MIT 18.03 First Order Autonomous Equation

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Part 2

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Problem 1: **[Autonomous Equations]** *This problem will use the Mathlet* `Phase Lines`*. Open the applet and understand its use and conventions. Click on [Phase Line] to see a representation of the phase line. Note the color coding: a green dot represents a stable or attracting equilibrium; a red dot represents an unstable or repelling equilibrium; and a blue dot represents a "semi-stable" equilibrium.*

The Kenyan government has a game preserve that, in the absence of hunting, supports an oryx population that follows the logistic equation with a stable population of one kilo-oryx (one thousand animals). Kenya wishes to investigate the effect of various hunting rates a on the oryx population.

(a) *This situation is well modeled by the top menu item in* `Phase Lines`*. Explain why this is a good model. The rest of this problem will use this equation.*

(b) *It appears that there is a pair of equilibria for some values of a, only one for at least one other value of a, and none for still other values of a. Calculate which values of a behave in which way. For each a find the critical values of y and in each case say whether the critical points involved are stable, unstable, or semi-stable.*

(c) *The Kenyan government hopes to allow 187.5 oryx to be killed in an average year. Determine the resulting stable population. If this strategy is adopted, what is the critical oryx population below which the population will crash (if the same harvest rate continues to be allowed)?*

(d) *For this value of a, there are five different behaviors possible for the oryx population. (Two solutions exhibit the "same behavior" if one is a time-translate of the other). Sketch one solution of each of the five types. Your sketch should make it clear what the behavior of the solution is as t gets small and as t gets large. Match each one up with a portion of the phase line.*

(e) *Invoke the Bifurcation Diagram for this autonomous equation. Move a along its slider to see the variety of behaviors of the phase line of as a varies. The green and red curve in the newly displayed bifurcation plane represents the equilibrium points for those equations for various values of a. Give an equation for that curve.*

a.

In the absence of hunting, the oryx population is well described by the logistic equation

$$\dot{y} = k_0\, y\left(1 - \frac{y}{p}\right),$$

where k_0 is the intrinsic growth rate and p is the carrying capacity. The problem states that the preserve supports a stable population of one kilo-oryx, so we take $p = 1$.

Including a constant hunting rate a, the population model becomes

$$\dot{y} = k_0\, y(1 - y) - a.$$

The Phase Line top-menu equation used in this problem is

$$\dot{y} = y(1 - y) - a,$$

which corresponds to the choice $k_0 = 1$. This is simply a rescaling of time and does not alter the qualitative behavior of the solutions.

Thus the equation is a standard logistic growth model with a constant harvesting term a, making it an appropriate model for studying the effect of hunting on the oryx population.

b.

Equilibria are obtained for values of y where

$$\dot{y} = y(1 - y) - a = 0 \implies -y^2 + y - a = 0$$

So there will pair of Equilibria for 2 solutions for the above equations, one equilibria for 1 solution and no equilibria for 0 Real solution.

Therefore,
Pair Equilibrium :-

$$(1)^2 - 4(-1)(-a) > 0 \implies a < 0.25$$

Critical Values of y are

$$y = \frac{-1 \pm \sqrt{(1)^2 - 4(-1)(-a)}}{2(-1)} = \frac{1 \pm \sqrt{1 - 4a}}{2}$$

- Below $\frac{1-\sqrt{1-4a}}{2}$ the Solution curves are Decreasing. Above it they are Increasing. So, this is an Unstable Critical Point.
- Below $\frac{1+\sqrt{1-4a}}{2}$ the Solution curves are Increasing. Above it they are Decreasing. So, this is an Stable Critical Point.

One Equilibrium :-

$$(1)^2 - 4(-1)(-a) = 0 \implies a = 0.25$$

Critical Value of y is

$$y = \frac{1}{2}$$

- Solution curves on both above and below this is Decreasing. So, this is a Semi-Stable Critical Point.

No Equilibrium :-

$$(1)^2 - 4(-1)(-a) < 0 \implies a > 0.25$$

c.

$$187.5 \implies a = 0.1875$$

This gives the Critical values to be

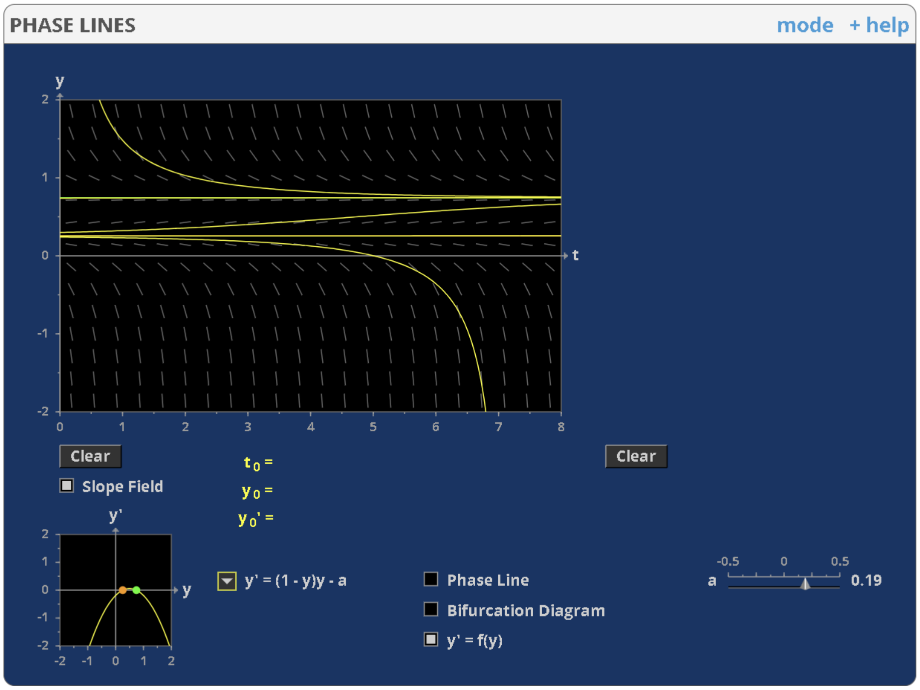
$$y = \frac{1 \pm \sqrt{1 - 4(0.1875)}}{2} = \frac{1}{4}\text{and}\frac{3}{4}$$

Here,

- The Stable Population now corresponds to $\frac{3}{4} \times 1000$ i.e. 750 Oryx.
- Critical Oryx Population below which the Population will Crash at this Harvest Rate is allowed comes out to be $\frac{1}{4} \times 1000$ i.e. 250 Oryx.

d.
The five distinct behaviors are:

- If $y(0) < \frac{1}{4}$, the population decreases monotonically toward extinction.
- If $y(0) = \frac{1}{4}$, the population remains constant (unstable equilibrium).
- If $\frac{1}{4} < y(0) < \frac{3}{4}$, the population increases toward $\frac{3}{4}$.
- If $y(0) = \frac{3}{4}$, the population remains constant (stable equilibrium).
- If $y(0) > \frac{3}{4}$, the population decreases toward $\frac{3}{4}$.



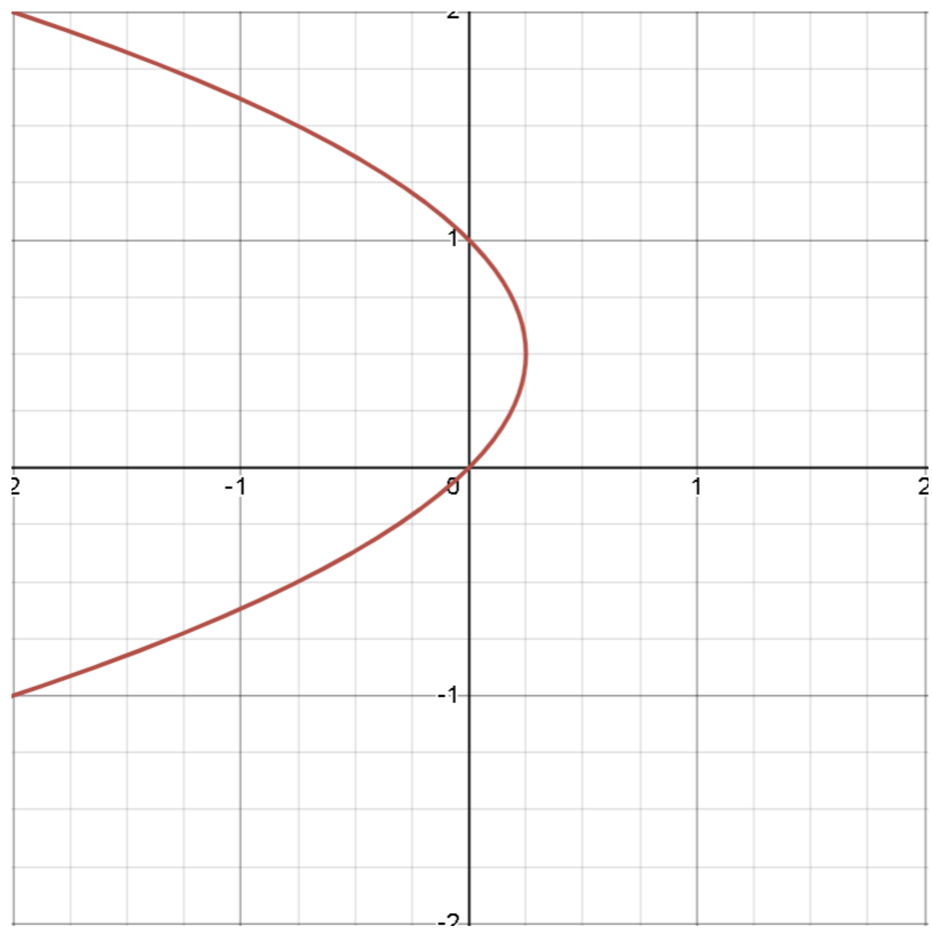
e.

Equilibria satisfy

$$y(1-y)-a=0 \quad \Longrightarrow \quad a=y(1-y)=y-y^2.$$

Thus the Bifurcation Diagram in the (a, y)-plane is given by the curve

$$a=y-y^2$$



Equivalently, solving for y as a function of a,

$$y^2 - y + a = 0 \quad \Longrightarrow \quad y(a) = \frac{1 \pm \sqrt{1-4a}}{2}, \qquad a \leq \frac{1}{4},$$

with the two branches meeting at $a = \frac{1}{4}$, $y = \frac{1}{2}$.