

MIT 18.03 Gain and Lag Phase

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Part 1

1.

Problem 1: a) Find the periodic solution to

$$x''' + x = 2 \cos t$$

in amplitude-phase form.

b) What is the gain and the phase lag ?

a.

Let

$$\begin{aligned} D &= \frac{d}{dt} \implies Dx = \frac{dx}{dt} = x' \\ D^3x &= x''' \end{aligned}$$

Periodic Solution is given by

$$\begin{aligned} x_p &= \left\{ \frac{1}{D^3 + 1} \right\} (2 \cos(t)) \\ &= \left\{ \frac{1}{D^3 + 1} \right\} (\operatorname{Re}[2e^{it}]) \\ &= \operatorname{Re} \left[\left\{ \frac{1}{D^3 + 1} \right\} (2e^{it}) \right] \\ &= \operatorname{Re} \left[\frac{2e^{it}}{-i + 1} \right] = \operatorname{Re} \left[\frac{2(1+i)e^{it}}{2} \right] = \operatorname{Re} \left[(1+i)e^{it} \right] \\ &= \cos(t) - \sin(t) = \sqrt{2} \cos(t - \arctan(1)) \end{aligned}$$

Therefore the Periodic Solution in Amplitude-Phase form is given by

$$x_p = \sqrt{2} \cos\left(t - \frac{\pi}{4}\right)$$

b.

$$\text{Gain} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$\text{Phase Lag} = \frac{\pi}{4}$$