3	2/9	4/3	4/9
3	1	2/3		

So this gives us

$$y(1) \approx 2/3 \approx 0.667$$

n-Step Euler Approximation

$$h = \frac{1-0}{n} = \frac{1}{n}$$

Manipulating the Euler Approximation formulas, we get

$$x_{n+1} = x_n + h \implies x_n = x_{n-1} + h = x_{n-1} + \frac{1}{n}$$

$$\implies x_n = x_{n-1} + \frac{1}{n}$$

$$\implies x_r = x_0 + \frac{r}{n}$$

$$\Sigma_{r=0}^{n-1} x_r = (n-1)x_0 + \frac{1}{n}\frac{n(n-1)}{2} = (n-1)(x_0 + 0.5)$$

$$y_{n+1} = y_n + hy'_n = y_n + 2hx_n \implies y_n = y_{n-1} + 2hx_{n-1} = y_{n-1} + \frac{2}{n}x_{n-1}$$

$$\implies y_n = y_{n-1} + \frac{2}{n}x_{n-1}$$

Consider the following additions for the sake simplifications:

$$\begin{aligned} (y_1 = y_0 + \frac{2}{n}x_0) \\ +(y_2 = y_1 + \frac{2}{n}x_1) \\ +(y_3 = y_2 + \frac{2}{n}x_2) \\ \cdots \\ +(y_n = y_{n-1} + \frac{2}{n}x_{n-1}) \end{aligned}$$

$$\implies y_n = y_0 + \frac{2}{n}\Sigma_{r=0}^{n-1}x_r = y_0 + \frac{2}{n}(n-1)(x_0 + 0.5)$$

Putting $x_0 = 0$ and $y_0 = 0$, we get

$$y_n = \frac{n-1}{n}$$

So if $n \to \infty$,

$$y_n = 1$$

Hence, the value converges to $y(1)$

b.

Error

$$e = y(1) - y_n = 1 - \frac{n-1}{n} = \frac{1}{n}$$

Here, the error is equal to h. So obviously proportional to h