MIT 18.03 Geometric Methods

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29th November 2025

Part 2

1.

Problem 1: *[Direction fields, isoclines] In this problem you will study solutions of the differential equation*

$$\frac{dy}{dx} = y^2 - x\,.$$

Solutions of this equation do not admit expressions in terms of the standard functions of calculus, but we can study them anyway using the direction field.

(a) *Draw a large pair of axes and mark off units from -4 to $+4$ on both. Sketch the direction field given by our equation. Do this by first sketching the isoclines for slopes $m = -1$, $m = 0$, $m = 1$, and $m = 2$. On this same graph, sketch, as best you can, a couple of solutions, using just the information given by these four isoclines.*

Having done this, you will continue to investigate this equation using one of the Mathlets. So invoke `http://math.mit.edu/mathlets/mathlets` *in a web browser and select* `Isoclines` *from the menu. (To run the applet from this window, click the little black box with a white triangle inside.) Play around with this applet for a little while. The Mathlets have many features in common, and once you get used to one it will be quicker to learn how to operate the next one. Clicking on "Help" pops up a window with a brief description of the applet's functionalities.*

Select from the pull-down menu our differential equation $y' = y^2 - x$. Move the m slider to $m = -2$ and release it; the $m = -2$ isocline is drawn. Do the same for $m = 0$, $m = 1$, and $m = 2$. Compare with your sketches. Then depress the mousekey over the graphing window and drag it around; you see a variety of solutions. How do they compare with what you drew earlier?

(b) *A separatrix is a curve such that above it solutions behave (as x increases) in one way, while below it solutions behave (as x increases) in quite a different way. There is a separatrix for this equation such that solutions above it grow without bound (as x increases) while solutions below it eventually decrease (as x increases). Use the applet to find its graph, and submit a sketch of your result.*

(c) *Suppose $y(x)$ is a solution to this differential equation whose graph is tangent to the $m = -1$ isocline: it touches the $m = -1$ isocline at a point (a, b), and the two curves have the same slope at that point. Find this point on the applet, and then calculuate the values of a and b.*

(d) *Now suppose that $y(x)$ is a solution to the equation for which $y(a) < b$, where (a, b) is the point you found in* **(c)**. *What happens to it as $x \to \infty$? I claim that its graph is asymptotic to the graph of $f(x) = -\sqrt{x}$. Explain why this is so. For large x, is $y(x) > f(x)$, $y(x) < f(x)$, or does the answer depend on the value of $y(a)$?*

The following observations will be useful in justifying your claims. Please explain as clearly as you can why each is true.
(i) The graph of $y(x)$ can't cross the $m = -1$ isocline at a point (x, y) with $x > a$.
(ii) If $c > a$ and $y(c)$ lies above the nullcline, then the graph of $y(x)$ continues to lie above the nullcline for all $x > c$.

(iii) If $c > a$ and $y(c)$ lies below the nullcline, then the graph of $y(x)$ will cross the nullcline for some $x > c$.

(e) *Suppose a solution $y(x)$ has a critical point at (c, d)—that is, $y'(c) = 0$ and $y(c) = d$. What can you say about the relationship between c and d? The applet can be very helpful here, but verify your answer.*

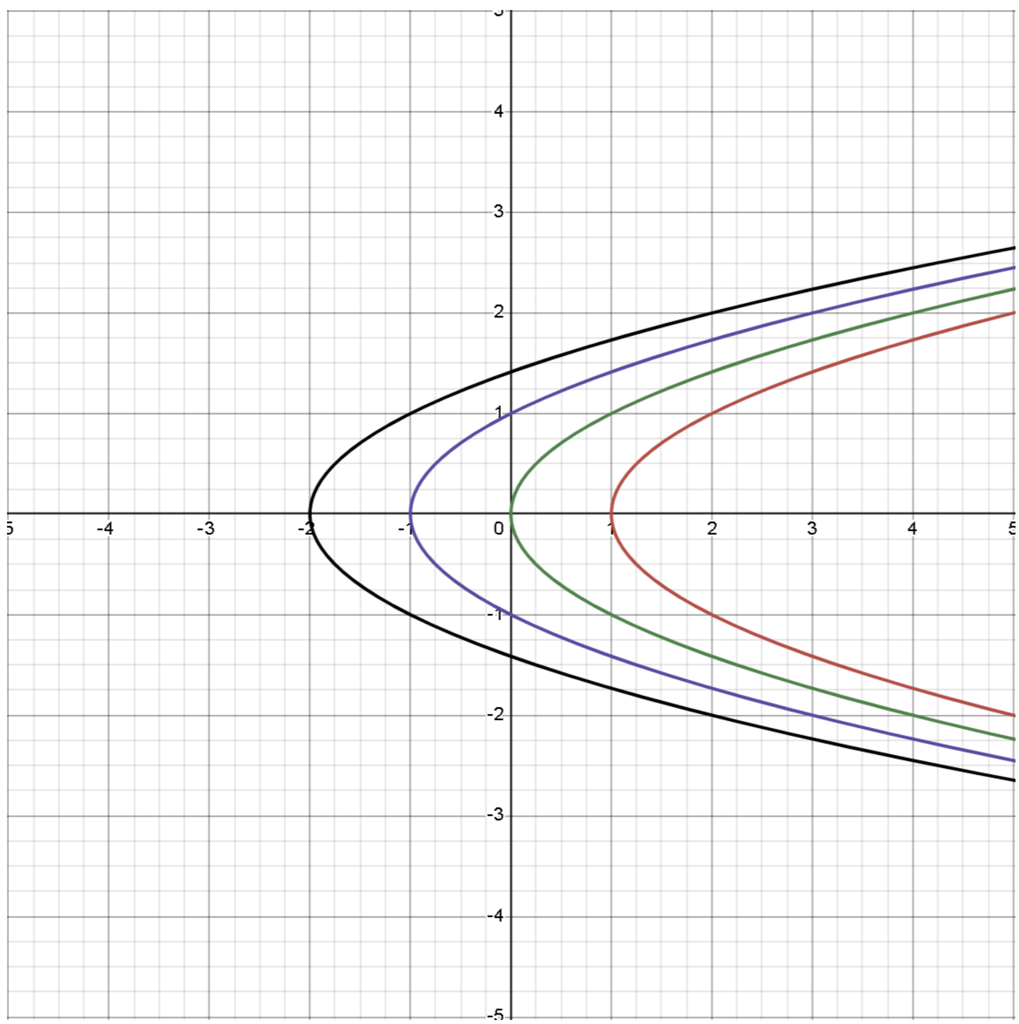
(f) *It appears from the applet that all critical points are local maxima. Is that true?*

a.
Starting with $y' = m$, we get

$$y^2 - x = m$$

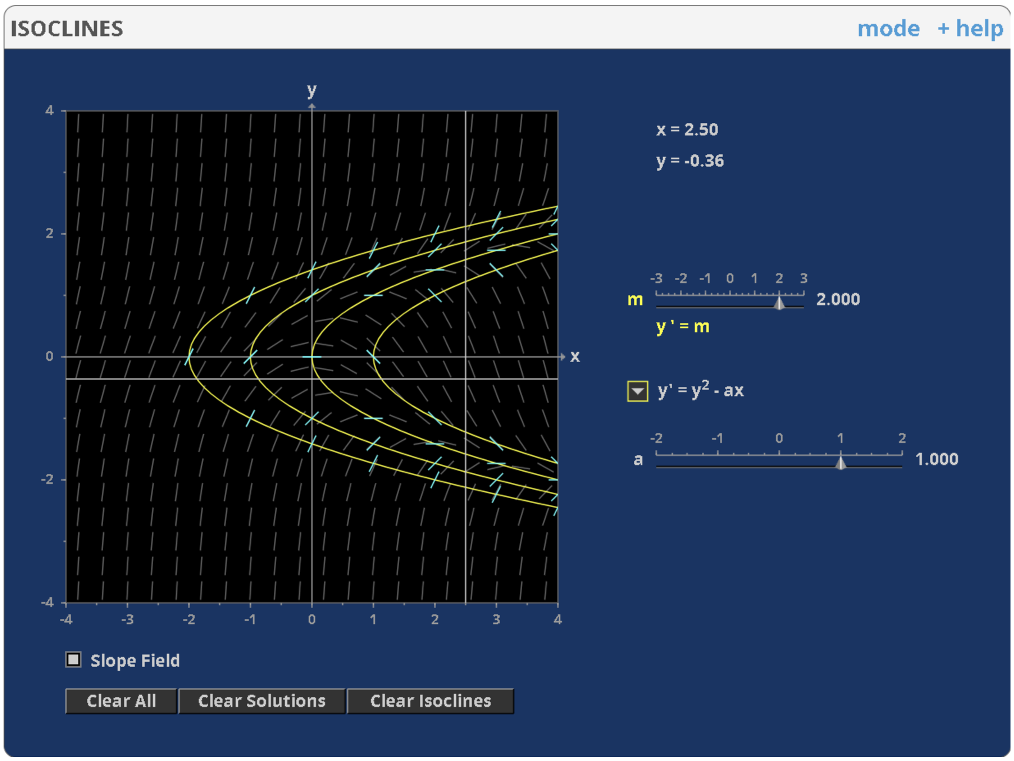
$$y^2 = x + m$$

So for various m we get various Parabolas. Their Sketch is given below:



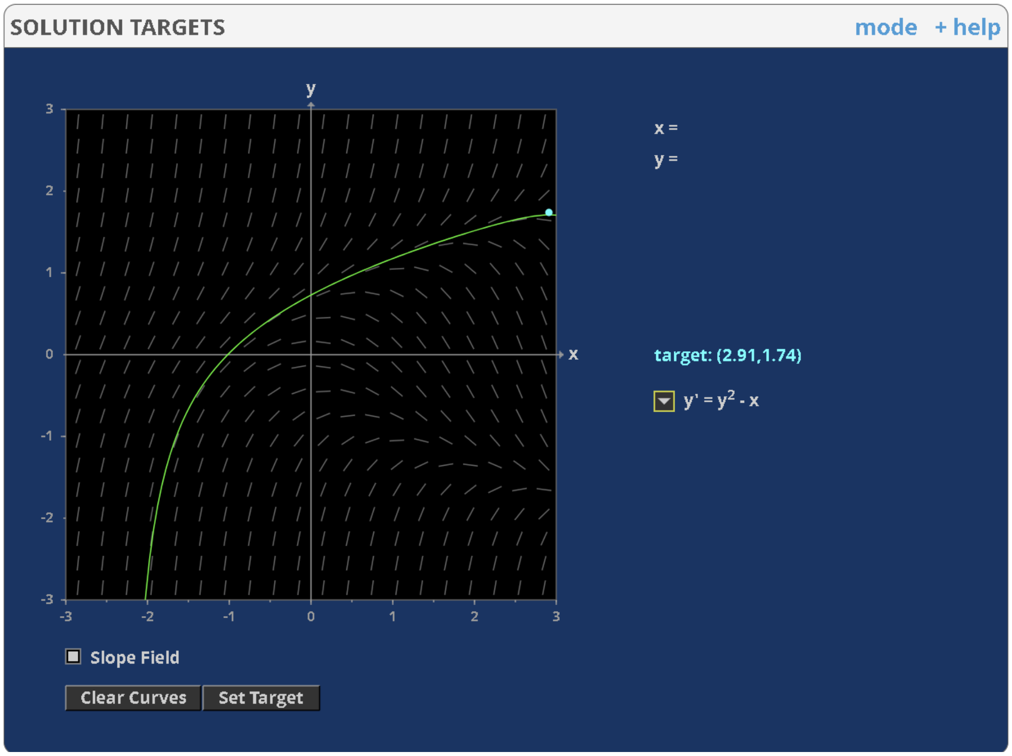
Here,

m	Colour
-1	Red
0	Green
+1	Purple
+2	Black



This exactly matches the Mathlet result.
This proves our Calculations were correct.

b.



c.

The easiest way to find two values a and b is to search for two independent algebric equations among them.

The first equation is obtained by the fact that (a, b) lies on $y^2 - x = -1$ i.e.

$$-a + b^2 + 1 = 0$$

Now let us consider a Function such that we represent the Isocline as (Function)=0. So,

$$F(x, y) = -x + y^2 + 1$$

So we get,

- On the Isocline $F(x, y) = 0$
- On one side of the Isocline $F(x, y) > 0$
- On another side of the Isocline $F(x, y) < 0$

So the solution curve should be such that when we move by it, $F(x, y)$ does not changes its sign at (a, b) i.e. it can be neither be increasing nor it can be decreasing, cause increasing or decreasing at (a, b) would imply that the solution curve crosses $F(x, y)$. So,

$$\frac{d}{dx}F(x,y)_{(x,y)=(a,b)} = 0$$

$$=> [\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x}]_{(x,y)=(a,b)} = 0$$

$$[-1 + 2yy']_{(x,y)=(a,b)} = 0$$

$$[-1 + 2y(y^2 - x)]_{(x,y)=(a,b)} = 0$$

$$-2ab + 2b^3 - 1 = 0$$

Solving,

$$-a + b^2 + 1 = 0 \text{ and } -2ab + 2b^3 - 1 = 0$$

we get,

$$a = \frac{5}{4} \textbf{ and } b = -\frac{1}{2}$$

So,

$$(a, b) = (\frac{5}{4}, -\frac{1}{2})$$

d.

Let $I_{-1} = \{(x, y) : y^2 - x = -1\}$ be the $m = -1$ isocline and $N = \{(x, y) : y^2 - x = 0\}$ the nullcline. Suppose $y(x)$ is a solution with $y(a) < b$ (so it starts below I_{-1}).

(i) If y first met I_{-1} at (x_1, y_1) then, since the curve started below I_{-1}, the crossing would be from below and the solution's tangent there would have to be strictly steeper (larger) than the tangent to the isocline. But the slope of the solution at any point of I_{-1} equals $y^2 - x = -1$, whereas the tangent slope of the isocline is $(dy/dx)_{I_{-1}} = 1/(2y)$, which (except at the unique tangency point (a, b) found in (c)) does not equal -1. Thus a first crossing from below leads to a contradiction, so the solution cannot cross I_{-1}.

(ii) If y first met the nullcline N at (x_2, y_2) then the solution slope at that point would be $y_2^2 - x_2 = 0$, whereas the slope of the nullcline curve is $(dy/dx)_N = 1/(2y_2) < 0$ on the lower branch. Crossing from above would require the solution to have slope more negative than the nullcline, contradicting $y' = 0$ at the crossing. Thus y cannot cross N in that manner either.

(iii) While y remains below N we have $y^2 - x < 0$, hence $y' > 0$, so the solution increases. The lower branch $y = -\sqrt{x}$ decreases with x, so an increasing solution below it must intersect the nullcline; equivalently, if the solution did not cross the nullcline it would attain a maximum off N, but every maximum occurs on N, contradiction. Hence the solution must eventually cross N.

Combining (i)–(iii) we see the solution (which started below I_{-1}) must meet the nullcline without ever having crossed I_{-1}; after this meeting it is steered into the attracting region of the lower nullcline branch. Setting $u = y + \sqrt{x}$ one obtains

$$u' = -2\sqrt{x}\,u + u^2 + \frac{1}{2\sqrt{x}},$$

By taking the threshold function $T(x) = 1/\sqrt{x}$ and comparing u' with $-2\sqrt{x}\,u$, one sees that $u(x) > T(x)$ makes $u' < 0$ and $u(x) < -T(x)$ makes $u' > 0$ for all large x, so the solution is forced into the tube $|u(x)| \le 1/\sqrt{x}$. Since $1/\sqrt{x} \to 0$, it follows that $u(x) \to 0$ and hence $y(x) \sim -\sqrt{x}$ as $x \to \infty$.

e.

We get our Equation as

$$y'(c) = 0 \text{ or } d^2 - c = 0$$

So the relation between c and d is given by

$$c = d^2$$

f.

Yes. Cause the y' changes sign from -ve to +ve as we move across the Isoline of $y' = 0$ from left to right.