MIT 18.03 First Order System

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2nd Order Linear ODE to a System of 1st Order Linear ODEs

Problem

Write the following equation as equivalent first-order systems.

$$y'' - x^2 y' + (1 - x^2)y = \sin(x)$$

Solution :-

$$y'' = x^2 y' + (x^2 - 1)y + \sin(x)$$

Let

$$y_1 = y \text{ and } y_2 = y'$$

Hence the Equivalent System of 1st Order Equations is :-

$$y_1' = y_2$$

$$y_2' = (x^2 - 1)y_1 + x^2 y_2 + \sin(x)$$

In Matrix Notation we can write it as

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ (x^2 - 1) & x^2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \sin(x) \end{bmatrix}$$

Matrix Method

Problem

Solve $X' = AX$ where A is

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ -2 & 1 & -1 \end{bmatrix}$$

Solution :-

Characteristic equation is given by

$$\det(A - \lambda I) = 0$$

$$\implies \begin{vmatrix} 1-\lambda & -1 & 0 \\ 1 & 2-\lambda & 1 \\ -2 & 1 & -1-\lambda \end{vmatrix} = 0$$

$$\implies -\lambda^3 + 2\lambda^2 + \lambda - 2 = 0$$

$$\implies (\lambda - 1)(\lambda - 2)(\lambda + 1) = 0$$

Hence the eigenvalues are

$$\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = -1$$

Eigenvector for $\lambda = 1$

$$(A - I)v = 0$$

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 1 & 1 \\ -2 & 1 & -2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\implies a_2 = 0, \quad a_1 = -a_3$$

Choosing $a_3 = 1$,

$$v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Eigenvector for $\lambda = 2$

$$(A - 2I)v = 0$$

$$\begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & 1 \\ -2 & 1 & -3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\implies a_2 = -a_1, \quad a_3 = -a_1$$

Choosing $a_1 = 1$,

$$v_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

Eigenvector for $\lambda = -1$

$$(A+I)v = 0$$

$$\begin{bmatrix} 2 & -1 & 0 \\ 1 & 3 & 1 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\implies a_2 = 2a_1, \quad a_3 = -7a_1$$

Choosing $a_1 = 1$,

$$v_3 = \begin{bmatrix} 1 \\ 2 \\ -7 \end{bmatrix}$$

General Solution

Hence the general solution of $X' = AX$ is

$$X(t) = c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 1 \\ 2 \\ -7 \end{bmatrix} e^{-t}$$

Matrix Exponential

Problem

(a) We have seen that a complex number $z = a + bi$ determines a matrix $A(z)$ in the following way: $A(a+bi) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. This matrix represents the operation of multiplication by z, in the sense that if $z(x+yi) = v + wi$ then $A(z)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} v \\ w \end{bmatrix}$. What is $e^{A(z)t}$? What is $A(e^{zt})$?

(b) Say that a pair of solutions $x_1(t), x_2(t)$ of the equation $m\ddot{x} + b\dot{x} + kx = 0$ is *normalized at* $t = 0$ if:

$$x_1(0) = 1, \quad \dot{x}_1(0) = 0$$
$$x_2(0) = 0, \quad \dot{x}_2(0) = 1$$

For example, find the normalized pair of solutions to $\ddot{x} + 2\dot{x} + 2x = 0$. Then find e^{At} where A is the companion matrix for the operator $D^2 + 2D + 2I$.

(c) Suppose that $e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $e^{2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ satisfy the equation $\dot{\mathbf{u}} = A\mathbf{u}$.

(i) Find solutions $\mathbf{u_1}(t)$ and $\mathbf{u_2}(t)$ such that $\mathbf{u_1}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{u_2}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(ii) Find e^{At}.

(iii) Find A.

Solution

(a) Let $z = a + bi$ with $a, b \in \mathbb{R}$. The associated matrix is

$$A(z) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

This matrix represents multiplication by z on $\mathbb{C} \cong \mathbb{R}^2$
Since

$$e^{A(z)t} = \sum_{n=0}^{\infty} \frac{(A(z)t)^n}{n!}$$

and $A(z)$ represents multiplication by z, it follows that $e^{A(z)t}$ represents multiplication by e^{zt}

Hence,

$$\boxed{e^{A(z)t} = \begin{pmatrix} e^{at}\cos(bt) & -e^{at}\sin(bt) \\ e^{at}\sin(bt) & e^{at}\cos(bt) \end{pmatrix}}$$

On the other hand,

$$e^{zt} = e^{(a+bi)t} = e^{at}(\cos bt + i \sin bt)$$

so

$$\boxed{A(e^{zt}) = \begin{pmatrix} e^{at}\cos(bt) & -e^{at}\sin(bt) \\ e^{at}\sin(bt) & e^{at}\cos(bt) \end{pmatrix}}$$

Thus,

$$\boxed{e^{A(z)t} = A(e^{zt})}$$

(b) Consider the equation

$$\ddot{x} + 2\dot{x} + 2x = 0$$

The characteristic equation is

$$r^2 + 2r + 2 = 0$$

with roots

$$r = -1 \pm i$$

Thus, a general solution is

$$x(t) = e^{-t}\left(C_1 \cos t + C_2 \sin t\right)$$

First solution. Impose $x_1(0) = 1$, $\dot{x}_1(0) = 0$

From $x_1(0) = 1$, we get $C_1 = 1$ Compute

$$\dot{x}(t) = e^{-t}\big[-C_1 \cos t - C_2 \sin t - C_1 \sin t + C_2 \cos t\big]$$

Thus

$$\dot{x}_1(0) = -C_1 + C_2 = 0 \implies C_2 = 1$$

Hence

$$\boxed{x_1(t) = e^{-t}(\cos t + \sin t)}$$

Second solution. Impose $x_2(0) = 0$, $\dot{x}_2(0) = 1$.
From $x_2(0) = 0$, we get $C_1 = 0$. Then

$$\dot{x}_2(0) = C_2 = 1$$

Hence

$$\boxed{x_2(t) = e^{-t} \sin t}$$

The companion matrix for $D^2 + 2D + 2I$ is

$$\boxed{A = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix}}$$

The fundamental matrix is

$$e^{At} = \begin{pmatrix} x_1(t) & x_2(t) \\ \dot{x}_1(t) & \dot{x}_2(t) \end{pmatrix}$$

Thus,

$$\boxed{e^{At} = e^{-t} \begin{pmatrix} \cos t + \sin t & \sin t \\ -\sin t & \cos t - \sin t \end{pmatrix}}$$

(c) We are given that

$$u_1(t) = e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad u_2(t) = e^{2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

satisfy $\dot{u} = Au$
(i) Let

$$X(t) = \begin{pmatrix} u_1(t) & u_2(t) \end{pmatrix} = \begin{pmatrix} e^{3t} & e^{2t} \\ e^{3t} & 2e^{2t} \end{pmatrix}$$

Then

$$X(0) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad X(0)^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

The solutions satisfying

$$u_1(0) = \begin{pmatrix}1\\0\end{pmatrix}, \quad u_2(0) = \begin{pmatrix}0\\1\end{pmatrix}$$

are the columns of

$$\boxed{X(t)X(0)^{-1} = \begin{pmatrix}2e^{3t} - e^{2t} & -e^{3t} + e^{2t}\\ 2e^{3t} - 2e^{2t} & -e^{3t} + 2e^{2t}\end{pmatrix}}$$

(ii) Therefore,

$$\boxed{e^{At} = X(t)X(0)^{-1} = \begin{pmatrix}2e^{3t} - e^{2t} & -e^{3t} + e^{2t}\\ 2e^{3t} - 2e^{2t} & -e^{3t} + 2e^{2t}\end{pmatrix}}$$

(iii) Finally,

$$A = \frac{d}{dt}e^{At}\bigg|_{t=0}$$

Differentiating and evaluating at $t = 0$ gives

$$\boxed{A = \begin{pmatrix}4 & -1\\ 2 & -1\end{pmatrix}}$$

Non-Linear Equation & Linearization

Problem

Problem 1: For the 2×2 autonomous system

$$\begin{aligned} x' &= x - 2y + \tfrac{1}{4}x^2 \\ y' &= 5x - y - y^2 \end{aligned} \; :$$

(a) Find the critical points.
Note: you will get a quartic polynomial; to help you solve it we'll tell you that one root is 0 and another is a positive integer no larger than 5. There are only two critical points, but you'll need to find the other two roots of this polynomial and show they don't give critical points.

(b) Find the linearized system at each critical point. Then carry out the proceedure described in the session on Linearization, culminating in one sketch which includes the trajectories near each of the critical points, and a *guess* at how they all fit together.

Problem 2: Classify as structurally stable or not stable each of the critical points found in the previous problem. Based on this what can you say for sure about the behavior of the trajectories near each critical point? How does this relate to your hand 'guess-sketch'?

Solution

1(a)

Critical points satisfy

$$x' = 0, \qquad y' = 0$$

Thus we solve

$$x - 2y + \frac{x^2}{4} = 0 \qquad 5x - y - y^2 = 0$$

From the second equation,

$$5x = y + y^2 \quad \Longrightarrow \quad x = \frac{y + y^2}{5}$$

Substituting into the first equation gives

$$\frac{y + y^2}{5} - 2y + \frac{1}{4}\left(\frac{y + y^2}{5}\right)^2 = 0$$

Multiplying through by 100 to clear denominators,

$$20(y + y^2) - 200y + (y + y^2)^2 = 0$$

Expanding,

$$y^4 + 2y^3 + 21y^2 - 180y = 0$$

or

$$y\big(y^3 + 2y^2 + 21y - 180\big) = 0$$

Hence $y = 0$ is one solution. Testing positive integers no larger than 5, we find that $y = 4$ is also a root. Factoring,

$$y^3 + 2y^2 + 21y - 180 = (y - 4)(y^2 + 6y + 45)$$

and since $y^2 + 6y + 45$ has negative discriminant, it has no real roots.

Therefore the only real solutions are

$$y = 0 \quad \text{and} \quad y = 4$$

The corresponding x-values are:

$$y = 0 \;\Rightarrow\; x = 0 \qquad y = 4 \;\Rightarrow\; x = 4$$

$$\boxed{\textbf{The critical points are } (0,0) \textbf{ and } (4,4)}$$

1(b)

We linearize the system

$$\begin{cases} x' = x - 2y + \dfrac{1}{4}x^2, \\ y' = 5x - y - y^2 \end{cases}$$

about each critical point.

Jacobian Matrix
The Jacobian is

$$J(x,y) = \begin{pmatrix} \dfrac{\partial x'}{\partial x} & \dfrac{\partial x'}{\partial y} \\ \dfrac{\partial y'}{\partial x} & \dfrac{\partial y'}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 + \dfrac{1}{2}x & -2 \\ 5 & -1 - 2y \end{pmatrix}$$

Linearization at $(0,0)$
Evaluating the Jacobian at $(0,0)$,

$$J(0,0) = \begin{pmatrix} 1 & -2 \\ 5 & -1 \end{pmatrix}.$$

The characteristic equation is

$$\det(J - \lambda I) = \begin{vmatrix} 1-\lambda & -2 \\ 5 & -1-\lambda \end{vmatrix} = \lambda^2 + 9 = 0.$$

Hence,

$$\lambda = \pm 3i.$$

The linearized system has purely imaginary eigenvalues, so the origin is a *center* for the linearization. Linear theory alone is inconclusive about stability, but nearby trajectories are closed curves to first order.

Linearization at $(4,4)$
Evaluating the Jacobian at $(4,4)$,

$$J(4,4) = \begin{pmatrix} 3 & -2 \\ 5 & -9 \end{pmatrix}.$$

The characteristic equation is

$$\det(J - \lambda I) = \begin{vmatrix} 3-\lambda & -2 \\ 5 & -9-\lambda \end{vmatrix} = \lambda^2 + 6\lambda - 17 = 0.$$

Solving,

$$\lambda = -3 \pm \sqrt{26}.$$

Since one eigenvalue is positive and the other is negative, $(4,4)$ is a *saddle point*.

Qualitative Phase Portrait

- Near $(0,0)$: the linearization predicts closed trajectories (center-type behavior). Linearization does not determine stability, but motion is locally oscillatory.
- Near $(4,4)$: the saddle has one stable and one unstable direction, with trajectories approaching along the stable manifold and repelled along the unstable manifold.

Combining these local behaviors suggests a global phase portrait in which trajectories circulate around the origin while being strongly influenced by the saddle structure near $(4,4)$.

$$\boxed{\textbf{The origin is a center (linearization), and } (4,4) \textbf{ is a saddle.}}$$

2

We classify the critical points found in Problem 1 according to structural stability.

Critical point $(0,0)$

From Problem 1(b), the Jacobian at $(0,0)$ has eigenvalues

$$\lambda = \pm 3i.$$

Since the eigenvalues are purely imaginary, the linearization predicts a center. Centers are *not structurally stable*, because arbitrarily small nonlinear perturbations can change the qualitative behavior (for example, into a spiral sink or source).

Thus, no definite conclusion can be drawn from linearization alone about the true nonlinear behavior near $(0,0)$.

Critical point $(4,4)$

At $(4,4)$, the Jacobian has eigenvalues

$$\lambda = -3 \pm \sqrt{26},$$

one positive and one negative.

Hence $(4,4)$ is a saddle point. Saddles are *structurally stable*, since small perturbations do not change the qualitative nature of the phase portrait.

Therefore, the behavior near $(4,4)$ is robust: trajectories approach along the stable manifold and are repelled along the unstable manifold.

Conclusion

- The critical point $(0, 0)$ is *not structurally stable.* - The critical point $(4, 4)$ is *structurally stable.*

This agrees with the qualitative sketch in Problem 1(b): the saddle behavior near $(4, 4)$ is reliable, while the behavior near the origin may change under small perturbations.