MIT 18.03 Linear Operator

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Part 1

3.

Problem 3: Find a particular solution to the DE

$$y^{(4)} - 2y'' + y = xe^x$$

Let

$$D = \frac{d}{dt} \implies Dx = \frac{dx}{dt} = x' \implies D^2x = x'' \& D^4x = x''''$$

So the given Equation becomes

$$(D^4 - 2D^2 + 1)y = xe^x$$

Let

$$(D^4 - 2D^2 + 1) = L(D) \implies L(D)y = xe^x$$

Therefore Particular Solution is given by

$$y_P = \left\{\frac{1}{L(D)}\right\}\left(xe^x\right) = e^x\left\{\frac{1}{L(D+1)}\right\}\left(x\right)$$

$$= e^x\left\{\frac{1}{(D+1)^4 - 2(D+1)^2 + 1}\right\}\left(x\right) = e^x\left\{\frac{1}{((D+1)^2 - 1)^2}\right\}\left(x\right)$$

$$= e^x\left\{(D^2 + 2D)^{-2}\right\}\left(x\right)$$

Let

$$u = \left\{(D^2 + 2D)^{-2}\right\}\left(x\right) \implies \left\{(D^2 + 2D)^2\right\}u = x$$

For (D^2+2D) we can set our Solution as (Ax^2+Bx+C)
Here (D^2+2D) applied twice.
So the expression of u will be of the form

$$u = (1+x)(Ax^2+Bx+C) = (A)x^3 + (A+B)x^2 + (B+C)x + (C)$$

Therefore we get

$$Du = (3A)x^2 + (2A+2B)x + (B+C)$$
$$\implies 2Du = (6A)x^2 + (4A+4B)x + (2B+2C)$$
$$D^2u = (6A)x + (2A+2B)$$

Let

$$v = (D^2+2D)u = (6A)x^2 + (10A+4B)x + (2A+4B+2C)$$

So

$$Dv = (12A)x + (10A+4B)$$
$$\implies 2Dv = (24A)x + (20A+8B)$$
$$D^2v = (12A)$$

This gives us

$$(D^2+2D)v = (24A)x + (32A+8B)$$

But

$$(D^2+2D)v = x$$
$$\implies (24A)x + (32A+8B) = x$$
$$\implies A = \frac{1}{24} \text{ and } B = -\frac{1}{6}$$

Since C is arbitrary, we can set $C=0$ for a Particular solution. In case any I.C.s were given, we could formulate a General Solution where the C.F. could have compensated for the C. So setting $C=0$ will cause no trouble at all.

Hence,

$$u = (1+x)\left(\frac{x^2}{24} - \frac{x}{6}\right)$$

Therefore, The Particular Solution is given by

$$y_P = e^x(1+x)\left(\frac{x^2}{24} - \frac{x}{6}\right)$$