MIT 18.03 ODE's with Periodic Input and Fourier Transform

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Part 2

1.

Problem 1: [Periodic solutions] Let $g(t)$ be the function which is periodic of period 2π, and such that $g(t) = t$ for $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$ and $g(t) = \pi - t$ for $\frac{\pi}{2} \le t \le \frac{3\pi}{2}$.

(a) Find a periodic solution to $\ddot{x} + \omega_0^2 x = g(t)$ (if there is one).

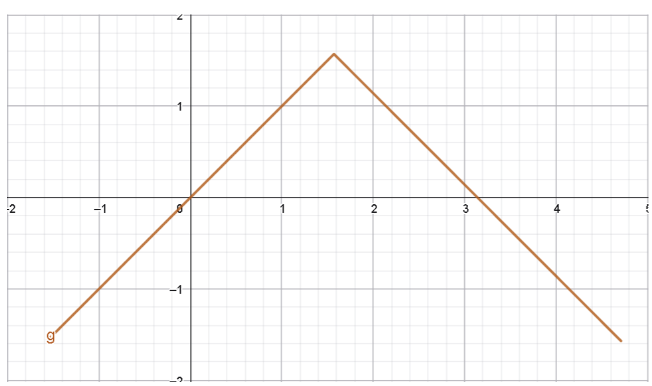
(b) For what (positive) values of ω_0 are there no periodic solution?

(c) Write ω_r for the smallest number you found in **(b)**. For ω_0 just less than ω_r, what is the solution like, approximately? How about for ω_0 just larger than ω_r?

(d) For what values of ω_0 are there more than one periodic solution?

(e) For the values of ω_0 found in **(d)**, are *all* solutions to $\ddot{x} + \omega_0^2 x = g(t)$ periodic?

First we have to find the corresponding Fourier Series of $g(t)$
$g(t)$ is having a plot of:



If we take the periodic version of $g(t)$ we get $g(t_0 + 2\pi) = g(t_0)$
In this question, based on given data, we can set $t_0 = -\frac{\pi}{2}$

We are given

$$g(t) = \begin{cases} t & \text{if } -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \\ \pi - t & \text{if } \frac{\pi}{2} \leq t \leq \frac{3\pi}{2} \end{cases}$$

Therefore in the Fourier Series of $g(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos(n\omega t) + b_n \sin(n\omega t) \Big]$ we get

$$\omega = \frac{\pi}{\pi} = 1$$

$$a_n = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{-\frac{\pi}{2}+2\pi} (g(t)\cos(n\omega t))dt = \frac{1}{\pi}\Big[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} t\cos(n\omega t))dt + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (\pi - t)\cos(n\omega t))dt \Big]$$

$$= \frac{1}{\pi}\Big[0 + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (\pi - t)\cos(nt)dt \Big]$$

$$\implies a_n = 0$$

$$b_n = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{-\frac{\pi}{2}+2\pi} (g(t)\sin(n\omega t))dt = \frac{1}{\pi}\Big[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} t\sin(n\omega t))dt + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (\pi - t)\sin(n\omega t))dt \Big]$$

$$= \frac{1}{\pi}\Big[2\int_{0}^{\frac{\pi}{2}} t\sin(nt))dt + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (\pi - t)\sin(nt)dt \Big]$$

$$\implies b_n = \frac{2}{\pi}\frac{(1 - (-1)^n)}{n^2}$$

Therefore we get

$$g(t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\Big[\frac{(1 - (-1)^n)}{n^2}\sin(nt) \Big]$$

a.

$$\ddot{x} + \omega_0^2 x = g(t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\Big[\frac{(1 - (-1)^n)}{n^2}\sin(nt) \Big]$$

$$\implies x_P = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(1 - (-1)^n)}{n^2(\omega_0^2 - n^2)}\sin(nt) \right]$$

b.

Note that in the expression of the Solution, there is a term $(1 - (-1)^n)$ in the Numerator. Now

$$(1 - (-1)^n) = 2 \text{ if } n \text{ is Odd}$$

$$(1 - (-1)^n) = 0 \text{ if } n \text{ is Even}$$

Therefore only Odd Harmonics are present in the Solution.

Note that the term $(\omega_0^2 - n^2)$ is present in the Denominator of the Solution. Now

$$(\omega_0^2 - n^2) = 0 \text{ if } \omega_0 = n$$

Hence No Bounded Periodic Solution occur i.e. Resonance occur for every

$$\omega_0 = 2m - 1 \text{ ; } m \in N$$

c.

For values of ω_0 such that $\omega_0 \neq 2m + 1$, $m \in \mathbb{Z}$, the denominator $(\omega_0^2 - n^2)$ never vanishes for the harmonics present in the forcing term. Hence a periodic solution exists.

From part (a), the periodic solution is

$$x_p(t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(1 - (-1)^n)}{n^2(\omega_0^2 - n^2)} \sin(nt).$$

Since $(1-(-1)^n) = 0$ for even n and 2 for odd n, only odd harmonics appear in the response. Thus the solution has the same period as the forcing function $g(t)$.

Moreover, the amplitude of each harmonic is inversely proportional to $n^2(\omega_0^2 - n^2)$, so higher harmonics are increasingly suppressed. Therefore, for all non-resonant values of ω_0, the system admits a bounded periodic solution dominated by the lower odd harmonics.

d.

Resonance occurs when one of the natural frequencies of the system coincides with a frequency present in the forcing function. From part (b), the forcing function contains only odd harmonics $n = 2m + 1$.

Therefore, when $\omega_0 = 2m + 1$, the denominator $(\omega_0^2 - n^2)$ vanishes for the corresponding harmonic, causing the amplitude of that mode to grow without bound. As a result, no bounded periodic solution exists and the solution contains a secular (linearly growing) term.

Physically, this means the system absorbs energy continuously from the forcing at that frequency, leading to unbounded oscillations.

e.

Although the natural frequency ω_0 may equal an even integer, resonance does not occur in this case. This is because the forcing function contains only odd harmonics, as shown by the factor $(1 - (-1)^n)$ in its Fourier series.

Since no even harmonics are present in the forcing, the system is never driven at frequencies $n = 2m$. Consequently, even if ω_0 is an even integer, no term in the forcing matches the natural frequency of the system, and resonance does not occur.

Thus resonance occurs only when ω_0 coincides with an odd harmonic of the forcing function.